Equilibrium Dynamics and Difference Equations*

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1 Equilibrium Dynamics

When we studied the generic dynamic programming problem in Stokey and Lucas we obtained the following first-order and envelope conditions

$$F_y(x_t, x_{t+1}) + \beta v'(x_{t+1}) = 0 \tag{1}$$

$$v'(x_t) = F_x(x_t, y_{t+1}) \tag{2}$$

We can combine the first-order and envelope conditions to get the Euler equation

$$F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) = 0 \tag{3}$$

We also derived the transversality condition

$$\lim_{t\to\infty} \beta^t F_x(x_t, x_{t+1}) \cdot x_t = 0. \tag{4}$$

In this section we are concerned with the following question: How can we use the equations (1) and (2), and possibly (3) or (4) to characterize the policy function h?

As Stokey and Lucas point out, in general we cannot say much, as the theorem of Boldrin and Montrucchio (see Thm. 6.1 of Stokey and Lucas) states that just about any smooth h can be obtained as the policy function to some dynamic programming problem.

One way to characterize the dynamics induced by h is to use first order approximations to the model in the neighborhood of a steady state. We will take this approach, which is fundamentally local, in subsequent sections. In this section I consider a different approach, which is to establish global stability of equilibrium dynamics around steady states.

Definition: A point $\bar{x}$ is globally stable if for all $x_0 \in X$ and $\{x_t\}_{t=0}^{\infty}$ such that $x_{t+1} = h(x_t)$, $\lim_{t\to\infty} x_t = \bar{x}$.

*Thanks to Rosen Valchev for pointing out a typo in the previous version of these notes.

You can soon see how hard proving global stability will be when you realize that even proving the existence of a unique steady state is difficult. It's quite obvious that a necessary condition for a point to be a steady state is that (3) hold for $x_t = x_{t+1} = x_{t+2} = \bar{x}$. Stokey and Lucas also point out that this is a sufficient condition if F is concave. Uniqueness of the steady state, on the other hand, is more elusive, but is a necessary condition for global stability. One approach to establishing global stability is to use a Liapounov function, if one is available for the problem in question. Lemma 4.2 in Stokey and Lucas shows how these functions could be useful.

Lemma: Let $X \subset R^l$ be compact and let $h : X \to X$ be continuous with $\bar{x} = h(\bar{x})$ for some $\bar{x}$. If there exists a continuous function $L : X \to R$ such that

(a) $L(x) \leq 0$, with equality iff $x = \bar{x}$,

(b) $L[h(x)] \geq L(x)$, with equality iff $x = \bar{x}$,

then $\bar{x}$ is a globally stable solution to (1) and (2).

You should consult the proof of this theorem in Stokey and Lucas. The problem, for many cases, is that it is not obvious how to form L, nor whether an L exists. Stokey and Lucas argue that one way to proceed is to check whether the following form of L satisfies the two conditions of the Lemma:

$$L(x) = (x - \bar{x}) \cdot [v'(x) - v'(\bar{x})]. \tag{5}$$

The reason this form of function is worth trying is that it automatically satisfies the Lemma's condition (a) when v is strictly concave.[1] It's then a matter of trying to verify (b). This turns out to be much harder. Stokey and Lucas motivate it as follows. They begin by noting that for any pair of points (x, y), (x', y') the strict concavity of F implies[2]

$$(x - x') \cdot [F_x(x, y) - F_x(x'y')] + (y - y') \cdot [F_y(x, y) - F_y(x'y')] \leq 0$$

with equality only when the two points are the same. We can therefore let (x, y) be $(x, h(x))$ and (x', y') be $(\bar{x}, \bar{x})$ so that

$$(x - \bar{x}) \cdot \{F_x [x, h(x)] - F_x(\bar{x}, \bar{x})\} + [h(x) - \bar{x}] \cdot \{F_y [x, h(x)] - F_y(\bar{x}, \bar{x})\} \leq 0. \tag{6}$$

The optimality conditions imply that $v'(x) = F_x[x, h(x)]$ and $F_y[x, h(x)] = -\beta v'[h(x)]$, hence

$$(x - \bar{x}) \cdot [v'(x) - v'(\bar{x})] - \beta [h(x) - \bar{x}] \cdot \{v' [h(x)] - v'(\bar{x})\} \leq 0. \tag{7}$$

[1] The strict concavity of v implies that $v'(x) < v'(\bar{x})$ for $x > \bar{x}$ and $v'(x) > v'(\bar{x})$ for $x < \bar{x}$, hence (a) is satisfied.

[2] We will usually need to assume F is concave to prove that v is concave. Also, the result holds for the same reason as given in the previous footnote.

Now notice that

$$L[h(x)] - L(x) = (h(x) - \bar{x}) \cdot [v'(h(x)) - v'(\bar{x})] - (x - \bar{x}) \cdot [v'(x) - v'(\bar{x})].$$

Obviously we could use (7) to show that $L[h(x)] - L(x) \geq 0$ if $\beta = 1$, but it is not. So we cannot get a generic result that will work for any concave F and v, but we are very close. If we could somehow start from

$$\beta(x - \bar{x}) \cdot \{F_x[x, h(x)] - F_x(\bar{x}, \bar{x})\} + (h(x) - \bar{x}) \cdot \{F_y[x, h(x)] - F_y(\bar{x}, \bar{x})\} \leq 0 \qquad (8)$$

instead of (6), we would have the desired result that $L[h(x)] - L(x) \geq 0$. So on a case by case basis one could try to show that (8) holds in order to demonstrate that L given in (5) is a Liapounov function.

In the end, I offer two proofs of the global stability of the dynamics of the neoclassical model. One of these is from Stokey and Lucas, while the other is based on Brock and Mirman (1972). Both proofs start by working out the steady state of the model.

Steady State Here I conjecture that the model has a steady state in which $c_t = c$ and $k_t = k$ for all t. In such a steady state, the Euler equation and resource constraint are

$$u'[g(k_t) - k_{t+1}] = \beta u'[g(k_{t+1}) - k_{t+2}]g'(k_{t+1})$$

$$c_t + k_{t+1} = g(k_t).$$

The steady state in which $k_t = k_{t+1} = k_{t+2} = k^*$ is the solution to

$$1 = \beta g'(k) \qquad (9)$$

$$c = g(k) - k. \qquad (10)$$

We can solve (9) for k^* once we note that $g'(k) = f'(k) + 1 - \delta$. If we rewrite (9) in terms of the rate of time preference, $\rho = \beta^{-1} - 1$ we have $f'(k) = \rho + \delta$. Since $\rho > 0$, $0 < \delta \leq 1$, $f'(k) > 0$, $f''(k) < 0$, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$, there is a unique, positive, finite solution to (9). Once we have solved for k^*, notice that we can easily obtain c^* from (10). We need to verify that the solution for c^* is positive. This requires verifying that $g(k^*) - k^* = f(k^*) - \delta k^* > 0$. Let $\bar{k}$ be the unique point at which $f(\bar{k}) = \delta\bar{k}$. That such a point exists follows from the fact that f is increasing, concave, has infinite slope at 0, and 0 slope at ∞. Obviously $f'(\bar{k}) < \delta$ and the point at which $f'(k) = \delta$ lies to the left of $\bar{k}$. Therefore k^* must also lie between 0 and $\bar{k}$ since $f'(k^*) = \rho + \delta$.

Stokey and Lucas' Proof Under assumption 4.3, 4.4, 4.7 and 4.8 (which are easily shown to be true for the growth model), Theorem 4.8 establishes that v is strictly concave and $h(k)$ is continuous. We also want to show that h is increasing. If you make enough of the regularity assumptions to ensure that the optimal k is always given by an interior solution where the first order condition holds then:

$$u'[g(k) - h(k)] = \beta v'\left[h(k)\right].$$

Pick k_0, $k_1 \in K_+ = (0, \bar{k}]$ with $k_1 > k_0$. We want to prove that $h(k_1) > h(k_0)$. Assume the contrary, so let $h(k_1) \leq h(k_0)$. Then since v is strictly concave

$$u'[g(k_0) - h(k_0)] = \beta v'\left[h(k_0)\right] \leq \beta v'[h(k_1)] = u'[g(k_1) - h(k_1)].$$

And since u is strictly concave this means

$$g(k_0) - h(k_0) \geq g(k_1) - h(k_1).$$

implying

$$g(k_0) - g(k_1) \geq h(k_0) - h(k_1) \geq 0.$$

But since g is strictly increasing this means $k_0 \geq k_1$, which is a contradiction.

With this result we can proceed to finish the proof. For any $k \in K_+$, the strict concavity of v implies

$$[k - h(k)]\left\{v'(k) - v'[h(k)]\right\} \leq 0 \tag{11}$$

with equality only at $k = h(k)$, i.e. only at the steady state.[3] The first order condition of the model states that $u'[g(k) - h(k)] = \beta v'\left[h(k)\right]$ and the envelope condition states that $v'(k) = u'[g(k) - h(k)]g'(k)$. I will use the shorthand $u'(c)$ to indicate $u'[g(k) - h(k)]$. If we substitute these conditions into (11) we get

$$[k - h(k)]\left[u'\left(c\right)g'(k) - u'(c)/\beta\right] \leq 0 \tag{12}$$

Since $u'(c)$ is strictly positive everywhere, we can take it out as a common factor in (12) and preserve the inequality:

$$[k - h(k)]\left[g'(k) - 1/\beta\right] \leq 0, \tag{13}$$

which, again, only holds with equality if $k = h(k)$. So for $k \neq k^*$ we have

$$[k - h(k)]\left[g'(k) - 1/\beta\right] < 0. \tag{14}$$

[3]This looks like the Liapounov function from equation (5), but Stokey and Lucas' proof does not use a Liapounov argument.

Since $g'(k) = \beta^{-1}$ at k^* and g is concave we know that $g'(k) - 1/\beta > 0$ for $k < k^*$ and $g'(k) - 1/\beta < 0$ for $k > k^*$. So if $k < k^*$ this tells us that $h(k) > k$. We also know that if $k > k^*$ then $h(k) < k$. With this result and the fact that $h(k)$ is increasing, Stokey and Lucas are able to draw their Figure 6.1 [equivalent to my Figure 2 below], and this guarantees the global stability of the model's dynamics in K_+.

