The Integral Theorems of Vector Calculus^{TW}

$$f = f(x, y, z) = f(\vec{\mathbf{x}}) \qquad \vec{\mathbf{x}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\vec{\mathbf{F}}(x, y, z) = M(x, y, z)\hat{\mathbf{i}} + N(x, y, z)\hat{\mathbf{j}} + P(x, y, z)\hat{\mathbf{k}}$$

$$\vec{\nabla} = \hat{\mathbf{i}}\partial_x + \hat{\mathbf{j}}\partial_y + \hat{\mathbf{k}}\partial_z$$

$$\text{grad } f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad \text{(vector)}$$

$$\text{div } \vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \qquad \text{(scalar)}$$

$$\text{curl } \vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \qquad \text{(vector)}$$

$$\boxed{\nabla \times \nabla f = \mathbf{0}} \qquad \boxed{\nabla \cdot (\nabla \times \vec{\mathbf{F}}) = 0}$$

Line integrals on parametric curve C: $t : a \to b$

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

$\vec{\mathbf{T}}$ unit Tangent vector, $\vec{\mathbf{n}}$ unit outward normal vector

$$\text{Work} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_C M\, dx + N\, dy + P\, dz = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{x}}(t)) \cdot \vec{\mathbf{x}}'(t)\, dt$$

$$\text{Flux}^{\textbf{(2D only!)}} = \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \oint_C -N\, dx + M\, dy$$

If $\vec{\mathbf{F}}(x, y, z) = (M, N, P)$ is a conservative vector field then a potential function $f(x, y, z)$ exists with

$$\boxed{\vec{\mathbf{F}} = \vec{\nabla} f}$$

Negative test: $\vec{\mathbf{F}}$ is conservative only if

$$\boxed{\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}} \qquad \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 \quad \text{(2D)} \right]$$

Positive test: If the partial integrals of components of $\vec{\mathbf{F}}$ all match, then it gives you f:

$$f(x, y, z) = \int M\, dx = \int N\, dy = \int P\, dz$$

Irrotational: $\vec{\mathbf{F}}$ is conservative and has no bad points

Path independence (**Fund. Thm of Line Integrals**):

if $\vec{\mathbf{F}}$ is irrot., then work on any curve from pt $\vec{\mathbf{A}}$ to pt $\vec{\mathbf{B}}$ is

$$\int_{\vec{\mathbf{A}}}^{\vec{\mathbf{B}}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = f(\vec{\mathbf{B}}) - f(\vec{\mathbf{A}})$$

2D Stokes' theorem (Green's Work Thm) $\vec{\mathbf{F}} = (M, N)$

$$\text{Work} = \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int\int_D (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \hat{\mathbf{k}}\, dA$$

$$\oint_C M\, dx + N\, dy = \int\int_D (N_x - M_y)\, dA$$

2D Divergence theorem (Green's Flux Thm) $\vec{\mathbf{F}} = (M, N)$

$$\text{Flux} = \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \int\int_D \vec{\nabla} \cdot \vec{\mathbf{F}}\, dA$$

$$\oint_C -N\, dx + M\, dy = \int\int_D (M_x + N_y)\, dA$$

D is the region inside C and has no singular points

Key ideas for using the theorems:

- Detours: Replace complicated curves with simpler curves with same endpoints!
- Closed curves: Inside determined by righthand rule: C is counterclockwise, Inside-out: C is clockwise.
- Outward normal for flux: Always pick $\vec{\mathbf{n}}$ pointing to the outside. To swap direction, in/out, $\vec{\mathbf{n}} \to -\vec{\mathbf{n}}$.
- Green's theorem: used to either **Replace** a difficult line integral with an easier double integral or **Replace** a difficult double integral with an easier line integral
- Singular points: they must be "cut out" of your region if you want to use results from the theorems.

Area integrals: Pick coordinates, Jacobian for dA, iterated integral, pick order of integration

Surface integrals: Jacobian, coordinates for dS, order of integration, parametric surfaces $x = x(u, v), y = y(u, v), z = z(u, v)$,

$$\vec{\mathbf{N}} = \frac{\partial \vec{\mathbf{x}}}{\partial u} \times \frac{\partial \vec{\mathbf{x}}}{\partial v} \qquad dS = |\vec{\mathbf{N}}|\, dv\, du$$

$$\text{Flux} = \int\int \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \int\int \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dv\, du$$

3D Stokes' theorem

$$\text{Work} = \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int\int_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS$$

Surface integral: $\vec{\mathbf{n}}$ is the "right-thumb" unit normal for edge curve C, gripped with fingers in the direction of C

Replace a difficult line integral with an easier surface int or

Replace a difficult surface integral with an easier line int

3D Divergence theorem

$$\text{Flux} = \int\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \int\int\int_D \vec{\nabla} \cdot \vec{\mathbf{F}}\, dV$$

Replace a difficult surface integral with an easier triple int or

Replace a difficult triple integral with an easier surface int

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List of unit normal vectors $\vec{\mathbf{n}}$

Parametric surfaces $\quad x = x(u, v), y = y(u, v), z = z(u, v)$

$$\vec{\mathbf{N}} = \vec{\mathbf{x}}_u \times \vec{\mathbf{x}}_v \qquad \longrightarrow \qquad \vec{\mathbf{n}} = \frac{\vec{\mathbf{N}}}{|\vec{\mathbf{N}}|}$$

Roof $z = g(x, y)$

$$\vec{\mathbf{n}} = \frac{(-g_x, -g_y, 1)}{\sqrt{1 + g_x^2 + g_y^2}}$$

Plane: $ax + by + cz = d$

$$\vec{\mathbf{n}} = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

Sphere: $\rho = a$

$$\vec{\mathbf{n}} = \frac{1}{a}(x, y, z)$$

Circular cylinder: $r = a$

$$\vec{\mathbf{n}} = \frac{1}{a}(x, y, 0)$$