

Chapter 2. Introduction to *Physiological Fluid Mechanics*

2.1. Introduction Fluids consist of liquids and gases. Fluids may be pure components or mixtures. Whereas solids are rigid and can be made into many shapes, fluids adopt the shape of the container in which they are placed. For the study of fluid mechanics, a more precise definition of a fluid is as follows (Deen, 1998). A fluid is a material that deforms continuously when subjected to a force applied tangential to a surface. Many common fluids we encounter, such as water and air, deform instantaneously. More complex fluids, such as polymeric solution and gels including many biological fluids, deform in a time dependent fashion. Some fluids exhibit solid-like behavior when the shear stress is less than a threshold, or yield, value. Many biological fluids consist of two phases with complex time-dependent behavior. These examples show that the applied force per unit area affects the rate of deformation of the fluid.

Fluid mechanics is the study of the motion of fluids in response to the application of stresses. By analytical, numerical or experimental solution of Newton's second law of motion, we can determine the velocity field and the forces acting on the fluid or solid surfaces for an imposed stress or flow. In turn, this information can be used to calculate the pressure drop occurring or the work required to move the fluid. In heat and mass transfer problems, the velocity field is used in the conservation relations describing the transfer of energy or solute motion.

Fluid motion is governed by Newton's second law of motion (also known as the equation of motion) and the conservation of mass. There are four fundamental variables, three velocity components corresponding to the orthogonal directions of the coordinate system used and pressure. Since the velocity is a vector quantity, the equation of motion has three components. The conservation of mass represents the final relation to be solved. In addition, velocities or stresses must be specified at the fluid boundaries.

This part of the book will focus on *physiological* fluid dynamics. Biological fluid dynamics represents the full range of fluid dynamic phenomena observed by living organisms. These include bird flight, swimming of fish, movement of bacteria as well as fluid flow in various circulatory systems such as blood and the phloem and xylem of plants. The book by Vogel (Vogel, 1981) is an excellent and readable introduction to this fascinating area. One goal of biological fluid dynamics is to explain the shape and function of organisms in terms of adaptation to the flow of air and water over the organism. In contrast, physiological fluid mechanics is more restrictive. Physiological fluid mechanics is concerned with fluid flow within organisms and the relation between fluid flow and physiological processes. Physiological fluid mechanics provides important insights into normal and pathological processes and the design of many bioengineering applications, from artificial kidneys to vascular grafts to engineered tissues. The distinction between biological and physiological fluid mechanics essentially deals with the class of problems analyzed.

The emphasis of this chapter is upon the development of the equations which describe fluid motion and forces exerted by and on moving fluids. Important yet relatively simple flow problems are developed for different geometries. The biomedical significance of such flows is emphasized. We then developed generalized approaches to analyze flow problems. In subsequent chapters, we discuss other techniques which allow us to simplify complex flow problems. These tools are the used to study a variety of flow problems.

2.2. Fluid Kinematics

Prior to specifying the forces acting on a system, we can develop a number of results about fluid motion alone. *Kinematics* is the description of such motion. When describing motion we need to specify a frame of reference. Several choices are available. One choice is to have a fixed reference frame and examine motion relative to this coordinate system. Alternatively, the reference frame can move at some speed and all motion is relative to the moving reference frame. For the most part, we will use a fixed reference frame, although occasionally the solution of some problems is easier with a moving reference frame.

With fluids we must also decide whether we wish to follow the motion of a differential fluid element, also known as a material point, or examine the change in motion at fixed points in a fluid. The former view is known as a *lagrangian* description of motion and is used extensively in solid mechanics. The latter approach is known as a *eulerian* description and is the most commonly used approach in fluid mechanics. For the most part, we will use the eulerian approach when developing and applying the conservation relations. The lagrangian approach will be used when we examine the motion of proteins and cells in a fluid.

2.2.1. Control Volumes In mechanics and thermodynamics, a system is an arbitrary quantity of mass of fixed identity. Everything else is known as the surroundings. In fluid mechanics individual elements of fluid are not usually studied. Instead, a control volume is used and the flow of mass, momentum, and energy is examined (Figure 2.1). Control volumes can be fixed in size or may change size with time. Further, control volumes can be fixed in space or move with some arbitrary velocity. Two useful control volumes are those which are of constant size and either fixed in

space or moving with the average speed of the fluid.

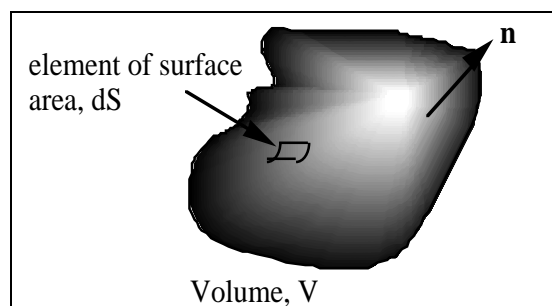


Figure 2.1. Control volume and un normal vector.

Specification of a control volume is arbitrary. Therefore, the control volume used is one which makes the analysis and solution of the problem easier. In general, the control volume has a volume

$V(t)$ and a surface $S(t)$. A unit outward normal vector, \mathbf{n} , can be defined at each point on the surface of the control volume. This vector is extremely useful when calculating stresses and velocities.

2.2.2. Velocity Field Knowing the fluid velocity provides a description of the flow field and enables one to calculate forces and stresses. As the above discussion highlights, we must carefully define the frame of reference used to define velocity.

Before providing mathematical descriptions of velocity, first consider a qualitative description of the difference between the lagrangian and eulerian descriptions of velocity. Imagine that you are wading in a stream and stop at various locations. You can feel the water moving past you. If the weather has been calm, the fluid motion is steady. If the weather is rough, the fluid might suddenly change speed. You have not changed location but the velocity changed. Now move downstream to a location where the stream width is the same but the depth has increased. The fluid is moving slightly slower. At each location you stop and observe the fluid motion, you apply a eulerian description. To observe the motion of the stream from a lagrangian viewpoint, get into a raft. The raft moves with the local speed of the stream. If the weather is calm the raft will move faster or slower depending on the local depth and width of the stream. The eulerian and lagrangian description apply not just to fluid motion, but any other property you might measure, such as temperature or the concentration of dissolved oxygen or a nutrient or pollutant.

To be more quantitative, let us first examine the motion of a differential fluid element using a lagrangian description (Figure 2.2). At time t , the particle is at position \mathbf{r} , which has coordinates x_1 , y_1 , and z_1 . At time $t + \Delta t$, the particle has moved to $\mathbf{r} + \Delta\mathbf{r}$ with coordinates x_2 , y_2 , and z_2 . The velocity of a fluid element is the time rate of change of the position of the fluid element.