Brock and Mirman's Proof Take as given the regularity assumptions made in the previous section and the results that v is concave and h is increasing. Next we can show that the policy function for consumption, $\boldsymbol{c}(k_t) = g(k_t) - h(k_t)$ is increasing. Pick some $k_0 < k_1$. Our second result means $h(k_0) < h(k_1)$. This, in turn, implies that $v'[h(k_0)] > v'[h(k_1)]$. And, from the first order condition this means

$$u'[\boldsymbol{c}(k_0)] = u'[g(k_0) - h(x_0)] > u'[g(k_1) - h(k_1)] = u'[\boldsymbol{c}(k_1)]$$

From the concavity of u this means $\boldsymbol{c}(k_0) < \boldsymbol{c}(k_1)$.

Now define

$$\mathcal{A}(k) \equiv u'[\boldsymbol{c}(k)] \tag{15}$$

$$\mathcal{B}(k) \equiv \beta\mathcal{A}(k)g'(k). \tag{16}$$

We need to plot $\mathcal{A}(k)$ and $\mathcal{B}(k)$ against k. To do this notice that $\mathcal{B}(k) > \mathcal{A}(k)$ for $k < k^*$ since $g'(k) > \beta^{-1}$ for $k < k^*$. Similarly, $\mathcal{B}(k^*) = \mathcal{A}(k^*)$ and $\mathcal{B}(k) < \mathcal{A}(k)$ for $k > k^*$. Also, both functions are downward sloping. To see this, notice that

$$\mathcal{A}'(k) = u''[\boldsymbol{c}(k_t)]\boldsymbol{c}'(k_t) < 0$$

since $u'' < 0$ and $\boldsymbol{c}' > 0$. Also

$$\mathcal{B}'(k) = \beta\mathcal{A}'(k)g'(k) + \beta\mathcal{A}(k)g''(k) < 0$$

since $\mathcal{A} > 0$, $\mathcal{A}' < 0$, $g' > 0$ and $g'' < 0$. The nonnegativity constraint on k' given k also implies that $\lim_{k\to 0} \boldsymbol{c}(k) = 0$ so that both functions asymptote towards ∞ as k goes to zero. Thus we can draw $\mathcal{A}(k)$ and $\mathcal{B}(k)$ as in Figure 1.

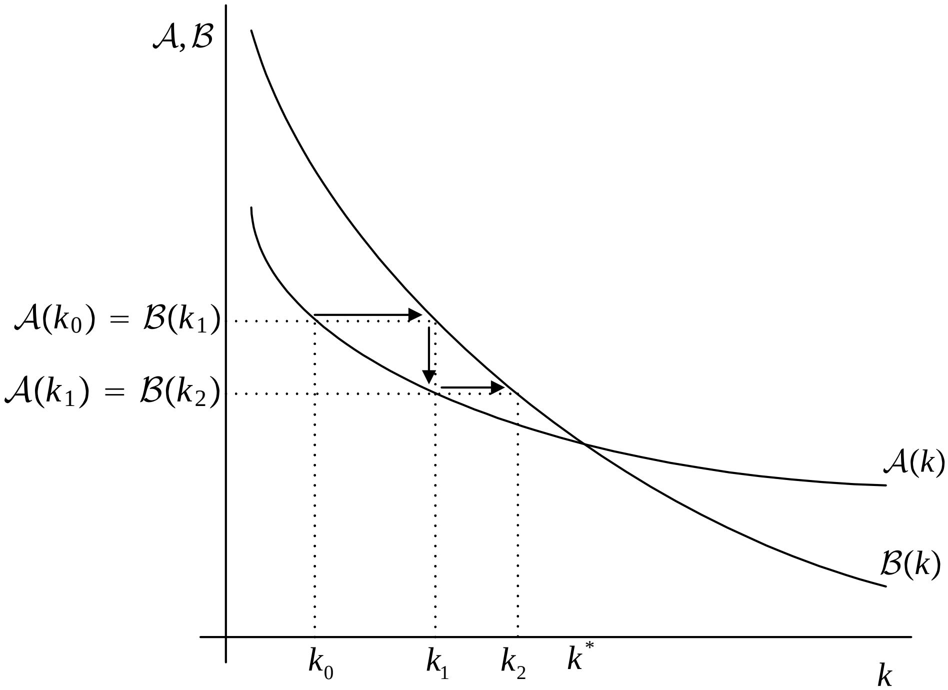
Now consider the dynamics of the capital stock starting from an arbitrary $k_0 < k^*$. Notice that the optimality conditions require that $u'(c_0) = \beta u'(c_1)[f'(k_1) + 1 - \delta]$, or, equivalently that

$$\mathcal{A}(k_0) = \mathcal{B}(k_1).$$

FIGURE 1

THE TRANSITION DYNAMICS OF THE NEOCLASSICAL MODEL

The Functions $\mathcal{A}$ and $\mathcal{B}$



From Figure 1, this implies that $k_0 < k_1 < k^*$. If $k_0 > k^*$ we have the opposite implication that $k^* < k_1 < k_0$. As we can see in the diagram, the dynamics clearly mean that k_t converges into the steady state at k^*. It is also clear that since $\boldsymbol{c}' > 0$, if $k_0 < k^*$ then $c_0 < c_1 < c^*$, if $k_0 > k^*$ then $c^* < c_1 < c_0$ and that c_t converges to its steady state value.

Figure 2 shows the results we have obtained regarding $h(k)$. First it is increasing. Second, it is above the 45 degree line for $k < k^*$ and below it for $k > k^*$. Also we know that $\lim_{k\to 0} h(k)$ must be zero since $\lim_{k\to 0} \boldsymbol{c}(k) = 0$ and $\lim_{k\to 0} g(k) = 0$.

A Phase Diagram Another diagram that is often used to illustrate dynamics is a phase diagram. A phase diagram for the system of difference equations

$$u'(c_t) = \beta u'(c_{t+1}) g'(k_{t+1}) \tag{17}$$

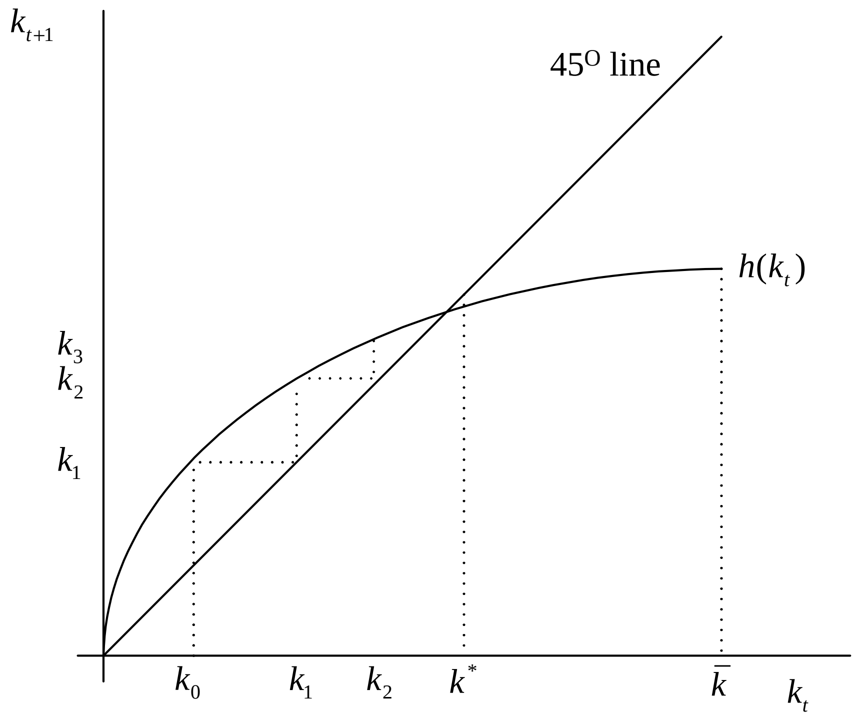
$$c_t + k_{t+1} = g(k_t). \tag{18}$$

plots c_t against k_t, the two variables governed by the difference equations. The phase diagram illustrates what would happen to c_t and k_t if we began from arbitrary values of k_0 *and* c_0 (as opposed to the optimal c_0 given k_0) and used the system of difference equations to solve for (k_1, c_1), then (k_2, c_2), etc. Notice that if you know c_t and k_t, then you can solve (18) for k_{t+1} and then solve (17) for c_{t+1}.

FIGURE 2

THE TRANSITION DYNAMICS OF THE NEOCLASSICAL MODEL

The Policy Function



Usually we begin with a diagram such as Figure 3 that demarcates regions in which c_t and k_t will be either increasing or decreasing. To draw this diagram, we ask the following questions:

1. When is $k_{t+1} > k_t$? From (18) we can see that $k_{t+1} > k_t$ whenever $c_t < g(k_t) - k_t$.

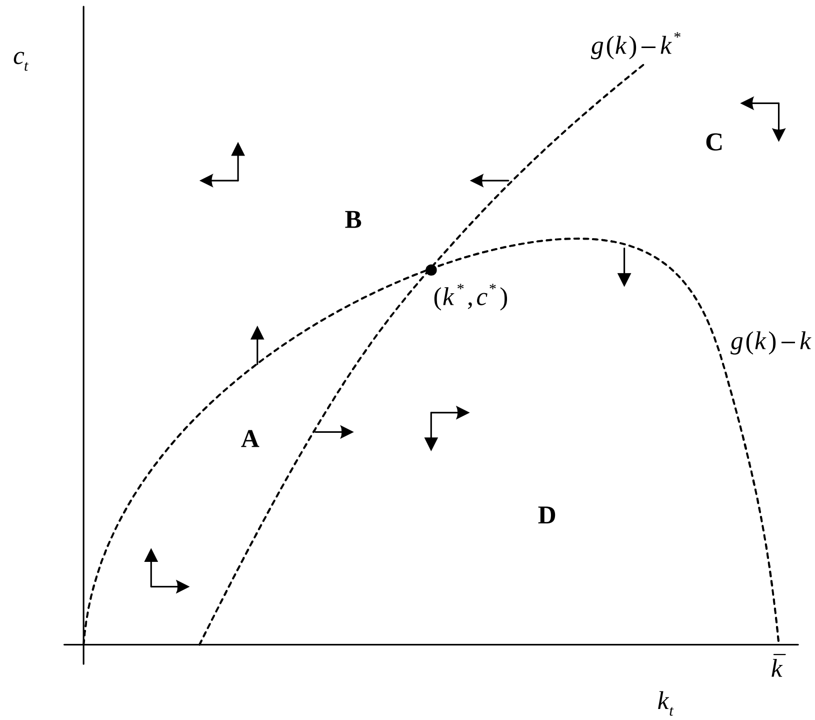
2. When is $c_{t+1} > c_t$? Notice that $c_{t+1} > c_t$ implies $u'(c_{t+1}) < u'(c_t)$. This, in turn, using (17), implies $g'(k_{t+1}) > \beta^{-1}$ and, hence, $k_{t+1} < k^*$. Unfortunately, we would like a condition that characterizes $c_{t+1} - c_t$ in terms of (k_t, c_t) *not* in terms of k_{t+1}. Notice, however, from (18), that $k_{t+1} < k^*$ is equivalent to $c_t > g(k_t) - k^*$.

