Tom's Math 103X Part 1 Review Sheet

Vector algebra

$$\vec{\mathbf{a}} = (a_1, a_2, a_3) = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} = |\vec{\mathbf{a}}|\hat{\mathbf{a}}$$

$$\vec{\mathbf{b}} = (b_1, b_2, b_3) \qquad \vec{\mathbf{c}} = (c_1, c_2, c_3)$$

$$k\vec{\mathbf{a}} = (ka_1, ka_2, ka_3)$$

Length: $|\vec{\mathbf{a}}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}} \geq 0$

Direction: unit vector $\hat{\mathbf{a}} = \frac{1}{|\vec{\mathbf{a}}|}\vec{\mathbf{a}} \qquad |\hat{\mathbf{a}}| = 1$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Vector geometry

"Tip to Tail" graphical vector addition: $\vec{\mathbf{a}} + \vec{\mathbf{b}}$

"Edge vectors" connect vertex positions: $\vec{\mathbf{a}} = \vec{Q} - \vec{P}$

Dot product: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0 \quad \leftrightarrow \quad \vec{\mathbf{a}} \perp \vec{\mathbf{b}} \qquad \theta = 90^\circ$$

Parallel vectors: $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$

$$\vec{\mathbf{a}} \parallel \vec{\mathbf{b}} \quad \leftrightarrow \quad \vec{\mathbf{a}} = k\vec{\mathbf{b}}$$

$$\vec{\mathbf{a}} \parallel \vec{\mathbf{b}} \quad \leftrightarrow \quad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$$

Cross product: $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$

$$(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \perp \vec{\mathbf{a}} \qquad (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \perp \vec{\mathbf{b}}$$

"Right-hand rule" for $\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta \qquad 0^\circ \leq \theta \leq 180^\circ$$

$$\vec{\mathbf{a}} \parallel \vec{\mathbf{b}} \quad \leftrightarrow \quad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}} \qquad \theta = 0^\circ, 180^\circ$$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = \text{area of parallelogram}$$

$$\tfrac{1}{2}|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = \text{area of triangle}$$

$$|\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}})| = \text{volume of parallelepiped}$$

Orthogonal decomposition: $\parallel, \perp$ projections

Components of $\vec{\mathbf{a}}$ parallel and perpendicular to $\vec{\mathbf{b}}$

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_{\parallel} + \vec{\mathbf{a}}_{\perp}$$

$$\vec{\mathbf{a}}_{\parallel} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{b}}|^2}\vec{\mathbf{b}} \qquad \text{then} \qquad \vec{\mathbf{a}}_{\perp} = \vec{\mathbf{a}} - \vec{\mathbf{a}}_{\parallel}$$

$\vec{\mathbf{a}}_{\parallel}$ = parallel or **tangent** to $\vec{\mathbf{b}}$

$\vec{\mathbf{a}}_{\perp}$ = perpendicular or **normal** to $\vec{\mathbf{b}}$

Distance problems: $\perp$ minimum distance can be found between two objects that are separate, they must have a common normal vector $\vec{\mathbf{n}}$. The $\perp$ distance is given by the length of $\vec{\mathbf{d}}_n$, where $\vec{\mathbf{d}} = \vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1$ is the vector between any two points on the two objects:

$$\perp \text{distance} = |\vec{\mathbf{d}}_n| = \frac{|\vec{\mathbf{d}} \cdot \vec{\mathbf{n}}|}{|\vec{\mathbf{n}}|}$$

Planes: $\mathcal{P} \perp \vec{\mathbf{n}} = (a, b, c)$ and contains point $\vec{\mathbf{x}}_0$

$$\vec{\mathbf{n}} \perp (\vec{\mathbf{x}} - \vec{\mathbf{x}}_0) \qquad \vec{\mathbf{v}}, \vec{\mathbf{w}} \parallel \mathcal{P}: \ \vec{\mathbf{n}} = \vec{\mathbf{v}} \times \vec{\mathbf{w}}$$

$$\vec{\mathbf{n}} \cdot (\vec{\mathbf{x}} - \vec{\mathbf{x}}_0) = 0 \qquad \vec{\mathbf{x}} = \vec{\mathbf{x}}_0 + \vec{\mathbf{v}}t + \vec{\mathbf{w}}s$$

Lines: $\ell \parallel \vec{\mathbf{v}} = (a, b, c)$ and contains point $\vec{\mathbf{x}}_0$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad \vec{\mathbf{x}} = \vec{\mathbf{x}}_0 + \vec{\mathbf{v}}t$$

Parametric curves

Position

$$\vec{\mathbf{x}}(t) = (x(t), y(t), z(t))$$

Velocity

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{x}}}{dt} = (x'(t), y'(t), z'(t))$$

Acceleration

$$\vec{\mathbf{a}}(t) = \frac{d^2\vec{\mathbf{x}}}{dt^2} = \frac{d\vec{\mathbf{v}}}{dt} = (x''(t), y''(t), z''(t))$$

Orthogonal decomposition of $\vec{\mathbf{a}}$: $\vec{\mathbf{T}} \cdot \vec{\mathbf{N}} = 0$

$$\vec{\mathbf{a}} = a_T\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$$

unit **Tangent** vector

$$\vec{\mathbf{T}} = \hat{\mathbf{v}} = \frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}}$$

unit **Normal** vector

$$\vec{\mathbf{N}} = \frac{1}{\left|\frac{d\vec{\mathbf{T}}}{dt}\right|}\frac{d\vec{\mathbf{T}}}{dt}$$

unit **Binormal** vector ($\overrightarrow{\text{Up}} = \overrightarrow{\text{Forward}} \times \overrightarrow{\text{Left}}$)

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

Tangential acceleration (speed-up)

$$a_T = \frac{d|\vec{\mathbf{v}}|}{dt} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

Normal acceleration (change of direction): $a_N \geq 0$

$$a_N = \kappa|\vec{\mathbf{v}}|^2 = |\vec{\mathbf{v}}|^2/\rho \qquad a_N\vec{\mathbf{N}} = \vec{\mathbf{a}} - a_T\vec{\mathbf{T}}$$

Curvature κ and Radius of Curvature ρ

$$\kappa = \frac{1}{\rho} = \frac{1}{|\vec{\mathbf{v}}|}\left|\frac{d\vec{\mathbf{T}}}{dt}\right| = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3} = \frac{a_N}{|\vec{\mathbf{v}}|^2}$$