Velocity has a direction and magnitude and vectors are \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 and \mathbf{i} , \mathbf{j} , and \mathbf{k} (Whitaker, 1981). therefore a vector quantity. For a rectangular coordinate system, the velocity is:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{e}_x v_x + \mathbf{e}_y v_y + \mathbf{e}_z v_z \quad (2.2.1a)$$

$$= \mathbf{e}_x \frac{dx}{dt} + \mathbf{e}_y \frac{dy}{dt} + \mathbf{e}_z \frac{dz}{dt} \quad (2.2.1b)$$

The vectors \mathbf{e}_x , \mathbf{e}_y , and \mathbf{e}_z are unit vectors which point parallel to the x , y and z coordinates, respectively. Other notations used for these unit

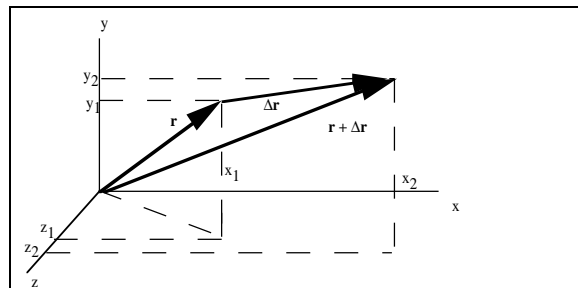


Figure 2.2. Motion of a differential fluid element.

In an Eulerian description, a velocity is associated with each location in the fluid. To see this, consider a differential cubic control volume fixed in space with origin at x, y, z (Figure 2.3). At a given instant in time, fluid elements enter the volume with velocity $\mathbf{v}(\mathbf{r}, t)$ and leave with velocities $\mathbf{v}(\mathbf{r} + \Delta\mathbf{r}, t)$. If the volume shrinks to a point, the velocity of a material point at that instant and at that location equals the local fluid velocity. Velocity is a function of location and time, i.e. $\mathbf{v} = \mathbf{v}(x, y, z, t)$.

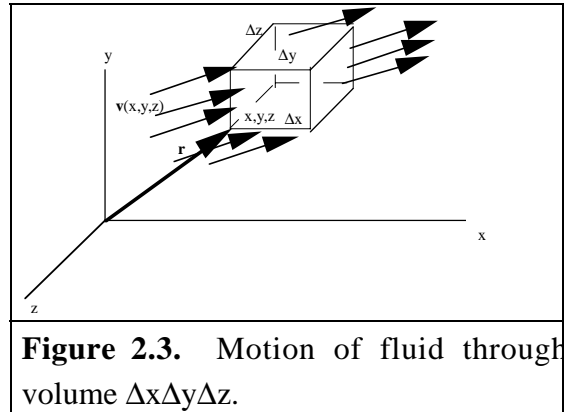


Figure 2.3. Motion of fluid through volume $\Delta x \Delta y \Delta z$.

For mixtures, the local fluid velocity can be defined in terms of either the mass average velocity \mathbf{v} or the molar average velocity \mathbf{v}^M . For a mixture containing i components each with velocity \mathbf{v}_i relative to fixed coordinates, these quantities are defined as:

$$\mathbf{v} = \sum_{i=1}^n \omega_i \mathbf{v}_i \qquad \mathbf{v}^M = \sum_{i=1}^n x_i \mathbf{v}_i \qquad (2.2.2)$$

where ω_i and x_i are the mass fraction and mole fraction, respectively. If each component is at a density ρ_i or concentration C_i then the mass and mole fractions are:

$$\omega_i = \frac{\rho_i}{\rho} \qquad x_i = \frac{C_i}{C} \qquad (2.2.3a,b)$$

The total density and concentration are summed over all components.

$$\rho = \sum_{i=1}^n \rho_i \qquad C = \sum_{i=1}^n C_i \qquad (2.2.4a,b)$$

For most applications with mixtures, the mass average velocity is used. This section of the text focuses on the motion of pure fluids or the bulk behavior of mixtures. In parts II and III we will examine the transport of individual components of dilute solutions.

2.2.3. Flow Rate Once the velocity is known, the average velocity and flow rate can be determined. The average velocity is the integral of the velocity which passes through a reference area (Figure 2.4):

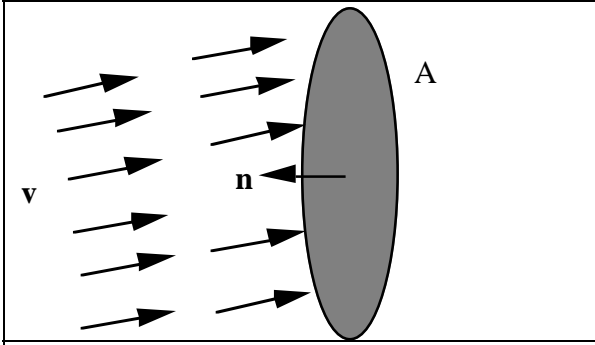


Figure 2.4. Area for calculating average velocity and flow rate.

$$\bar{v} = \frac{1}{A} \int_A \mathbf{v} \cdot \mathbf{n} dA \quad (2.2.5)$$

where \mathbf{n} is a unit vector normal to the surface A . The flow rate represents the amount of fluid which crosses a unit area of the flow field. The volumetric flow rate Q is:

$$Q = \bar{v} A = \int_A \mathbf{v} \cdot \mathbf{n} dA \quad (2.2.6)$$

The mass flow rate M is the integral of the product of the local density times the velocity:

$$M = \int_A \rho \mathbf{v} \cdot \mathbf{n} dA \quad (2.2.7)$$

2.2.4. Acceleration Acceleration is the rate of change of velocity. Fluid acceleration results when the velocity changes with time or space, i.e. $\mathbf{a} = \mathbf{a}(x,y,z,t)$. The total derivative of the velocity is:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial \mathbf{v}}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial \mathbf{v}}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial \mathbf{v}}{\partial z} = \frac{\partial \mathbf{v}}{\partial t} + v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \quad (2.2.8)$$

Using vector notation, the second through fourth terms on the right hand side of equation 2.2.8 can be written in a more compact form which is independent of the particular coordinate system (see Appendix).

$$\mathbf{v} \cdot \nabla \mathbf{v} = v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \quad (2.2.9a)$$

Expressions for the gradient operator ∇ in cylindrical and spherical coordinates are given in the Appendix. Substituting equation 2.2.9a into equation 2.2.8 yields:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \quad (2.2.9b)$$

The acceleration of a fluid element has two terms. The first term on the right hand side of equation 2.2.9b arises from the change in velocity with time. This term is referred to as the local acceleration. The second term is unique to fluid elements and represents fluid acceleration arising from a change in the velocity field (convective acceleration). For example, if fluid flowing down a tube passes through a narrow constriction, the fluid velocity must increase in order to maintain a constant flow rate. Physiologically, such a situation might arise during blood flow when there is a constriction or stenosis in a blood vessel due to a calcified valve or atherosclerosis. Conversely, a local expansion of the vessel, known as an aneurysm, can produce fluid deceleration. These conditions can cause significant complications.