To summarize, if $c_t < g(k_t) - k_t$, k_t is increasing, and if $c_t > g(k_t) - k^*$, c_t is increasing. So our diagram is divided into four regions, **A** (where c_t and k_t both increase), **B** (where c_t increases but k_t decreases), **C** (where c_t and k_t both decrease), and **D** (where c_t decreases but k_t increases). Hence the directions of the little arrows I have drawn into the diagram.

We know that if c_t is chosen optimally, the path of (k_t, c_t) will lie in region **A** or **C**. Why? Because in the previous section we showed that if $k_0 < k^*$, the capital stock and consumption increase monotonically to their steady state values (this only happens in region **A**). On the other hand if $k_0 > k^*$, the capital stock and consumption decrease monotonically to their steady state values (this only happens in region **C**).

FIGURE 3

CONSTRUCTING THE PHASE DIAGRAM FOR THE NEOCLASSICAL MODEL, STEP 1

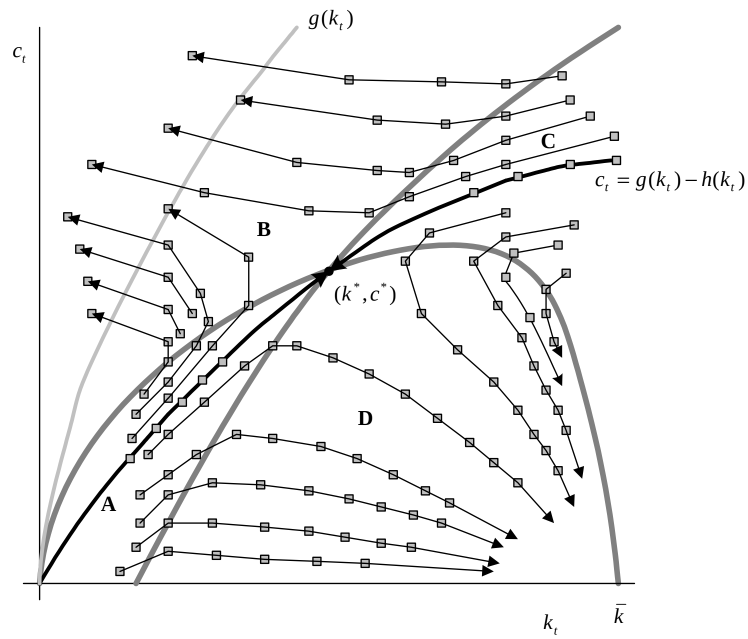


Obviously this means that one of the paths for (k_t, c_t) (these are called phase trajectories), which are illustrated in Figure 4, corresponds to $c_t = g(k_t) - h(k_t)$ and converges in towards the steady state. How did I draw in the other paths? The reason the paths in Figure 4 are not smooth is that this we are illustrating the dynamics of a difference equation not a differential equation, which would have smooth phase trajectories.

***Paths Starting in* B** Suppose our arbitrary choice of (k_0, c_0) is in **B**. Clearly, the path of subsequent values of (k_t, c_t) moves up and to the left (NW). But we can also show that the path will eventually violate the nonnegativity constraint on k_t or make c_t undefined. To see this, notice that (18) implies $k_{t+1} - k_t = g(k_t) - k_t - c_t$. The size of the decrease in the capital stock is equal to the distance c_t is above the curve $g(k_t) - k_t$. Clearly that distance increases as we move along a path that leads to the NW, so the decreases in k_t become larger at each step. This implies that eventually the rule for solving for k_{t+1} will imply $k_{t+1} < 0$ (this happens if $c_t > g(k_t)$) violating feasibility, or $k_{t+1} = 0$ (this happens if $c_t = g(k_t)$), making it impossible to solve for the value of c_{t+1}.

FIGURE 4

THE PHASE DIAGRAM FOR THE NEOCLASSICAL MODEL



***Paths Starting in* D** If our arbitrary choice of (k_0, c_0) is in **D** the path of subsequent values of (k_t, c_t) clearly moves down and to the right (SE). However, we can also show that any path that starts in **D** must converge towards $k_t = \bar{k}$, $c_t = 0$. To see this, notice that it is obvious that any path starting in **D** cannot escape into regions **A** and **B**, because those regions cannot be reached by SE movement that begins inside **D**. But we can also show that the dynamics cannot escape into region **C** or the boundary between **C** and **D**. To see this, imagine a (k_t, c_t) pair in **D**. Define $\bar{k}_t$ as the largest solution to $c_t = g(\bar{k}_t) - \bar{k}_t$. Notice that since $(k_t, c_t) \in$**D**, $\bar{k}_t > k_t$. But I can also show that $\bar{k}_t > k_{t+1}$. To see this notice that

$$\begin{aligned} k_{t+1} &< \bar{k}_t \Leftrightarrow g(k_t) - c_t < \bar{k}_t \\ &\Leftrightarrow g(k_t) - \left[g(\bar{k}_t) - \bar{k}_t\right] < \bar{k}_t \\ &\Leftrightarrow g(k_t) < g(\bar{k}_t) \end{aligned}$$

which is clearly true since $\bar{k}_t < k_t$. With $k_t < k_{t+1} < \bar{k}_t$ and $c_{t+1} < c_t$ we have the result that (k_{t+1}, c_{t+1}) lies inside **D**. The reason the dynamics must converge to $(\bar{k}, 0)$ is that they always move to the SE (so they cant stall anywhere inside **D**) and, because the next solution for k_{t+1} is always positive, the next solution for c_{t+1} will always be positive as well, so the dynamics cannot go across the x-axis.

An interesting fact about paths that begin in **D** is that they violate the transversality condition of the model because they all converge to $(\bar{k}, 0)$. (Here you should refer to the notes on dynamic programming where transversality conditions are discussed.) The transversality condition is

$$\lim_{t\to\infty} \beta^t u'(c_t) g'(k_t) k_t = 0.$$

Notice that for any path that begins in **D** there is some finite T for which $\hat{k} < k_t < \bar{k}$ for $t \geq T$, where $\hat{k}$ is the point where $g(k) - k$ is maximized; i.e. $g'(\hat{k}) = 1$. For any $t > T$ we have

$$\begin{aligned} u'(c_t) &= u'(c_{t-1})/[\beta g'(k_t)] \\ &= u'(c_{t-2})/[\beta^2 g'(k_t) g'(k_{t-1})] \\ &= \cdots = u'(c_T)/[\beta^{(t-T)} g'(k_t) g'(k_{t-1}) \cdots g'(k_{T+1})] \end{aligned}$$

Since $k_t > \hat{k}$ for $t \geq T$ this means $g'(k_t) < g'(\hat{k}) = 1$ for $t \geq T$, and that $g'(k_t) < 1$ for $t \geq T$. Hence $u'(c_t) > \beta^{-(t-T)} u'(c_T)$ for $t \geq T$. This means $\beta^t u'(c_t) > \beta^T u'(c_T)$ for $t \geq T$. Therefore $\beta^t u'(c_t) g'(k_t) k_t > \beta^T u'(c_T) g'(\bar{k}) \hat{k}$ for $t \geq T$. Hence

$$\lim_{t\to\infty} \beta^t u'(c_t) g'(k_t) k_t > \beta^T u'(c_T) g'(\bar{k}) \hat{k} \neq 0.$$

***Paths Starting in* A** If we start with a point $(k_0, c_0) \in$**A** there are three possibilities: the subsequent values of (k_t, c_t) (i) could stay in **A**, (ii) move into **B** or (iii) move in to **D**. We can easily eliminate the fourth possibility that the path moves into **C**: since $c_t > g(k_t) - k^*$ for any point in **A**, this means $k_{t+1} = g(k_t) - c_t < k^*$, implying $(k_{t+1}, c_{t+1}) \notin$**C**.

One path that clearly lies in **A**, and stays in **A**, is the optimal path along which $c_t = g(k_t) - h(k_t)$. We can eliminate the possibility that any other phase trajectory starting in **A** stays in **A**. Any phase trajectory that stays in **A** converges to (k^*, c^*).[4] But any such path satisfies (i) the difference equations (17) and (18) and (ii) the transversality condition.[5] From Theorem 4.15 in Stokey and Lucas this would imply that this other path was also optimal. But we know there is a unique optimal path. Therefore only the optimal path stays in **A**.

Paths that start out above the optimal path eventually get into set **B**. To see this let $k_1^* = h(k_0)$ and $c_0^* = g(k_0) - k_1^*$. If $c_0 > c_0^*$ the phase trajectory stays strictly above the optimal path. To see why notice that $c_0 > c_0^*$ implies $k_1 < k_1^*$. Hence, $u'(c_0) < u'(c_0^*)$ and $g'(k_1) > g'(k_1^*)$. Thus, if we solve for c_1 along the phase trajectory using (17) we will necessarily need $u'(c_1) < u'(c_1^*)$, where c_1^* is the value of c_1 that would prevail along the

[4] The other possibilities for paths that stay in A would be (i) that the path converges to a point in A other than (k, c) or (ii) the path cycles around in A without ever converging to any point. The first possibility is impossible because there is only one steady state ot the system (17)–(18). The second possibility is also not possible. We know the dynamics in A go in one direction: to the NE, and do not cycle.

