Math 103X: Multiple integrals^{TW}

Given: Integrand function $f(x,y,z)$, domain of integration D. Evaluate: $\int_D f\,dD$, $\int\int_D f\,dD$ or $\int\int\int_D f\,dD$.

1) Pick coordinate system: xyz (Rectangular), $r\theta z$ (Cylindrical) or $\rho\theta\phi$ (Spherical), or uvw (General Jacobian),

domain element dD becomes: $dA = d\text{Area}$ $dV = d\text{Volume}$ $ds = d\text{Arclength}$ $dS = d\text{SurfaceArea}$

2) Sketch the domain, write down the boundaries of domain to get the limits of integration.
3) Pick the order of integration, do the iterated integrals, starting from the innermost, one variable at a time.

Rectangular	Cylindrical	Spherical	General
$x = x$	$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$x = x(u,v,w)$
$y = y$	$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$y = y(u,v,w)$
$z = z$	$z = z$	$z = \rho\cos\phi$	$z = z(u,v,w)$

Area (2D) $\vec{\mathbf{T}}(u,v) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ $x = x(u,v),\ y = y(u,v)$

$$dA = J\,du\,dv \qquad J = \left|\frac{\partial\vec{\mathbf{T}}}{\partial u}\times\frac{\partial\vec{\mathbf{T}}}{\partial v}\right| = \left\|\begin{matrix} x_u & y_u \\ x_v & y_v\end{matrix}\right\|$$

Rectangular Area $dA = dy\,dx = dx\,dy$

Polar Area $dA = r\,dr\,d\theta$

Volume (3D) $\vec{\mathbf{T}}(u,v,w) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

$x = x(u,v,w),\ y = y(u,v,w),\ z = z(u,v,w)$

$$dV = J\,du\,dv\,dw$$

$$J = \left|\frac{\partial\vec{\mathbf{T}}}{\partial w}\cdot\left(\frac{\partial\vec{\mathbf{T}}}{\partial u}\times\frac{\partial\vec{\mathbf{T}}}{\partial v}\right)\right| = \left\|\begin{matrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w\end{matrix}\right\|$$

Rectangular Volume $dV = dz\,dy\,dx$

Cylindrical Volume $dV = r\,dz\,dr\,d\theta$

Spherical Volume $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Arclength (1D) Parametric curves $\vec{\mathbf{x}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$

$$ds = |\vec{\mathbf{x}}'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

Graph of function: $x = t,\ y = f(x),\ z = 0$

$$ds = \sqrt{1 + f'(x)^2}\,dx$$

Surface Area (2D) Parametric surfaces $\vec{\mathbf{T}}(u,v) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$ [1]

$$dS = \left|\frac{\partial\vec{\mathbf{T}}}{\partial u}\times\frac{\partial\vec{\mathbf{T}}}{\partial v}\right|du\,dv = \left\|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & z_u \\ x_v & y_v & z_v\end{matrix}\right\|du\,dv$$

Graph of function: $x = u, y = v, z = f(x,y)$

$$\vec{\mathbf{T}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + f(x,y)\hat{\mathbf{k}}$$

$$dS = \sqrt{1 + f_x^2 + f_y^2}\,dy\,dx$$

Area in xy plane: $x = u,\ y = v,\ z = 0$

$$dS = dA$$

Circular Cylinder surface, radius $r = a$:

- Using Cylindrical coordinates: $z = u, \theta = v$

$$dS = a\,dz\,d\theta$$

Sphere surface, radius $\rho = a$:

- Using Spherical coordinates: $\theta = u, \phi = v$

$$dS = a^2\sin\phi\,d\phi\,d\theta$$

Cone surface, angle $\phi = \alpha$:

- Using Spherical coordinates: $\theta = u, \rho = v$

$$dS = \rho\sin\alpha\,d\rho\,d\theta$$

- Using Cylindrical coordinates: $\theta = u, r = v, z = r/\tan\alpha$

$$dS = r\csc\alpha\,dr\,d\theta$$

[1]Note: The "||stuff||" in the general formula for dS means "the length of the cross product vector given by the determinant." For the Jacobian in dA and dV the double bars mean the absolute value of the determinant.