Example 2.1. For the following velocity field which describes high speed inviscid flow through a conical converging channel, compute the fluid acceleration.

$$v_x = \frac{V_o}{\left(1 + \frac{r_L - r_o}{r_o} \left(\frac{x}{L}\right)\right)^2} \quad (2.2.10)$$

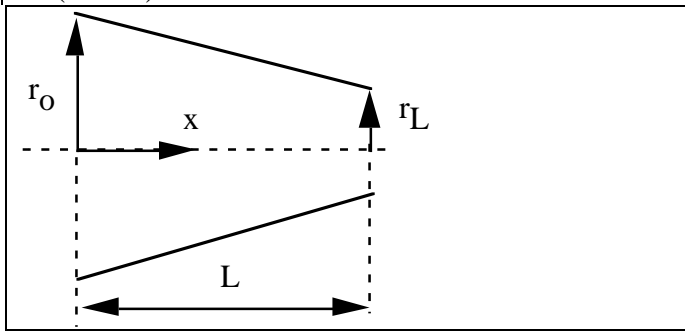


figure 2.5. Flow through a conical channel.

Since there is only one component to velocity and flow is steady, equation 2.2.8 reduces to:

$$\mathbf{a} = v_x \frac{\partial \mathbf{v}}{\partial x} = \mathbf{e}_x v_x \frac{\partial v_x}{\partial x} \quad (2.2.11)$$

From equation 2.2.10, the derivative is:

$$\frac{\partial v_x}{\partial x} = \frac{-2V_o \left(\frac{r_L - r_o}{r_o}\right)}{L \left(1 + \frac{r_L - r_o}{r_o} \left(\frac{x}{L}\right)\right)^3} \quad (2.2.12)$$

$$\mathbf{a} = \mathbf{e}_x \frac{-2V_o^2 \left(\frac{r_L - r_o}{r_o}\right)}{L \left(1 + \frac{r_L - r_o}{r_o} \left(\frac{x}{L}\right)\right)^5} \quad (2.2.13)$$

The negative sign indicates that the fluid decelerates. For $r_L/r_o = 2$, the velocity at $x=L$ is one-fourth the inlet velocity and the acceleration is 1/32 the value at the inlet.

2 . Streamlines, Streamtubes and Streaklines Once the flow field is known there are several ways to display the velocity field. One approach is to display the velocity vectors as arrows (Figure 2.6). In this example the size of vector is proportional to the magnitude.

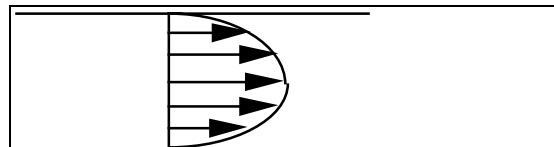


Figure 2.6. Vector representation of one-dimensional flow between parallel plates.

This method of display works well for one dimensional-flows and some simple two dimensional flows. Alternatively, for complex flows, one can display a curve which is everywhere tangent to the velocity vectors (Figure 2.7). Such a curve is known as a streamline. Since the streamlines are always tangent to the velocity vector, there is no flow normal to the streamline. This result is very useful when developing control volumes to analyze complex flows.

The streamlines can be represented mathematically as

$$\frac{d\mathbf{r}}{ds} = \mathbf{v} \quad (2.2.14a)$$

where s is the arclength along the streamlines. For each component of the velocity, equation 2.2.14a can be expressed as:

$$\frac{dx}{ds} = v_x \quad \frac{dy}{ds} = v_y \quad \frac{dz}{ds} = v_z \quad (2.2.14b)$$

By rearranging equation 2.2.14a and 2.2.14b, we obtain the following relation among the velocity components along a streamline:

$$ds = \frac{dr}{v} = \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \quad (2.2.15)$$

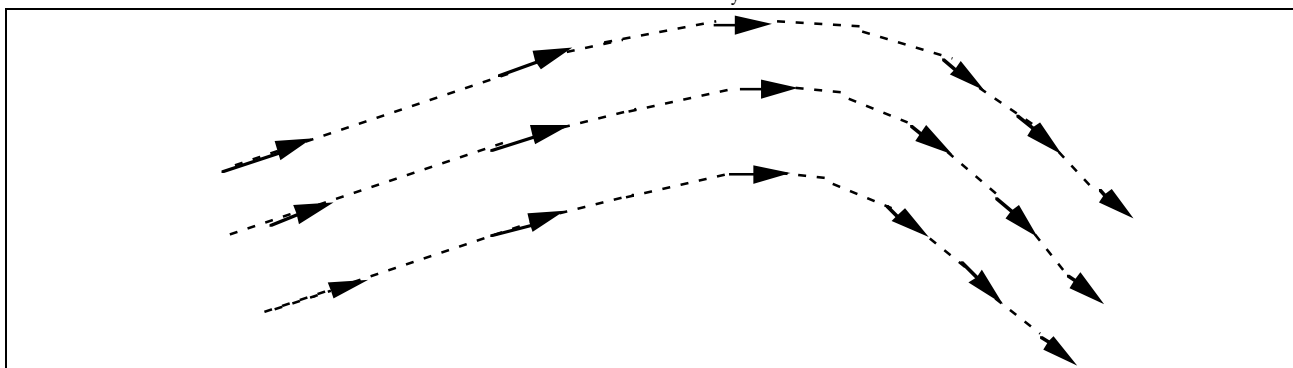


Figure 2.7. Streamlines (dotted lines) are everywhere tangent to the velocity vector.

The following example illustrates how to compute the streamlines. Consider the following two-dimensional velocity field.

$$v_x = y \left(1 - \frac{x}{L} \right) \quad v_y = \frac{y^2}{2L} \quad (2.2.16a)$$

where $0 \leq x \leq L$. Inserting equation 2.2.16a into 2.2.15 yields:

$$\frac{dx}{y \left(1 - \frac{x}{L} \right)} = \frac{2L dy}{y^2} \quad (2.2.16b)$$

Rearranging yields:

$$\frac{dx}{L \left(1 - \frac{x}{L} \right)} = \frac{2 dy}{y} \quad (2.2.16c)$$

Integrating equation 2.2.16c yields:

$$\ln \left(1 - \frac{x}{L} \right) = -2 \ln y + C' \quad (2.2.16d)$$

or
$$y = \sqrt{\frac{C'}{\left(1 - \frac{x}{L} \right)}} \quad (2.2.16e)$$

A family of streamlines is plotted in Figure 2.8 for various values of C .