[5] Along such a path $\lim_{t\to\infty} u'(C_t)[f'(K_t) + 1 - \delta]K_t = u'(C)K/\beta$, so that $\lim_{t\to\infty} \beta^t u'(C_t)[f'(K_t) + 1 - \delta]K_t = 0$.

optimal path. Thus $c_1 > c_1^*$ and the (k_1, c_1) pair along the phase trajectory lies strictly to the NW of (k_1^*, c_1^*). Since our trajectory path stays strictly above the optimal path, leaves **A** (shown above) and does not go into **C** (shown above) it must go into **B**. A symmetric argument implies that paths that start out below the optimal path eventually get into set **D**.

Paths Starting in **C** If we start with a point $(k_0, c_0) \in$**C** there are three possibilities: the subsequent values of (k_t, c_t) (i) could stay in **C**, (ii) move into **B** or (iii) move in to **D**. Symmetric to the argument above the dynamics never move into **A**. Also symmetric to the argument above, the only phase trajectory that stays in **C**, is one that starts on the optimal path. Paths that start out above the optimal path remain above it and move into **B**, paths that start out below it remain below it and eventually move into **D**.

2 Solving First Order Scalar Difference Equations

Here we will discuss first order deterministic and stochastic difference equations. The discussion draws heavily on Sargent (1987), Chapter 9 and to some extent Chapter 11, which should be consulted by the reader for a much more thorough treatment.

2.1 Deterministic Difference Equations

Consider the first-order difference equation

$$y_t = \lambda y_{t-1} + bx_t + a \qquad \forall t. \tag{19}$$

Here y_t is treated as an unknown sequence which we would like to characterize. The sequence $\{x_t\}_{t=-\infty}^{\infty}$ is known and determinstic.

Difference equations can be written using *lag operator* notation. The operator L lags a variable in the following sense. If we define a new sequence $\{y_t\}$ in terms of the sequence $\{x_t\}$ and let $y_t = Lx_t$, then $y_t = x_{t-1}$. So we could simply write $x_{t-1} = Lx_t$. In general $x_{t-k} = L^k x_t$.

In lag operator notation we rewrite (19) as

$$(1 - \lambda L)y_t = bx_t + a. \tag{20}$$

A solution that is immediately apparent is

$$y_t = (1 - \lambda L)^{-1}(bx_t + a). \tag{21}$$

One problem with this solution is that we have not yet discussed polynomials in the lag operator, such as $1-\lambda L$, and how they are inverted. Naively it would seem that there are two possibilities, the backward inverse,

$$(1-\lambda L)^{-1} = 1 + \lambda L + \lambda^2 L^2 + \cdots, \tag{22}$$

and the forward inverse,

$$(1-\lambda L)^{-1} = -\lambda^{-1}L^{-1}(1 + \lambda^{-1}L^{-1} + \lambda^{-2}L^{-2} + \cdots). \tag{23}$$

Apart from the issue of how $(1-\lambda L)^{-1}$ should be defined, there is another problem with (21), which is that there is a more *general* solution:

$$y_t = (1-\lambda L)^{-1}(bx_t + a) + \lambda^t c, \tag{24}$$

where c is an arbitrary constant. This solution is easily verified given that $L\lambda^t = \lambda^{t-1}$.

So there are infinitely many solutions to (19). How do we narrow ourselves down to one solution? There are three common ways to do this: (i) requiring that the $\{y_t\}$ sequence is bounded for all $\{x_t\}$ sequences that are bounded, (ii) treating one of the values of the $\{y_t\}$ sequence as a known scalar, (iii) requiring the $\{y_t\}$ sequence to converge to some finite value either as $t \to \infty$, or as $t \to -\infty$.

Mapping Bounded Sequences to Bounded Sequences One way to obtain a unique solution for the sequence $\{y_t\}$ is to require that the solution to (19) should be bounded for any bounded sequence $\{x_t\}$. Later we will see a theoretical argument for this assumption in the context of an economic model. For now we simply impose the restriction that polynomials in the lag operator should map bounded sequences to bounded sequences. Let the set of all bounded sequences be $\mathcal{B}$. Clearly, $L : \mathcal{B} \to \mathcal{B}$ since $\{Lx_t\} \in \mathcal{B}$ for any $\{x_t\} \in \mathcal{B}$. It is also clear that $(1-\lambda L) : \mathcal{B} \to \mathcal{B}$ for any finite value of λ. But the representation of $(1-\lambda L)^{-1}$ given by (22) only maps from $\mathcal{B}$ to $\mathcal{B}$ when $|\lambda| < 1$, while the representation of $(1-\lambda L)^{-1}$ given by (23) only maps from $\mathcal{B}$ to $\mathcal{B}$ when $|\lambda| > 1$.

Hence, when $|\lambda| < 1$, the requirement that $\{y_t\} \in \mathcal{B}$ for any $\{x_t\} \in \mathcal{B}$ implies that there is a unique solution for y_t given by

$$\begin{aligned} y_t &= (1 + \lambda L + \lambda^2 L^2 + \cdots)(bx_t + a) \\ &= b\sum_{j=0}^{\infty} \lambda^j x_{t-j} + \frac{a}{1-\lambda}. \end{aligned} \tag{25}$$

The boundedness assumption also requires that we set $c = 0$ in (24), since $\lim_{t\to-\infty} \left|c\lambda^t\right| \to \infty$ for any $c \neq 0$.

When $|\lambda| > 1$, the assumption that $\{y_t\} \in \mathcal{B}$ for any $\{x_t\} \in \mathcal{B}$ implies that there is a unique solution for y_t given by

$$\begin{aligned} y_t &= -\lambda^{-1}L^{-1}(1+\lambda^{-1}L^{-1}+\lambda^{-2}L^{-2}+\cdots)(bx_t+a) \\ &= -b\sum_{j=0}^{\infty}\lambda^{-(j+1)}x_{t+1+j}+\frac{a}{1-\lambda}. \end{aligned} \tag{26}$$

We again set $c = 0$ in (24), since $\lim_{t\to\infty}\left|c\lambda^t\right| \to \infty$ for any $c \neq 0$.

When $|\lambda| = 1$, $\{y_t\}$ is unbounded for at least some bounded $\{x_t\}$ sequences. Hence, in this case, the boundedness restriction cannot be used to find a unique solution for y_t.

Conditioning on a Known Value Suppose we know one of the elements of the sequence $\{y_t\}$, say y_0. This assumption is also motivated by theory. Frequently in macroeconomics we work with models in which a planner or economic agent maximizes a dynamic objective function given some initial level of resources. This initial level of resources at least partially determines the solution to the optimization problem. We will see an example of this below.

Given a value of y_0, we may generate the solution for the rest of the $\{y_t\}$ sequence recursively using (19):

$$y_t = \begin{cases} \lambda y_{t-1} + bx_t + a & \text{if } t > 0 \\ \lambda^{-1}y_{t+1} - \lambda^{-1}(bx_{t+1}+a) & \text{if } t < 0. \end{cases} \tag{27}$$

We can also work through the recursions to get the solutions

$$y_t = \begin{cases} \lambda^t y_0 + \sum_{j=0}^{t-1}\lambda^j\left(bx_{t-j}+a\right) & \text{if } t > 0 \\ \lambda^t y_0 - \sum_{j=0}^{-t-1}\lambda^{-(j+1)}(bx_{t+j+1}+a) & \text{if } t < 0. \end{cases} \tag{28}$$

These solutions work for all values of λ and are unique given a value of y_0.

Convergence in the Limit There are some examples in which we know something about the limiting properties of the y_t sequence and there is only one solution for y_t that has these limiting properties. To take a simple example, suppose we have $a = 0$ and $x_t = 0$ for all t, so that the general solution is simply $y_t = \lambda^t c$. If $|\lambda| > 1$ yet we know that $\lim_{t\to\infty} y_t = 0$ then the unique solution satisfying that convergence property is $c = 0$ and $y_t = 0$ for all t. On the other hand if we knew $\lim_{t\to-\infty} y_t = 0$ we would not be able to pin down c. Similarly if $|\lambda| < 1$ and we know that $\lim_{t\to-\infty} y_t = 0$ then the unique solution satisfying that convergence property is $c = 0$ and $y_t = 0$ for all t. On the other hand if we knew $\lim_{t\to\infty} y_t = 0$ we would not be able to pin down c.

2.2 Stochastic Difference Equations

Now consider the difference equation:

$$E_{t-1}y_t = \lambda y_{t-1} + bE_{t-1}x_t + a \qquad \forall t, \tag{29}$$

where E_{t-1} is the expectations operator conditional on information available at time $t-1$. Now x_t is a stochastic process, and presumably the solution for y_t will also be a stochastic process. Another way of writing (29) is

$$y_t = \lambda y_{t-1} + bE_{t-1}x_t + a + \epsilon_t, \tag{30}$$

where $\epsilon_t = y_t - E_{t-1}y_t$ is a white noise process. It is also convenient to use the notation $z_{t-1} = E_{t-1}x_t$ and rewrite (31) as:

$$y_t = \lambda y_{t-1} + bz_{t-1} + a + \epsilon_t. \tag{31}$$

You might think that we could simply think of (31) as a version of (19) with bx_t being replaced by $bz_{t-1} + \epsilon_t$. Thus you might expect the general solution to be

$$y_t = (1-\lambda L)^{-1}(bz_{t-1} + \epsilon_t + a) + \lambda^t c, \tag{32}$$

with c being an arbitrary constant and there being two possible forms of $(1-\lambda L)^{-1}$ given by (23) and (22) respectively.

The solution implied by (22) and (32), which is

$$y_t = \sum_{j=0}^{\infty} \lambda^j (bz_{t-j-1} + \epsilon_{t-j} + a) + \lambda^t c, \tag{33}$$

is valid. This is easily verified by noting that given (33) we have

$$\begin{aligned} E_{t-1}y_t &= \sum_{j=0}^{\infty} \lambda^j (bz_{t-j-1} + a) + \sum_{j=1}^{\infty} \lambda^j \epsilon_{t-j} + \lambda^t c \\ \lambda y_{t-1} &= \sum_{j=1}^{\infty} \lambda^j (bz_{t-j-1} + a) + \sum_{j=1}^{\infty} \lambda^j \epsilon_{t-j} + \lambda^t c. \end{aligned}$$

Hence

$$E_{t-1}y_t - \lambda y_{t-1} = bz_{t-1} + a,$$

which is the original difference equation.

