

The Envelope Theorem

LaTeX file: *Envelope-nb-all* — Daniel A. Graham <daniel.graham@duke.edu>, June 22, 2005

An example without constraints

Here we consider a simple maximization problem without constraints, with a single choice variable and a single parameter. The object is to maximize

In: $f = (a - (x - a)^2) / 2;$

In: $x_{best} = \text{Solve}[D[f, x] == 0, x]$

Out: $\{x \rightarrow a\}$

Substituting this solution, $x(a)=a$, into the objective function gives

In: $F = (f /. x_{best})[[1]]$

Out: $\frac{a}{2}$

as the optimized value of the objective function. Now the envelope theorem states that the derivative of F with respect to the parameter a

In: $D[F, a]$

Out: $\frac{1}{2}$

and the partial derivative of f with respect to a , evaluated at the optimal x

In: $D[f, a] /. x_{best}$

Out: $\left\{\frac{1}{2}\right\}$

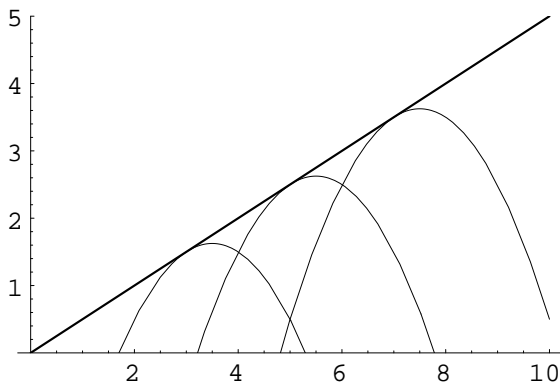
In: π_p

Out: π_p

are exactly the same.

To see why this is the case lets plot $F(a)$ and $f(x,a)$ for several fixed choices of x :

In: $\text{Plot}[\{F, f /. x \rightarrow 3, f /. x \rightarrow 5, f /. x \rightarrow 7\}, \{a, 0, 10\}, \text{PlotRange} \rightarrow \{0, 5\}, \text{PlotStyle} \rightarrow \{\{\text{Thickness}[0.004]\}, \{\text{Thickness}[0.004]\}, \{\text{Thickness}[0.004]\}, \{\text{Thickness}[0.004]\}\}]$



Note that F is, in fact, the envelope of the various f curves. The derivative of F corresponds to the slope of F and the partial derivatives of f evaluated at the optimal x corresponds to the slope of an f curve where it is tangent to the envelope F .

An example with constraints

Here we consider a slightly more complicated example in which the objective is to maximize

In: $f = x^2$;

with respect to x and subject to a constraint that must be less than or equal to a . The Lagrangian for this problem is

In: $l = x^2 + \lambda(a - x)$;

and the obvious solution is

In: $x_{\text{best}} = x \rightarrow a$;
 $\lambda_{\text{dabest}} = \lambda \rightarrow 2a$;
 $L = l / \{x_{\text{best}}, \lambda_{\text{dabest}}\}$

Out: a^2

Note that the derivative of L with respect to a is

In: $D[L, a]$

Out: $2a$

and the partial derivative of the l with respect to a evaluated at the optimal x and λ is

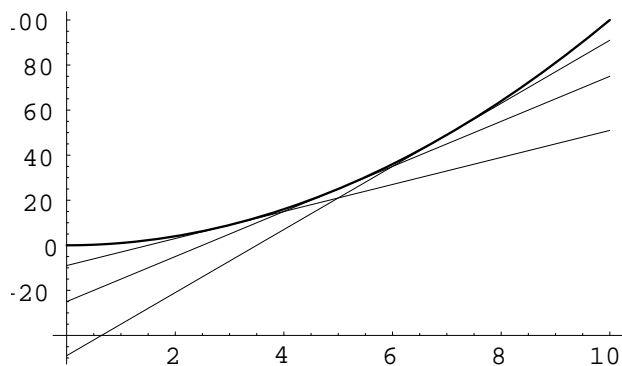
In: $D[l, a] / \{x_{\text{best}}, \lambda_{\text{dabest}}\}$

Out: $2a$

which, as the envelope theorem predicts, is exactly the same.

As before, the plot of L and l for particular (optimal) choices of x and λ illustrates why this result is called the envelope theorem.

In: $\text{Plot}[\{L, l /. \{x \rightarrow 3, \lambda \rightarrow 6\}, l /. \{x \rightarrow 5, \lambda \rightarrow 10\}, l /. \{x \rightarrow 7, \lambda \rightarrow 14\}\}, \{a, 0, 10\}, \text{PlotStyle} \rightarrow \{\{\text{Thickness}[0.004]\}, \{\text{Thickness}[\dots]\}$



Once again, the upper curve, L , is the envelope of the curves that correspond to l evaluated at combinations of x and λ which would be optimal at alternative values of a . The envelope theorem again states that where a lower curve is tangent to the upper curve, the slopes must be equal.