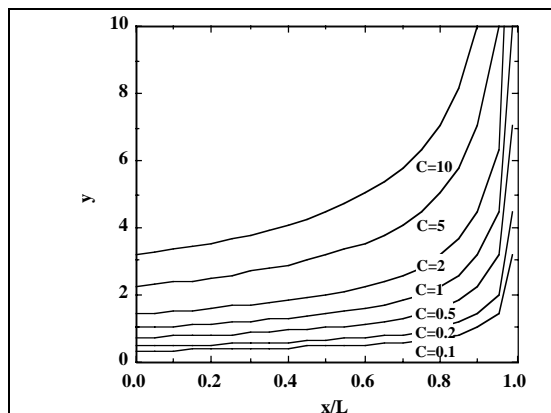


Figure 2.8. Streamlines represented equation 2.2.16e.

Numerous techniques are available to visualize the flow. Examples include the injection of dyes, smoke, or bubbles or the illumination of particles. These visualization techniques trace *pathlines* if the motion of an individual particle is followed or *streaklines* if a continuous path is examined. Pathlines can be produced by injecting individual bubbles or dye for a brief interval. Streaklines are produced by continuous injection of dye. Pathlines and streaklines represent the trajectory of particles (particle paths). Even if the agent used to visualize flow has the same density as the fluid, streaklines and streamlines are identical only for steady flow. The trajectories are given by:

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \quad (2.2.17)$$

For the flow field given by 2.2.16a,

$$\frac{dx}{y\left(1 - \frac{x}{L}\right)} = dt = \frac{2Ldy}{y^2} \quad (2.2.18)$$

which is identical to equation 2.2.16b. Now consider the following unsteady flow.

$$v_x = V_o \frac{x}{L} \frac{1}{1+at} \quad v_y = V_o \frac{y}{L} \quad (2.2.19)$$

The streamlines are given by:

$$\frac{(1+at)dx}{x} = \frac{dy}{y} \quad (2.2.20)$$

Time is not a variable, so integration and rearrangement yields:

$$y = C_2 \left(\frac{x}{C_1} \right)^{(1+at)} \quad (2.2.21)$$

Particle pathlines are obtained by solution of the following two equations:

$$\frac{dx}{dt} = V_o \frac{x}{L} \frac{1}{1+at} \quad \frac{dy}{dt} = V_o \frac{y}{L} \quad (2.2.22)$$

Integrating these expressions yields:

$$x = C_1(1+at)^{La/V_o} \quad y = C_2 \exp\left(\frac{V_o t}{L}\right) \quad (2.2.24a,b)$$

Equation 2.2.21 describes a family of curves at each time whereas equation 2.2.24a,b yields individual points x,y at each time.

2.3. Conservation Relations A complete analysis of any transport process requires specification of the conservation relations, constitutive equations and boundary conditions. These equations can be specified in either an integrated form that describes the average behavior or in a differential form which describes behavior at each point. We will use both approaches although we will emphasize the differential form. Solution of the resulting equations can be performed analytically, numerically or experimentally. Such solutions can be used to address a fundamental scientific issue, such as the mechanism of transport of molecules in a tissue, or a design issue, such as the particle size and amount of enzyme needed for a bioreactor. For flow problems we need to specify the conservation of mass and the conservation of linear momentum.

In this section, we will state the conservation relations in words and describe the individual terms. In subsequent sections we will develop balances for some simple but important situations. Then, in sections 2.9 and 2.10 we will generalize the conservation of mass and the conservation of linear momentum by deriving differential relations valid in three dimensions for different coordinate systems. We will use a similar approach in Chapter 6 for solute transport.

2.3.1. Conservation of Mass Consider a control volume of constant size and fixed in space. A fluid mixture passes through the borders of the volume. Within the volume chemical reactions can occur. Such a dynamic situation can lead to changes in the concentration of the various components of the mixture. If we examine an individual solute, we can state the conservation relation in words.

$$\left[\begin{array}{l} \text{Rate of accumulation} \\ \text{of } i \text{ in control volume} \end{array} \right] = \left[\begin{array}{l} \text{Transport of } i \text{ into} \\ \text{control volume} \end{array} \right] - \left[\begin{array}{l} \text{Transport of } i \text{ out} \\ \text{of control volume} \end{array} \right] + \left[\begin{array}{l} \text{Gain or loss of } i \text{ due} \\ \text{to chemical reaction} \end{array} \right] \quad (2.3.1)$$

In this section, only nonreacting systems are considered and the third term on the right hand side is zero. We will relax this assumption in parts B and C of the text. The system under study can be a single pure component or a mixture. At this point we will not focus on the individual components of the system, but rather on the total mass entering or leaving the control volume. Since we are examining the total mass, entry into and exit from the control volume can only occur by fluid flow. As a result, the mass balance can be restated as:

$$\left[\begin{array}{l} \text{Rate of accumulation of} \\ \text{mass in control volume} \end{array} \right] = \left[\begin{array}{l} \text{Flow of mass into} \\ \text{control volume} \end{array} \right] - \left[\begin{array}{l} \text{Flow of mass out} \\ \text{of control volume} \end{array} \right] \quad (2.3.2)$$

2.3.2. Momentum Balances Specification of the velocities and forces in a flowing system requires solution Newton's second law of motion. Before presenting a complete statement of Newton's second

law for a fluid, we will examine some specific idealized applications using momentum balances. The momentum balances can be expressed in words as:

$$\left[\begin{array}{c} \text{Sum of forces} \\ \end{array} \right] = \left[\begin{array}{c} \text{Rate of momentum flow} \\ \text{into control volume} \end{array} \right] - \left[\begin{array}{c} \text{Rate of momentum flow} \\ \text{out of control volume} \end{array} \right] + \left[\begin{array}{c} \text{Rate of momentum} \\ \text{accumulation} \end{array} \right] \quad (2.3.3)$$

In general, momentum is the product of mass times velocity ($m\mathbf{v}$). In fluid mechanics, momentum is expressed as an intensive variable (i.e. independent of the mass or volume) and equals the product of density times velocity ($\rho\mathbf{v}$). For a mixture component the mass average velocity (equation) is used.

2.3.3. Forces Forces which act on the control volume can be divided into body forces or surface forces. Body forces, such as gravity and electromagnetic fields, act on the entire control volume. For example, the net gravity force, \mathbf{F}_g , equals

$$\mathbf{F}_g = \int_{V(t)} \rho \mathbf{g} dV \quad (2.3.4)$$

Electromagnetic fields will affect the motion of electrolytes in solution or the flow of ferromagnetic fluids. Electrolyte transport is considered in Chapter 6.