The solution implied by (23) and (32), which is

$$y_t = -\sum_{j=0}^{\infty} \lambda^{-(j+1)} (bz_{t+j} + \epsilon_{t+1+j} + a) + \lambda^t c,$$

is easily seen to be invalid, since $E_{t-1}y_t$ has no terms involving the future values of ϵ_t but λy_{t-1} does. The fact that y_t must be determined as of date t imposes an additional restriction. It turns out that the correct forward-looking solution is

$$y_t = -\sum_{j=0}^{\infty} \lambda^{-(j+1)}(bE_t z_{t+j} + a) + \lambda^t c. \tag{34}$$

Once again we face the dilemma that there are infinitely many solutions to (29). We could use either representation (33) or (34). Furthermore, both (33) and (34) represent an infinite number of possible solutions because c could be any constant. Additionally, (33) represents an infinite number of possible solutions because ϵ_t could be *any* white noise process.

The question is whether and how we can narrow down the set of possible solutions. One way to do this is to implement a variant of the boundedness condition that is based on the property of covariance stationarity. In order to do this a definition and a couple of theorems from time series analysis will be useful.

Definition: A stochastic process $\{y_t\}$ is *covariance stationary* if $E(y_t) = \mu$ and $\text{Var}(y_t) = \gamma_0 < \infty$ and $E[(y_t - \mu)(y_{t-s} - \mu)] = \gamma_s$ for all t.

Wold Decomposition Theorem. Any covariance stationary stochastic process $\{x_t\}$ can be represented as $x_t = y_t + z_t$ where z_t is linearly deterministic [i.e. $\text{var}(z_t|z_{t-1}, z_{t-2}, \dots) = 0$] and y_t is purely non-deterministic with an MA(∞) representation $y_t = \psi(L)\epsilon_t$, with $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

Stationarity of ARMA Processes. An ARMA process $\phi(L)y_t = \theta(L)\epsilon_t$ is covariance stationary if and only if $\theta(L)$ is square-summable, i.e. $\sum_{j=0}^{\infty} \theta_j^2 < \infty$ and the roots of $\phi(Z) = 0$ lie strictly outside the unit circle.

We can restrict the set of possible solutions by requiring that y_t be covariance stationary for any covariance stationary x_t. Covariance stationarity of y_t immediately requires that $c = 0$ no matter which of the two representations, (33) and (34), we use because otherwise the mean of the process y_t would be time varying. We will assume that x_t is purely non-deterministic, allowing us to write $x_t = \theta(L)e_t$ with $\theta(L)$ being square-summable and e_t being white noise. Notice that $z_t = E_t x_{t+1} = \tilde{\theta}(L)e_t$, where $\tilde{\theta}_j = \theta_{j+1}$ and, therefore, $\tilde{\theta}(L)$ is square-summable.

Consider the case where $|\lambda| < 1$. If we impose $c = 0$, the solution (33) can be rewritten as

$$y_t - \lambda y_{t-1} = bz_{t-1} + \epsilon_t + a = b\tilde{\theta}(L)e_{t-1} + \epsilon_t + a. \tag{35}$$

This makes it clear that the backward looking solution represents y_t as an ARMA(1,∞) processes which is covariance stationary because $\tilde{\theta}(L)$ and the root of $1 - \lambda Z = 0$ is $Z =$

λ^{-1}, which lies outside the unit circle. With $c = 0$ the forward looking solution is $y_t = -\sum_{j=0}^{\infty} \lambda^{-(j+1)}(bE_t z_{t+j} + a)$. Clearly the forward looking solution, (34), can only be finite if $a = 0$. But even when $a = 0$, there will be many covariance stationary x_t processes for which the forward looking solution is ill-defined.[6] Hence, when $|\lambda| < 1$ the restriction that y_t should be covariance stationary for any covariance stationary x_t will lead us to use

$$\begin{aligned} y_t &= \lambda y_{t-1} + bE_{t-1}x_t + \epsilon_t + a \\ &= \sum_{j=0}^{\infty} \lambda^j (bE_{t-j-1}x_{t-j} + \epsilon_{t-j} + a) \end{aligned} \tag{36}$$

as the solution for y_t. Although the solution is covariance stationary it remains non-unique, since ϵ_t could be any white noise process. As we will see in the context of a theoretical example, below, it is sometimes possible to eliminate the multiplicity because the variable y_t is known to be in the time $t-1$ information set. In this case, the ϵ_t terms drop out of (36) and the solution for y_t reduces to

$$y_t = \lambda y_{t-1} + bE_{t-1}x_t + a = \sum_{j=0}^{\infty} \lambda^j (bE_{t-j-1}x_{t-j} + a). \tag{37}$$

Now consider the case where $|\lambda| > 1$. It is immediately obvious that y_t will not be covariance stationary if we use the backward-looking solution as described by (35). This is because the root of the autoregressive polynomial is λ^{-1} and this lies inside the unit circle. On the other hand consider the forward looking solution with $c = 0$ is $y_t = -\sum_{j=0}^{\infty} \lambda^{-(j+1)}(bE_t z_{t+j} + a)$. Given that $z_t = \tilde{\theta}(L)e_t$ we have $E_t z_{t+j} = \sum_{k=0}^{\infty} \tilde{\theta}_{j+k} e_{t-k}$ so we can write

$$\begin{aligned} y_t &= \frac{a}{1-\lambda} - b\sum_{j=0}^{\infty} \lambda^{-(j+1)} \sum_{k=0}^{\infty} \tilde{\theta}_{j+k} e_{t-k} \\ &= \frac{a}{1-\lambda} - b\lambda^{-1} \sum_{j=0}^{\infty} (\sum_{k=0}^{\infty} \tilde{\theta}_{j+k} \lambda^{-k}) e_{t-j}. \end{aligned}$$

I am missing a little proof—which I think can be based on the convolution formula—that the sequence $c_j = \sum_{k=0}^{\infty} \theta_{j+k+1} \lambda^{-k}$ is square summable. This would deliver that this representation of y_t is covariance stationary. Given this result, when $|\lambda| > 1$ the restriction that y_t should be covariance stationary for any covariance stationary x_t would lead us to use

$$y_t = -\sum_{j=0}^{\infty} \lambda^{-(j+1)} (bE_t x_{t+j+1} + a), \tag{38}$$

[6] Consider, for example, the case where $x_t = \rho x_{t-1} + \epsilon_t$ with $|\rho| < 1$. This means $z_t = \rho x_t$ and that $E_t z_{t+j} = \rho^{j+1} x_t$. With $a = 0$ $y_t = -b\sum_{j=0}^{\infty} (\rho/\lambda)^{(j+1)} x_t$. This is not finite when $|\lambda| < |\rho|$.

as the solution for y_t.

If $|\lambda| = 1$ neither solution can be shown to be covariance stationary for all possible covariance stationary x_t processes, but it is common, in this case, to work with the backward looking solution which is simply an ARIMA(0,1,∞).

3 Solving Deterministic Models

The basic principles of solving for the equilibrium dynamics of both deterministic and stochastic models are easily illustrated using the determinstic neoclassical growth model. After reviewing the optimality conditions from the model, I describe how to linearize the first order conditions to generate a system of linear difference equations that will then need to be solved. I then demonstrate how three different solution methods can be applied to these equations: (i) the Blanchard and Kahn (1980) approach, (ii) the shooting method and (iii) the method of undetermined coefficients.

When we solved the social planner's problem for the neoclassical model we obtained the following optimality conditions:

$$u'(c_t) = \beta u'(c_{t+1}) g'(k_{t+1}), \qquad t \geq 0 \tag{39}$$

$$c_t = g(k_t) - k_{t+1}, \qquad t \geq 0. \tag{40}$$

3.1 Linear Approximation

Given standard assumptions about u and f (and therefore g) equations (39) and (40) are nonlinear and we cannot determine the form of the policy function h, which yields the optimal value of k_{t+1} as a function of k_t.[7] A standard approach to working with equations like (39) and (40) is to approximate them with linear equations.

The first step is to find the steady state of the model. This can be done by removing time susbcripts from the variables in (39) and (40) and solving for c and k:

$$1 = \beta g'(k) \tag{41}$$

$$c = g(k) - k \tag{42}$$

Given a functional form for f, (41) can be solved for k, and, given this solution, (42) can be solved for c (we did this above).

In the next step we first totally differentiate (39) and (40) in c_t, c_{t+1}, k_t and k_{t+1} at their steady state values c, c, k, k:

$$u''(c)dc_t = \beta u''(c) g'(k) dc_{t+1} + \beta u'(c) g''(k) dk_{t+1},$$

[7] A notable exception is the case where $f(k) = Ak^{\alpha}$, $u(c) = \ln c$ and $\delta = 1$.

$$dc_t = g'(k)dk_t - dk_{t+1}.$$

We can simplify the first equation since $\beta g'(k) = 1$ in the steady state. We can also rewrite both equations in terms of $\hat{c}_t = dc_t/c$ and $\hat{k}_t = dk_t/k$:[8]

$$[u''(c)c]\,\hat{c}_t = [u''(c)c]\,\hat{c}_{t+1} + [\beta u'(c)g''(k)k]\,\hat{k}_{t+1},$$

$$[c]\,\hat{c}_t = [g'(k)k]\,\hat{k}_t - [k]\,\hat{k}_{t+1}.$$

We then divide the first equation by $u'(c)$ and the second equation by k:

$$\left[\frac{u''(c)c}{u'(c)}\right]\hat{c}_t = \left[\frac{u''(c)c}{u'(c)}\right]\hat{c}_{t+1} + [\beta g''(k)k]\,\hat{k}_{t+1},$$

$$\left[\frac{c}{k}\right]\hat{c}_t = [g'(k)]\hat{k}_t - \hat{k}_{t+1}.$$

If we define the coefficient of relative risk aversion at the steady state as $\sigma \equiv -u''(c)c/u'(c)$, and a parameter $\mu \equiv -\beta g''(k)k > 0$ and note that $g'(k) = \beta^{-1}$, then the two equations can be written more compactly as:

$$\hat{c}_t = \hat{c}_{t+1} + (\mu/\sigma)\hat{k}_{t+1}, \tag{43}$$

$$\frac{c}{k}\hat{c}_t = \beta^{-1}\hat{k}_t - \hat{k}_{t+1}. \tag{44}$$

In vector form, these two linear equations can be expressed as a first-order difference equation:

$$\begin{pmatrix} \mu/\sigma & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta^{-1} & -(c/k) \end{pmatrix}\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} 1/\beta & -c/k \\ -\mu/(\sigma\beta) & 1+\mu c/(\sigma k) \end{pmatrix}\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}. \tag{45}$$

Write the difference equation (45) as

$$x_t = Mx_{t-1}. \tag{46}$$

We can use (46) to approximately solve the neoclassical model using one of three methods: (i) the Blanchard and Kahn approach, (ii) the shooting method and (iii) the method of undetermined coefficients.. Each of these methods relies on specific properties of M, and things we know about the neoclassical model.

Although we will not do so in these notes, it can be shown that the neoclassical growth model has a unique solution.[9] That is, given an initial value of k_0, there are unique choices for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ that solve the social planner's problem. Also, the optimal path has the property that $\lim_{t\to\infty}(c_t, k_t) = (c, k)$. In terms of the linearized model, this means $\lim_{t\to\infty} x_t = 0$.

[8] Notice that the notation $\hat{x}_t = dx_t/x$ indicates the percentage deviation of x_t from its steady state value, x, because the total derivative was taken in the neighborhood of x.

[9] See, for example, chapters 1–4 of Stokey and Lucas (1989).

3.2 Blanchard and Kahn's Method

Assuming that it is possible to diagonalize M, we begin by letting $M = V\Lambda V^{-1}$, where V is a matrix whose columns are the eigenvectors of M normalized to have unit length, and Λ is a diagonal matrix with the corresponding eigenvalues arranged along the diagonal.[10] Define $\tilde{x}_t = V^{-1}x_t$. Premultiplying (46) by V^{-1} we get $V^{-1}x_t = V^{-1}V\Lambda V^{-1}x_{t-1}$, or

$$\tilde{x}_t = \Lambda\tilde{x}_{t-1}. \tag{47}$$

Thus we get a pair of scalar difference equations:

$$\tilde{x}_{1t} = \lambda_1\tilde{x}_{1t-1} \qquad \text{and} \qquad \tilde{x}_{2t} = \lambda_2\tilde{x}_{2t-1}. \tag{48}$$

Above, we saw that the general solutions to the two difference equations are

$$\tilde{x}_{1t} = c_1\lambda_1^t \text{ and } \tilde{x}_{2t} = c_2\lambda_2^t, \tag{49}$$

with c_1 and c_2 being arbitrary constants.

The next step is to pin down the values of c_1 and c_2. To do so we will rely on things we know about the optimal paths of the elements of x_t rather than $\tilde{x}_t$. First, we know the value of $\hat{k}_0$ because k_0 is given. Second, we know that the correct solution has the property that $\lim_{t\to\infty} x_t = 0$. Since $x_t = V\tilde{x}_t$ we can write

$$x_{1t} = v_{11}c_1\lambda_1^t + v_{12}c_2\lambda_2^t \qquad x_{2t} = v_{21}c_1\lambda_1^t + v_{22}c_2\lambda_2^t. \tag{50}$$

The fact that $\hat{k}_0$ (equivalently, x_{10}) is known puts one restriction on the constants c_1 and c_2. They must jointly satisfy $\hat{k}_0 = x_{10} = v_{11}c_1 + v_{12}c_2$. With this one restriction, the solutions for x_{1t} and x_{2t} become:

$$\begin{aligned} x_{1t} &= cv_{11}\lambda_1^t + \left(\hat{k}_0 - cv_{11}\right)\lambda_2^t && (51) \\ x_{2t} &= cv_{21}\lambda_1^t + \left(\hat{k}_0 - cv_{11}\right)\frac{v_{22}}{v_{12}}\lambda_2^t && (52) \end{aligned}$$

We now check whether the fact that $\lim_{t\to\infty} x_t = 0$ allows us to pin down c. Without loss of generality, we will assume that $|\lambda_1| \le |\lambda_2|$. There are three cases to consider.

1. If $|\lambda_1| \ge 1$ and $|\lambda_2| \ge 1$ there is no solution for x_t such that $\lim_{t\to\infty} x_t = 0$. Even if you set $c = 0$ to eliminate the $cv_{..}\lambda_1^t$ terms in (51) and (52) you would be left with the explosive terms $\hat{k}_0\lambda_2^t$ and $(v_{22}/v_{12})\hat{k}_0\lambda_2^t$. These would vanish in the limit only if $\hat{k}_0 = 0$ and, therefore, $x_t = 0$ for all t; that is, if the system was at the steady state at time 0. On the

[10] An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors. The case where M cannot be diagonalized is discussed briefly below.

other hand, if you assumed $c = \hat{k}_0/v_{11}$ to eliminate the second terms in each solution, then you would be left with the explosive $cv_{\cdot\cdot}\lambda_1^t$ terms.

2. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ there are arbitrarily many solutions for x_t such that $\lim_{t\to\infty} x_t = 0$. Clearly $\lim_{t\to\infty} x_t = 0$ regardless of the value of c. So we are left with an infinite number of solutions given by (51) and (52).

3. If $|\lambda_1| < 1$ and $|\lambda_2| \geq 1$ there is a unique solution.[11] We must set $c = \hat{k}_0/v_{11}$ so that the λ_2^t terms drops out of the solutions (51) and (52). This implies $x_{1t} = \hat{k}_0\lambda_1^t$ and $x_{2t} = (v_{21}/v_{11})\hat{k}_0\lambda_1^t$. This completely characterizes the paths of $\hat{k}_t$ and $\hat{c}_t$ and tell us that $\hat{c}_t = (v_{21}/v_{11})\hat{k}_t$ along these paths.

There is an alternative approach to solving the model. The fact that $c = \hat{k}_0/v_{11}$ is equivalent to $c_2 = 0$. If we substitute this into (49) we find $\tilde{x}_{2t} = 0$. The definition $\tilde{x}_t = V^{-1}x_t$ tells us that $\tilde{x}_{2t} = v^{21}x_{1t} + v^{22}x_{2t}$, where v^{ij} is the ij element of V^{-1}. Knowing that $\tilde{x}_{2t} = 0$ implies that $x_{2t} = -(v^{21}/v^{22})x_{1t}$ or $\hat{c}_t = -(v^{21}/v^{22})\hat{k}_t$. It is easy to verify that $v_{21}/v_{11} = -(v^{21}/v^{22})$ so that the two ways of proceeding get to the same answer.

Appendix Proof that $0 < \lambda_1 < 1 < \lambda_2$. Let $\theta = \mu c/(\sigma k)$ and note that $\theta > 0$. The eigenvalues of M are the solutions to

$$\psi(\lambda) = (\beta^{-1} - \lambda)(1 + \theta - \lambda) - \theta\beta^{-1} = 0$$

We have $\psi'(\lambda) = 2\lambda - (1 + \theta + \beta^{-1})$ and $\psi''(\lambda) = 2$ so that ψ is a convex quadratic function. Notice that $\psi(\lambda)$ is minimized at $\bar{\lambda} = \left(1 + \theta + \beta^{-1}\right)/2$ and $\psi(\bar{\lambda}) < 0$. This means that the roots are real and distinct with $\lambda_1 < \bar{\lambda} < \lambda_2$. We immediately obtain $\lambda_2 > 1$ since $\bar{\lambda} > 1$. Also, $\psi(1) = -\theta$. Since ψ is monotonically decreasing for $\lambda < \bar{\lambda}$ this means $\lambda_1 < 1$.

3.3 The Phase Diagram and the Shooting Method

The shooting method is an approach to solving the original nonlinear representation of the model, given by equations (39) and (40). If one knew the optimal value of c_0 one could solve recursively for the remaining choices, $\{c_t, k_t\}_{t=1}^{\infty}$. Given k_t and c_t, (39) can be solved for k_{t+1}. Then c_{t+1} can be obtained from (40).

Since c_0 is unknown, an approach to solving for $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is to guess the value of c_0, for some $T > 0$, compute $\{c_t, k_t\}_{t=1}^{T}$ using (39) and (40) and then modify the guess for c_0 depending on what happens to the recursive simulation. The only subtlety in the shooting method is how, if at all, the guess for c_0 should be modified after each simulation. Generally speaking, the method for modifying the guess relies on knowledge of the model's phase diagram.

[11] It turns out that this is the relevant case for the neoclassical model because $0 < \lambda_1 < 1 < \lambda_2$.

Although we will not construct it here, the phase diagram for the nonlinear version of the model has the property that for any guess c_0, $\lim_{t\to\infty}(c_t, k_t) = (c, k)$, if and only if the guess for c_0 is correct.[12] Otherwise (c_t, k_t) diverges from the steady state. In particular, if the guess for c_0 is too large, the path for the capital stock will eventually violate the non-negativity constraint. On the other hand, if the guess for c_0 is too small, $\lim_{t\to\infty}(c_t, k_t) = (0, \bar{k})$ where $\bar{k}$ is the solution of the equation $f(\bar{k}) = \delta\bar{k}$. These facts can be used to construct a shooting algorithm in which the guess for c_0 is adjusted downward when it becomes clear that the guess for c_0 is too large, and adjusted upward when it becomes clear than the guess is too small.

Just as it can be used to solve the original model, the shooting method can be used to solve the linearized model. We can construct the phase diagram for the linearized model in Figure 5. There are two paths in Figure 5 along which $\hat{c}_t$ and $\hat{k}_t$ are proportional to one another. These are the optimal path, corresponding to $c = \hat{k}_0/v_{11}$, and a suboptimal path, corresponding to $c = 0$.