Forces which act on control volume surfaces are known as stresses. Stresses have units of force per unit area (e.g. N/m^2 or dyne/cm^2). Stresses are tensorial quantities (Appendix) and have two directional components associated with them (Figure 2.9). Stresses are represented as σ or σ_{ij} where the index i refers to the plane on which the stress acts and the index j refers to the direction in which the stress acts. The indices i and j can take the values for the three orthogonal directions ($x, y,$ and z in Figure 2.9).

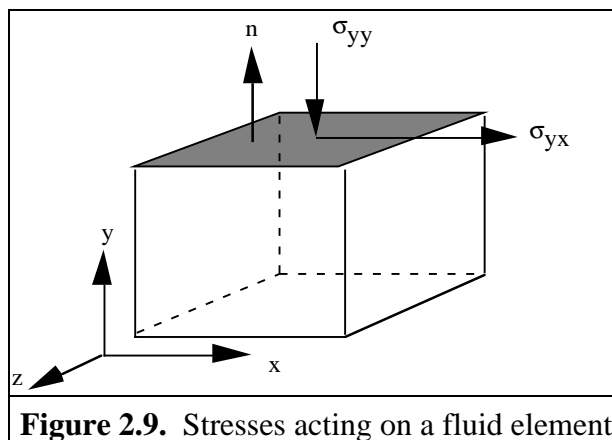


Figure 2.9. Stresses acting on a fluid element.

Stresses may act normal to a control volume surface or tangential to a control volume surface. Tangential stresses are also known as shear stresses. Thus, σ_{yy} is a normal stress acting on a plane of constant y in the y direction. Likewise σ_{yx} is a shear stress acting on a plane of constant y in the x direction.

The sign convention used in this book is that the stress is exerted by the fluid in the direction in which the outward normal unit vector points. Since n points outward in the positive y direction from the control volume surface in Figure 2.9 and σ_{yy} are σ_{yx} positive. The stress changes sign when the unit normal faces in the negative direction. A consequence of this sign convention is that compressive

stresses, such as pressure, are negative because these stresses point in the direction opposite of that of the unit normal. Tensile stresses are positive since they act in the direction of the unit normal.

There are several important properties of the stress tensor valid for most fluids which we will state without proof. Interested students should consult a reference on fluid mechanics for a proof of these statements (Aris 1962; Batchelor 1970; Deen 1998).

1. Stresses and torques acting on a material point are in equilibrium.
2. The stress tensor is symmetric, $\sigma_{ij} = \sigma_{ji}$.

The stress tensor contains nine components. Symmetry of the stress tensor indicates that only six of the nine components are needed. These results apply to most fluid except those that produce internal force couples, such as ferrofluids.

A fluid at rest cannot support a shear stress. Pressure is the only force that acts on a fluid at rest. Pressure is compressive and acts normal to a surface. At a point, pressure is uniform in all directions (i.e. isotropic). The pressure is a function of temperature and volume. Because of this distinction between stresses which can be supported at rest and under motion, the stress is divided into two components, the pressure stress and the deviatoric or shear stress (τ).

$$\sigma = -P\mathbf{I} + \tau \tag{2.3.5}$$

where \mathbf{I} is the identity matrix with diagonal terms of magnitude one and off diagonal terms of magnitude 0. Since we refer to pressure as a positive quantity, the negative sign is introduced to be consistent with the sign convention that pressure stresses are compressive.

2.3.4. Boundary Conditions In order to solve the conservation of linear momentum, we need to specify boundary conditions for either the stress or the velocity. The specific boundary conditions depend upon whether the boundary is a solid-fluid interface or a fluid-fluid interface. Let us first consider a solid-fluid interface. Unless the force applied at a surface is specified, we usually cannot state anything about the state of stress. If the stress or force applied to a solid surface is specified, then the boundary condition is the stresses are continuous across an interface. Thus, normal to an interface

$$\sigma_{ii}|_1 = \sigma_{ii}|_2 \tag{2.3.6a}$$

or
$$(\tau_{ii} - P)|_1 = (\tau_{ii} - P)|_2 \tag{2.3.6b}$$

where the indices refer to one of the three orthogonal coordinates which is normal to the surface. Tangential to the surface, the stresses are also equal.

$$\tau_{ij}|_1 = \tau_{ij}|_2 \tag{2.3.6c}$$

where the direction i is normal to the interface and j is tangent to the interface.

At the fluid-solid boundary, the "no-slip" condition is almost always used. This boundary condition is that the fluid velocity tangent to the solid surface equals the velocity of the solid surface.

$$V_{\text{solid}} = v_j \quad (2.3.6d)$$

This condition was first postulated in the nineteenth century without experimental proof. Subsequent experiments indicate that this relation is valid for almost all viscous fluid. Attractive forces between the fluid and solid result in strong adhesion at the surface. The solid-fluid attractive forces are stronger than those between fluid molecules such that when a stress is applied to the fluid, fluid molecules above the layer contacting the surface move whereas the fluid molecules attached to the surface are stationary.

If the solid is impermeable, then there is no normal component to the velocity. If the solid is porous, fluid motion will exist normal to the interface. The porous media consists of a fluid or void phase through which flow occurs and a solid phase. The media is characterized by a porosity or void volume ϕ which is the volume fraction of the media through which fluid can flow. Fluid velocity in porous media is characterized by a superficial velocity (v_s) which is the ratio of the flow divided by the cross-sectional area of the solid normal to the direction of flow. This area includes both solid and fluid phases. The superficial velocity is less than the actual interstitial velocity. At the interface, this boundary condition is:

$$v_s = \phi v_i \quad (2.3.6e)$$

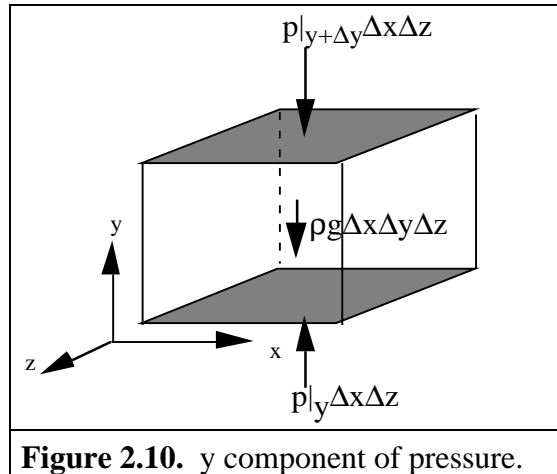
Many biological tissues are porous. Flow through porous tissues is important in the delivery of high molecular weight solutes and the movement of lymph.