As we saw in the previous section, if $c = \hat{k}_0/v_{11}$, $\hat{c}_0 = (v_{21}/v_{11})\hat{k}_0$, and for $t > 0$, $\hat{c}_t = (v_{21}/v_{11})\hat{k}_t$. This is the unique stable path in Figure 5, which implies convergence to the steady state.[13]

The other linear path in Figure 5 is the one corresponding to $c = 0$. When $c = 0$, $\hat{c}_0 = (v_{22}/v_{12})\hat{k}_0$, and for $t > 0$, $\hat{c}_t = (v_{22}/v_{12})\hat{k}_t$.[14]

The remaining paths, as we have see, are explosive, and correspond to the solutions, (51) and (52), for other values of c. These paths are not linear because $\hat{c}_t$ and $\hat{k}_t$ diverge at different rates along them. To draw these paths it is useful to note that (45) implies that $\hat{k}_{t+1} > \hat{k}_t$ if $\hat{c}_t < (\beta^{-1} - 1)(k/c)\hat{k}_t$ and that $\hat{c}_{t+1} > \hat{c}_t$ if $\hat{c}_t > \beta^{-1}(k/c)\hat{k}_t$. Thus, we can define the regions **A**, **B**, **C**, **D** in Figure 5, in which $\hat{c}_t$ and $\hat{k}_t$ are both increasing (**A**), $\hat{c}_t$ is increasing but $\hat{k}_t$ is decreasing (**B**), $\hat{c}_t$ and $\hat{k}_t$ are both decreasing (**C**), and $\hat{c}_t$ is decreasing while $\hat{k}_t$ is decreasing (**D**).

The optimal path, $\hat{c}_t = (v_{21}/v_{11})\hat{k}_t$, lies entirely within regions **A** and **C**. The steady state $(0, 0)$ is a saddle point, with the optimal solution being a saddle path. All other paths are divergent.

It is now clear how the phase diagram can be used to design the shooting algorithm. Given $\hat{k}_0$ start with a guess for $\hat{c}_0$. Then simulate (45) forward. If the path enters region **B** stop and revise the guess for $\hat{c}_0$ downward. If the path enters region **D** stop, and revise the guess for $\hat{c}_0$ upward. Keep doing this until the path gets within a tolerance limit of $(0, 0)$.

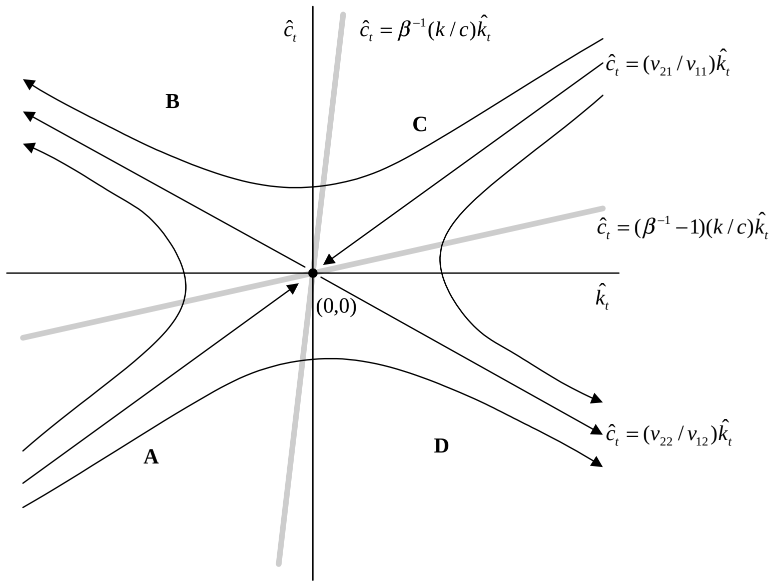
[12] See my notes on the neoclassical growth model for a discussion of the uniqueness of the model's solution and for a complete discussion of the phase diagram.

[13] The reader should verify that for the neoclassical model, $v_{21}/v_{11} > 0$.

[14] The reader should verify that for the neoclassical model, $v_{22}/v_{12} < 0$.

FIGURE 5

THE PHASE DIAGRAM FOR THE LINEARIZED NEOCLASSICAL MODEL



While the shooting method is obviously feasible, there is really no reason to use it once we have linearized the model. The reason is that the exact solution to the linearized model is as easy to characterize as the phase diagram using the Blanchard and Kahn approach.

A few more words about the phase diagram are in order. We obtained Figure 5 because the neoclassical model is one for which $0 < \lambda_1 < 1 < \lambda_2$. The roots are real, distinct, one is stable, while the other is unstable. For more general matrices M, a system $x_t = Mx_{t-1}$ can have many types of phase diagrams. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ all paths will lead into the steady state. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$ all paths will lead away from the steady state. If V is diagonal (in either direction), all the paths will be straight lines. If the roots are complex the paths will spiral around the steady state, either convergently, explosively or perpetually. Nonetheless, in all of these cases, as long as M is diagonalizable, the most general solution takes the form (**??**). This is true even for complex roots. The one case we have not considered is the case where M cannot be diagonalized.

When M cannot be Diagonalized This case can arise, but does not necessarily do so, when M has a repeat eigenvalue, λ. Even when M is not diagonalizable, it can still be written in the Jordan form $M = PJP^{-1}$ where P is a specific invertible matrix and

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Define $\tilde{x}_t = P^{-1}x_t$. Premultiplying (46) by P^{-1} we get

$$\tilde{x}_t = J\tilde{x}_{t-1}.$$

Thus we get a pair of equations:

$$\tilde{x}_{1t} = \lambda \tilde{x}_{1t-1} + \tilde{x}_{2t-1} \text{ and } \tilde{x}_{2t} = \lambda \tilde{x}_{2t-1}.$$

These have the general solution

$$\tilde{x}_{1t} = c_1 \lambda^t + c_2 t \lambda^{t-1} \text{ and } \tilde{x}_{2t} = c_2 \lambda^t.$$

Therefore

$$\begin{aligned} x_{1t} &= p_{11}\left(c_1\lambda^t + c_2 t\lambda^{t-1}\right) + p_{12}c_2\lambda^t \\ x_{2t} &= p_{21}\left(c_1\lambda^t + c_2 t\lambda^{t-1}\right) + p_{22}c_2\lambda^t. \end{aligned}$$

Once again we could pin the value of c_2 down using the fact that $x_{10} = \hat{k}_0$. However, in the next step we would find either (i) no bounded solution for any value of c_1, if $|\lambda| > 1$, or (ii) a bounded solution for every value of c_1, if $|\lambda| < 1$.

3.4 Method of Undetermined Coefficients

Another approach to solving the model is to guess that the optimal solution is of the form $\hat{c}_t = \kappa \hat{k}_t$ for some scalar κ. As we have seen, above, this conjecture is correct. If the conjecture is substituted into (45) we have

$$\begin{pmatrix} 1 \\ \kappa \end{pmatrix} \hat{k}_{t+1} = \begin{pmatrix} 1/\beta & -c/k \\ -\mu/(\sigma\beta) & 1 + \mu c/(\sigma k) \end{pmatrix} \begin{pmatrix} 1 \\ \kappa \end{pmatrix} \hat{k}_t$$

or

$$\begin{pmatrix} 1 \\ \kappa \end{pmatrix} \hat{k}_{t+1} = \begin{pmatrix} \beta^{-1} - \kappa c/k \\ -\mu/(\sigma\beta) + \kappa[1 + \mu c/(\sigma k)] \end{pmatrix} \hat{k}_t.$$

Dividing the second equation through by κ:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \hat{k}_{t+1} = \begin{pmatrix} \beta^{-1} - \kappa c/k \\ -\mu/(\kappa\sigma\beta) + 1 + \mu c/(\sigma k) \end{pmatrix} \hat{k}_t. \tag{53}$$

For the two equations implicit in (53) to both be valid requires that

$$\beta^{-1} - \kappa c/k = -\mu/(\kappa\sigma\beta) + 1 + \mu c/(\sigma k).$$

This means κ is a solution to the quadratic equation

$$\kappa^2 c/k + \kappa[1 - \beta^{-1} + \mu c/(\sigma k)] - \mu/(\sigma\beta) = 0. \tag{54}$$

There are two solutions, κ_1 and κ_2. Suppose we choose one of these solutions, say κ_1. We can verify if it is the correct solution by going back to one of the two equations, say

$\hat{k}_{t+1} = (\beta^{-1} - \kappa_1 c/k)\hat{k}_t$. Notice that for any $\hat{k}_0$, we will get $\lim_{t\to\infty} \hat{k}_t = 0$ if and only if $|\beta^{-1} - \kappa_1 c/k| < 1$. We could verify whether we had the correct solution by checking whether this inequality was true. If it was not we would use the solution corresponding to κ_2.

There's another, more straightforward, way of thinking about solving for κ. Notice that we could also guess a solution of the form $\xi_1 \hat{k}_t + \xi_2 \hat{c}_t = 0$ for all t, and try to solve for ξ_1 and ξ_2. We can write this guess as $\boldsymbol{\xi}' x_t = 0$, for all t, where $\boldsymbol{\xi}' = \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}$. If we premultiply (46) by $\boldsymbol{\xi}'$ we get $\boldsymbol{\xi}' x_{t+1} = \boldsymbol{\xi}' M x_t$. By assumption $\boldsymbol{\xi}' x_{t+1} = 0$ so this means $\boldsymbol{\xi}' M x_t = 0$. We also have $\boldsymbol{\xi}' x_t = 0$. For both of these equalities to hold for all possible x_t, the elements of the vector $\boldsymbol{\xi}'$ must be proportional to the elements of the vector $\boldsymbol{\xi}' M$, with the proportion being given by some scalar λ.[15] I.e. there is some $\lambda \neq 0$ for which $\lambda \boldsymbol{\xi}' = \boldsymbol{\xi}' M$. If we transpose both sides of the equation we get $M' \boldsymbol{\xi} = \lambda \boldsymbol{\xi}$. But now we can see that finding $\boldsymbol{\xi}$ is a matter of finding the eigenvalues and eigenvectors of M'.

There are two eigenvectors of M', $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, so which one should we pick? Assume, as we did above, that M can be diagonalized as $M = V \Lambda V^{-1}$. Notice that since $M' = (V^{-1})' \Lambda V'$, M' has the same eigenvalues as M, and its eigenvectors are the columns of $(V^{-1})'$. I.e. $(V^{-1})' = \begin{pmatrix} \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 \end{pmatrix}$.