2.4. Fluid Statics

2.4.1. Static Equilibrium Although most of the transport problems encountered in biomedical engineering will involve moving fluids, there are some important results under static conditions which are relevant to biomedical engineers. Under static conditions, the only forces acting on a fluid element are pressure and gravity. Thus:

$$\mathbf{F}_g + \mathbf{F}_{\text{pressure}} = 0 \quad (2.4.1)$$

Applying the force balance to a cubic control volume (Figure 2.10) yields:



$$\text{x direction} \quad (p|_x - p|_{x+\Delta x})\Delta y\Delta z = 0 \quad (2.4.2a)$$

$$\text{y direction} \quad -\rho g\Delta x\Delta y\Delta z + (p|_y - p|_{y+\Delta y})\Delta x\Delta z = 0 \quad (2.4.2b)$$

$$\text{z direction} \quad (p|_z - p|_{z+\Delta z})\Delta x\Delta y = 0 \quad (2.4.2c)$$

The sign convention for pressure is opposite that cited above for stress. Since pressure is compressive, it is positive when it acts on a surface in which the outward normal points in the negative direction and negative when it acts on a surface in which the outward normal points in the positive direction. Gravity acts in the negative y direction. Dividing each of the force balances by a volume element, $\Delta x \Delta y \Delta z$, yields:

$$\frac{(p|_x - p|_{x+\Delta x})}{\Delta x} = 0 \quad (2.4.3a)$$

$$-\rho g + \frac{(p|_y - p|_{y+\Delta y})}{\Delta y} = 0 \quad (2.4.3b)$$

$$\frac{(p|_z - p|_{z+\Delta z})}{\Delta z} = 0 \quad (2.4.3c)$$

Taking the limit as the volume shrinks to zero and using the definition of the differential, yields the following differential balances:

$$-\frac{dp}{dx} = 0 \quad (2.4.4a)$$

$$-\rho g - \frac{dp}{dy} = 0 \quad (2.4.4b)$$

$$-\frac{dp}{dz} = 0 \quad (2.4.4c)$$

Equations 2.4.4a-2.4.4c can be written in vector notation as follows:

$$\rho \mathbf{g} - \nabla p = 0 \quad (2.4.5)$$

The force balance represented by equation 2.4.5 is used in manometry, determination of forces on surfaces and submerged objects.

A useful application of this analysis is to calculate the pressure distribution in a column of fluid (Figure 2.11). Equations 2.4.4a-2.4.4c apply. Equations 2.4a and 2.4c indicate that pressure is uniform in the x and z directions. Rearranging equation 2.4.4b:

$$\frac{dp}{dy} = -\rho g \quad (2.4.6)$$

Assuming constant density, equation 2.4.6 can be integrated to yield, assuming $p = p_o$ at $y = 0$:

$$p = p_o - \rho gy \quad (2.4.7)$$

Since $y < 0$, the pressure increases with increasing depth in the fluid. The pressure difference with respect to atmospheric pressure, $p - p_o$, is known as the gage pressure. For water with a density of 1000 kg/m^3 , the pressure doubles every 10.34 meters. The blood pressure difference in a human 1.8 m tall is 0.1836 atmospheres or 140 mm Hg. Normal systolic blood pressure is only 120 mm Hg due to the location of the heart.

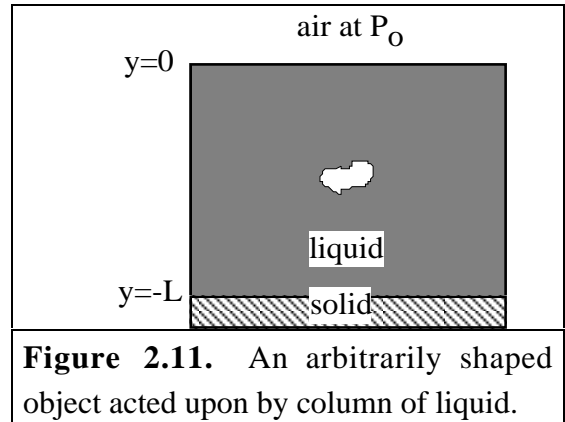


Figure 2.11. An arbitrarily shaped object acted upon by column of liquid.

Knowing the distribution of forces in the fluid, one can calculate the force of the fluid acting on an arbitrarily shaped body by a uniform column of liquid in equilibrium with air as shown in Figure 2.11. The particle has a density ρ_p and a volume V . The fluid exerts a surface force due to the pressure on the submerged object. To determine the net surface force, we use the force balance for a fluid given by equation 2.4.5. Integrating over the volume V of the solid yields:

$$\rho g V = \int_V \nabla p dV \quad (2.4.8)$$

The surface force on the object is the integral of the stress over the surface of the object.

$$\mathbf{F}_p = \int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS \quad (2.4.9)$$

Since the fluid is static ($\boldsymbol{\tau} = 0$), $\boldsymbol{\sigma} = -p\mathbf{I}$. The product $\mathbf{I} \cdot \mathbf{n}$ equals \mathbf{n} . Thus, the pressure force is:

$$\mathbf{F}_p = - \int_S p \mathbf{n} dS \quad (2.4.10)$$

The divergence theorem (Appendix) relates surface and volume integrals.

$$\int_S p \mathbf{n} dS = \int_V \nabla p dV \quad (2.4.11)$$

Thus, by substituting equations 2.4.10 and 2.4.11 into 2.4.9, the pressure force is

$$\mathbf{F}_p = -\rho\mathbf{g}V \quad (2.4.12)$$

This pressure force is known as the buoyant force (\mathbf{F}_b) and is in the direction opposite to the gravity force. Buoyancy was first discovered by Archimedes in the third century B.C. If the object has a density less than that of water, it will not be completely immersed in the liquid. For that case, the surface and volume integrations are only over the submerged portion of the object ($V = V_{\text{submerged}}$).

DNA, proteins and other molecules dissolved in solution as well as suspended cells can be considered immersed objects with a specific density. The density changes once the molecules are hydrated. The resulting buoyant force is important in calculating the speed of settling of molecules and cells and the separation of molecules during ultracentrifugation, a process which produces centrifugal forces substantially greater than gravity and allows separation of molecules with small density differences.

2.4.2. Surface Tension The interface between two immiscible liquids, a liquid and a gas, or a liquid and a solid is different than the bulk liquid. In the bulk liquid phase, the molecules of the liquid are surrounded by other liquid molecules. Attractive forces on molecules are, on average, the same in all directions. At the interface, however, the environment surrounding individual molecules undergoes a sharp discontinuity. At a liquid-gas interface below the boiling point, the attractive forces among liquid molecules is much greater than any attractive force between the liquid molecules and the gas. A drop of water placed on a sheet of Teflon will tend to form a round shape because the surface of the Teflon is nonpolar and hydrophobic, repelling water molecules. In both these examples, the surface is under tension due to the discontinuity of attractive forces at the surface.

Surface tension also arises with soap film bubbles, liposomes and cell membranes. Surface tension has units of force per unit length or energy per unit area. Because surface tension also has units of energy per unit area, an alternate, and equally valid view of surface tension, is that surface tension represents the work to create new surface. Typical air-liquid surface tensions are presented in Table 2.1.

Table 2.1. Air-Liquid Surface Tension at 20 °C. From (White, 1986)

Liquid	Surface Tension, N/m
ethanol	0.0228
mercury	0.484
methanol	0.0225
water	0.0728

A pressure difference across an interface will produce a curved surface. Pressure is higher on the concave side. Consider a spherical drop of radius R as shown in Figure 2.12. The external pressure is p_0 and the

pressure inside the drop is p_i . A force balance yields:

$$(p_i - p_o)\pi R^2 - \gamma 2\pi R = 0 \quad (2.4.13)$$

Defining $\Delta p = p_i - p_o$, rearrangement of equation 2.4.13 yields:

$$\Delta p = \frac{2\gamma}{R} \quad (2.4.14)$$

For an arbitrary curved surface, the following, more general, relation holds.

$$\Delta p = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (2.4.15)$$

where R_1 and R_2 are the principal radii of curvature.

Equation 2.4.15 is known as the Law of Laplace. Nonspherical shapes arise when a drop is on a surface. For a sphere, $R_1 = R_2$, and equation 2.4.15 equals 2.4.14. For a planar surface, R_1 and R_2 go to infinity and the pressure difference equals zero.

The air-liquid surface tension is affected by the presence of surfactants. Surfactants are molecules that contain polar and nonpolar segments. They are also known as amphipathic molecules which means that the "molecule consists of parts, each of which has an affinity for a different phase" (p. 254 of (Hiemenz, 1977)). The molecules are usually abbreviated RX where R is a long chain alkyl group and X is a polar or ionic group such as -OH, -COOH, SO_3^- , and NR_3^+ . These molecules have low solubility in water due to the alkyl chain. At the air-water interface the polar groups are in contact with the aqueous phase and the alkyl chain is in contact with air. The exact orientation depends upon the surface density of the molecules.

Surface tension effects are significant in lung physiology and pathology. Alveoli are spheres connected to terminal bronchi in the lung. Alveoli are the site of gas exchange from the lung to the blood. Alveoli size varies widely. In the absence of surfactant, the pressure to inflate smaller alveoli is greater than the pressure to inflate larger alveoli. As a result, smaller alveoli collapse at low lung volumes, causing larger alveoli to become overdistended, a condition known as atelectasis (Rhoads and Tanner, 1995). The surfactant is a lipoprotein-rich phospholipid and dipalmitoylphosphatidylcholine (DPPC) is the primary component (Figure 2.13). The surface tension decreases as the area decreases. This area decrease occurs during heavy exhalation. This decrease arises because surfactant molecules are squeezed out of the monolayer and form

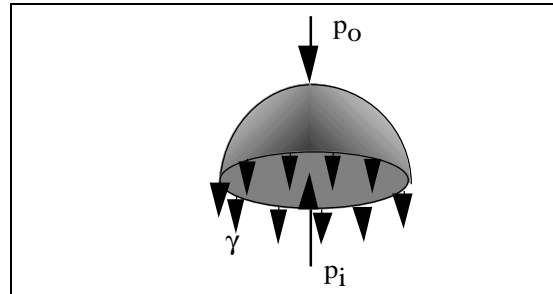


Figure 2.12. Forces acting on a spherical surface.

micelles. When the alveoli expand during inhalation, surface tension rises as surfactant molecules are added to the monolayer. Since smaller alveoli go through a larger change in relative area during expiration, the surface tension of these alveoli decreases more than the surface tension of large alveoli and collapse does not occur. Another role of surfactant is to reduce lung edema.

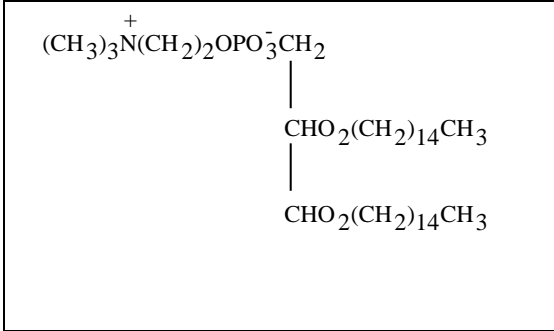


Figure 2.13. Structure of dipalmitoyl-phosphatidylcholine

In developing fetuses, surfactant is not produced until the last six weeks before normal delivery. Premature infants can develop infant respiratory distress syndrome due to lack of surfactant. Infants with this problem have labored breathing, edema, and atelectasis. Death can result if lung damage is too severe. Efforts to reduce complications involve adding synthetic surfactants.

2.4.3. Membrane and Cortical Tension Although the primary role of the membrane is to regulate solute transport, the cell membrane can also serve to reduce interfacial tension. The lipid component of the membrane is amphipathic. The cell membrane, however, cannot be viewed simply as a surfactant. Proteins are embedded in the membrane. These proteins function as receptors and channels to transport molecules in and out of the cell. A layer of sugar molecules, known as the glycocalyx, coats the outer surface of the membrane. Beneath the membrane is a cytoskeletal network known as the cortex. This cortex provides some mechanical rigidity to the membrane and is a site of signal transduction (Alberts et al., 1994).

Blood consists of red cells, white cells (or leukocytes), platelets and plasma. Red cells are the most abundant and account for many of the unique properties of blood flow (Section 2.11). The concentration of cells in blood is shown in Table 2.2. Red cells are very elastic and can readily deform into capillaries. Leukocytes are substantially more rigid. After stimulation of leukocytes with a variety of agents (e.g. chemotactic agents and cytokines) the cells become more rigid. Increased leukocyte rigidity can produce occlusion of small vessels under certain pathological conditions.