Since $x_t = M x_{t-1}$, $x_t = M^t x_0 = V \Lambda^t V^{-1} x_0$ or

$$\begin{aligned} x_t &= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_2' \end{pmatrix} x_0 \\ &= \begin{pmatrix} v_{11}\lambda_1^t(\boldsymbol{\xi}_1' x_0) + v_{12}\lambda_2^t(\boldsymbol{\xi}_2' x_0) \\ v_{21}\lambda_1^t(\boldsymbol{\xi}_1' x_0) + v_{22}\lambda_2^t(\boldsymbol{\xi}_2' x_0) \end{pmatrix}. \end{aligned}$$

If we choose $\boldsymbol{\xi}_1$ for our solution, we have $\boldsymbol{\xi}_1' x_0 = 0$ so that

$$x_t = \begin{pmatrix} v_{12}\lambda_2^t(\boldsymbol{\xi}_2' x_0) \\ v_{22}\lambda_2^t(\boldsymbol{\xi}_2' x_0) \end{pmatrix}.$$

Notice that this means $\lim_{t\to\infty} x_t = 0$ if and only if $|\lambda_2| < 1$. Similarly, if we choose $\boldsymbol{\xi}_2$ for our solution, we have $\boldsymbol{\xi}_2' x_0 = 0$ so that

$$x_t = \begin{pmatrix} v_{11}\lambda_1^t(\boldsymbol{\xi}_1' x_0) \\ v_{21}\lambda_1^t(\boldsymbol{\xi}_1' x_0) \end{pmatrix}.$$

In this case $\lim_{t\to\infty} x_t = 0$ if and only if $|\lambda_1| < 1$. Without loss of generality let $|\lambda_1| \leq |\lambda_2|$.

Since our criterion for choosing the correct solution is that $\lim_{t\to\infty} x_t = 0$ this means:

- if $|\lambda_1| \geq 1$ and $|\lambda_2| \geq 1$ neither solution will have the desired property

[15] Notice that if we define a vector $\theta = M'\xi$, one of the equations says $\xi_1 \hat{k}_0 + \xi_2 \hat{c}_0 = 0$, while the other says that $\theta_1 \hat{k}_0 + \theta_2 \hat{c}_0 = 0$. So one equation implies that $\hat{c}_0 = -(\xi_1/\xi_2)\hat{k}_0$ while the other implies that $\hat{c}_0 = -(\theta_1/\theta_2)\hat{k}_0$. So $\theta_1/\theta_2 = \xi_1/\xi_2$ or, in other words, $\theta_1 = \lambda \xi_1$ and $\theta_2 = \lambda \xi_2$ for some λ.

- if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then either solution will work and the choice will be ambiguous.
- if $|\lambda_1| < 1$ and $|\lambda_2| \geq 1$ then clearly we would need to use $\boldsymbol{\xi}_2$ as our solution.

As we saw before, for the neoclassical model the last condition holds. So there is a unique solution, corresponding to the eigenvector $\boldsymbol{\xi}_2$ of M' that corresponds to the larger eigenvalue. As we noted above, the eigenvectors of M' are the columns of $(V^{-1})'$. Hence they are the rows of V^{-1}. This means that $\boldsymbol{\xi}_2 = \begin{pmatrix} v^{21} & v^{22} \end{pmatrix}'$. So $v^{21}\hat{k}_t + v^{22}\hat{c}_t = 0$, or $\hat{c}_t = -(v^{21}/v^{22})\hat{k}_t$. This is the same as the solution we obtained using the Blanchard and Kahn approach.

4 Linear Approximation in the Stochastic Model

Now imagine that we have the stochastic model where output is given by $z_t f(k_t)$ and z_t is some stationary stochastic process. Now the optimality conditions are:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \left[z_{t+1} f'(k_{t+1}) + 1 - \delta \right] \right\} \tag{55}$$

$$c_t = z_t f(k_t) + (1-\delta)k_t - k_{t+1}. \tag{56}$$

It is straightforward to show that the linearized representation of these equations is

$$\hat{c}_t = E_t \hat{c}_{t+1} + \frac{\mu}{\sigma}\hat{k}_{t+1} - \frac{\mu_Z}{\sigma} E_t \hat{z}_{t+1}$$

$$\frac{c}{k}\hat{c}_t = \frac{y}{k}\hat{z}_t + \beta^{-1}\hat{k}_t - \hat{k}_{t+1},$$

where μ, σ, c and k are defined as before, $\mu_Z = \beta f'(k) z$ and $y = zf(k)$.

In vector form, these two linear equations can be expressed as a first-order difference equation:

$$\begin{pmatrix} \mu/\sigma & 1 \\ 1 & 0 \end{pmatrix} E_t \begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \beta^{-1} & -(c/k) \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} + \begin{pmatrix} \mu_Z/\sigma \\ 0 \end{pmatrix} E_t \hat{z}_{t+1} + \begin{pmatrix} 0 \\ y/k \end{pmatrix} \hat{z}_t$$

or

$$E_t \begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} \beta^{-1} & -\frac{c}{k} \\ -\beta^{-1}\frac{\mu}{\sigma} & 1 + \frac{\mu c}{\sigma k} \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\mu_Z}{\sigma} \end{pmatrix} E_t \hat{z}_{t+1} + \begin{pmatrix} \frac{y}{k} \\ -\frac{\mu}{\sigma}\frac{y}{k} \end{pmatrix} \hat{z}_t.$$

For the moment it is probably easiest to write this system as

$$E_t x_{t+1} = M x_t + \zeta_t$$

where

$$\zeta_t = \begin{pmatrix} 0 \\ \frac{\mu_Z}{\sigma} \end{pmatrix} E_t \hat{z}_{t+1} + \begin{pmatrix} \frac{y}{k} \\ -\frac{\mu}{\sigma}\frac{y}{k} \end{pmatrix} \hat{z}_t.$$

Notice that the matrix M is unchanged. Once again, we could diagonalize the system and write it as

$$E_t\tilde{x}_{t+1} = \Lambda\tilde{x}_t + \tilde{z}_t$$

where $\tilde{x}_t = V^{-1}x_t$ and $\tilde{z}_t = V^{-1}\zeta_t$. In this case we end up with two scalar stochastic difference equations

$$\begin{aligned} E_t\tilde{x}_{1t+1} &= \lambda_1\tilde{x}_{1t} + \tilde{z}_{1t} \\ E_t\tilde{x}_{2t+1} &= \lambda_2\tilde{x}_{2t} + \tilde{z}_{2t}. \end{aligned}$$

We would like to find solutions for $\tilde{x}_{1t}$ and $\tilde{x}_{2t}$ that are (i) consistent with $\hat{k}_{t+1}$ being determined at time t and (iii) imply that x_t is covariance stationary.

There is a very direct way of finding the solution which is due to King, Plosser and Rebelo (2002). They suggest solving the second equation forward using (38):

$$\tilde{x}_{2t} = -\sum_{j=0}^{\infty}\lambda_2^{-(j+1)}E_t\tilde{z}_{2t+j} = -\sum_{j=0}^{\infty}\lambda_2^{-(j+1)}\begin{pmatrix} v^{21} & v^{22} \end{pmatrix}E_t\zeta_{t+j}$$

Then they suggest using the fact that $\tilde{x}_{2t} = v^{21}x_{1t} + v^{22}x_{2t}$ to solve for $x_{2t} = -(v^{22})^{-1}v^{21}x_{1t} + (v^{22})^{-1}\tilde{x}_{2t}$. Therefore

$$x_{2t} = -(v^{22})^{-1}v^{21}x_{1t} - (v^{22})^{-1}\sum_{j=0}^{\infty}\lambda_2^{-(j+1)}\begin{pmatrix} v^{21} & v^{22} \end{pmatrix}E_t\zeta_{t+j}$$

Then take the original difference equation, which states that $x_{1t+1} = m_{11}x_{1t} + m_{12}x_{2t} + \zeta_{1t}$ and rewrite it as

$$\begin{aligned} x_{1t+1} &= \left[m_{11} - m_{12}(v^{22})^{-1}v^{21}\right]x_{1t} + \zeta_{1t} - m_{12}(v^{22})^{-1}\sum_{j=0}^{\infty}\lambda_2^{-(j+1)}\begin{pmatrix} v^{21} & v^{22} \end{pmatrix}E_t\zeta_{t+j} \\ &= \lambda_1 x_{1t} + \zeta_{1t} - m_{12}(v^{22})^{-1}\sum_{j=0}^{\infty}\lambda_2^{-(j+1)}\begin{pmatrix} v^{21} & v^{22} \end{pmatrix}E_t\zeta_{t+j} \end{aligned}$$

The King, Plosser Rebelo code typically works with the assumption that $E_t\hat{z}_{t+j} = \rho^j z_t$ in which case

$$\zeta_t = \begin{pmatrix} \frac{y}{k} \\ \frac{\mu_Z}{\sigma}\rho - \frac{\mu}{\sigma}\frac{y}{k} \end{pmatrix}\hat{z}_t = \psi\hat{z}_t$$

and

$$E_t\zeta_{t+j} = \psi\rho^j\hat{z}_t.$$

Hence

$$x_{2t} = -\frac{v^{21}}{v^{22}}x_{1t} - \frac{\lambda_2}{\lambda_2 - \rho}\begin{pmatrix} \frac{v^{21}}{v^{22}} & 1 \end{pmatrix}\psi\hat{z}_t$$

$$x_{1t+1} = \lambda_1 x_{1t} + \left[\begin{pmatrix} 1 & 0 \end{pmatrix} - m_{12} \frac{1}{\lambda_2 - \rho} \begin{pmatrix} \frac{v^{21}}{v^{22}} & 1 \end{pmatrix} \right] \psi \hat{z}_t$$

It's important to keep in mind that the King, Plosser, Rebelo approach only works in specific circumstances. Recall that we first solved for $\tilde{x}_{2t}$ forward and then inverted the equation $\tilde{x}_{2t} = v^{21} x_{1t} + v^{22} x_{2t}$ to solve for the non-predetermined variables, x_{2t}, in terms of the predetermined variables x_{1t} and the variable just solved for, $\tilde{x}_{2t}$. The invertibility of the latter mapping depends on the dimension of x_2 and $\tilde{x}_2$ being the same. I.e. the number of non-predetermined variables must be the same as the number of explosive roots. If number of explosive roots is smaller, the system of equations will have a non-unique solution. If the number of explosive roots is bigger, there will be no covariance stationary solution. The flipside of this analysis is that the number of predetermined variables, which we solved for in the second step, was equal to the number of stable roots. If there are more stable roots than predetermined variables, the model will have multiple solutions. If there are less stable roots than predetermined variables, there will be no covariance stationary solution.

There is a way of getting these results using the same logic as was used to obtain the bounded solutions in the deterministic model, except here we use the covariance stationarity property and the fact that k_t is predetermined. This could be a good homework question.

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