Table 2.2. Blood cell concentrations. From (Rhoads and Tanner, 1995)

Blood Cell Type	Range, cells/ml
Erythrocytes	4.2-6.5 x 10 ⁹
Leukocytes	
neutrophils	4-10 x 10 ⁶
lymphocytes	2.5-5.0 x 10 ⁶
monocytes	1-10 x 10 ⁵
eosinophils	0-5 x 10 ⁵
basophils	0-1 x 10 ⁵
platelets	1.5-3.5 x 10 ⁸

The mechanical properties of red cells and leukocytes can be analyzed using micropipet techniques (Figure 2.14). A pipet suction pressure p_p is applied to draw the cell into the pipet. The external pressure in the bath is p_o and the pressure within the cells is p_i . Under a sufficiently high suction pressure, and for red cells in the presence of a low osmolality solution, the cell outside the pipet will adopt a nearly spherical shape. Under these conditions the membrane is under isotropic tension and equation 2.4.15 applies (Waugh and Hochmuth, 1995) where $\Delta p = p_o - p_p$.

The law of Laplace has been used to determine the bursting strength and area expansivity modulus of red cells and lipid vesicles and the cortical tension of leukocytes. Red cell membranes and phospholipid vesicles are resistant to significant area dilation. Red cells and phospholipid vesicles rupture when the tension exceeds 10N/m , suggesting that red cell rupture is determined largely by lipids in the membrane (Fung, 1993).

Analysis of the mechanical properties of white blood cells has revealed that the membrane cortex supports a tension. For a neutrophil this tension is approximately $0.024\text{-}0.035\text{ mN/m}$ (Waugh and Hochmuth, 1995). This cortical tension causes the cells to maintain a spherical shape in solution. When a neutrophil is drawn into a pipet and the tension exceeds the cortical tension, the cell flows freely in the pipet.

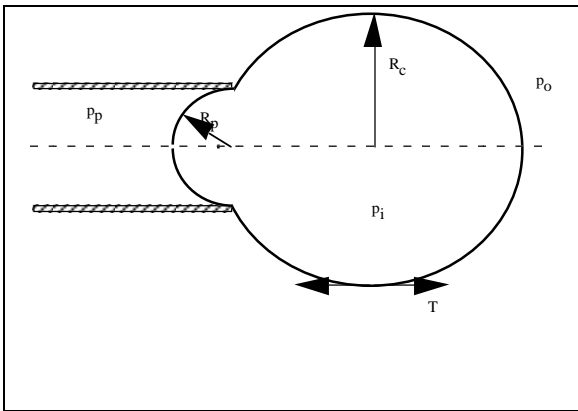


Figure 2.14. Schematic of a cell drawn into a micropipet.

2.5. Constitutive Relations

2.5.1 Newton's Law of Viscosity In order to use Newton's second law of motion to determine the velocity field and forces acting on a fluid, we need to determine the variables which affect shear stress. Such a relationship is known as a constitutive equation. Unlike a conservation relationship which is valid for all materials, a constitutive relationship is not universal and applies to a limited class of fluids. Experimental measurements are needed to derive constitutive relationships.

Consider a control volume that deforms with the fluid as shown in Figure 2.15. Application of a shear stress to a fluid will produce deformation and motion of the fluid. If a shear stress is applied to an elastic solid, the location of material points will change over a brief period. The deformation will be fixed as long as the stress is applied. For a solid, the stress is a function of the deformation. If, however, a stress is applied to the fluid, the location of material points will continue to change with time. For a fluid the shear stress is thus a function of the rate of deformation.

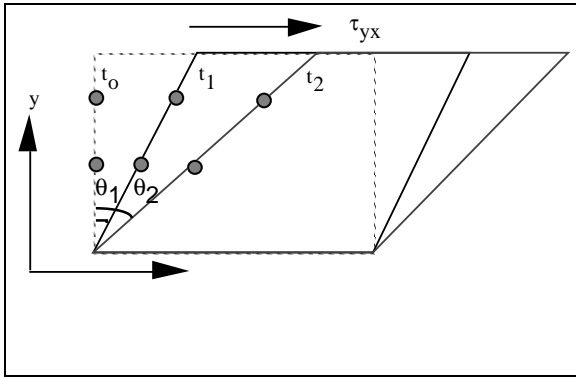


Figure 2.15. Continuous deformation of a fluid element exposed to a shear stress.

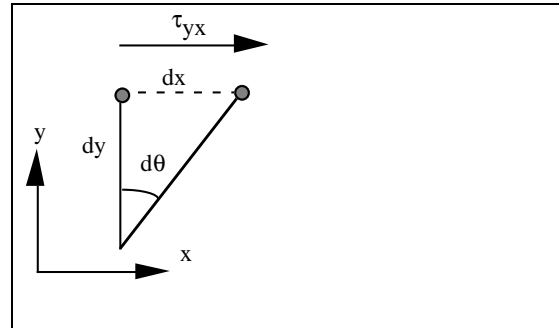


Figure 2.16. Location of a material point after a time interval dt in a fluid exposed to a shear stress τ_{yx} .

The rate of deformation equals the velocity gradient. To see this, consider the change in the location of material points in a small time differential dt after application of a shear stress. The rate of deformation is simply $\frac{d\theta}{dt}$ where θ is the angular change in the edge of the control volume shown in Figure 2.16. From the geometry of the deformation:

$$\tan(d\theta) = \frac{dx}{dy} \quad (2.5.1)$$

For differential changes in the angle, $\tan(d\theta) \approx d\theta$ (2.5.2)

Further, $dx = dv_x dt$. Thus,

$$d\theta \approx \frac{dx}{dy} = \frac{dv_x dt}{dy} \quad (2.5.3)$$

and the rate of deformation is equal to the velocity gradient:

$$\frac{d\theta}{dt} = \frac{dv_x}{dy} \quad (2.5.4)$$

The velocity gradient is often referred to as the shear rate, $\dot{\gamma}_x$. Thus,

$$\tau_{yx} = f(\dot{\gamma}_x) \quad (2.5.5)$$

Determination of the functional relationship requires experiments. This simplest possible relationship, obeyed by many fluids is a linear relation between shear stress and shear rate.

$$\tau_{yx} = \mu \dot{\gamma}_x = \mu \frac{dv_x}{dy} \quad (2.5.6)$$

where the coefficient of proportionality is known as the viscosity. Fluids which obey this relationship are known as Newtonian fluids. Equation 2.5.6 is often referred to as Newton's law of viscosity. This

term is misleading, since this expression is not universal as are conservation equations and the laws of thermodynamics.

2.5.2. Non-Newtonian Rheology Rheology is the branch of mechanics which studies the deformation of fluids. The distinction between Newtonian and non-Newtonian behavior can be explained in terms of the shear stress and the shear rate. In general, the shear stress and shear rate are related in terms of the apparent viscosity, η_{app} :

$$\tau_{yx} = \eta_{app}(T, P, \dot{\gamma}_x) \dot{\gamma}_x \tag{2.5.7}$$

Methods to determine the relationship between shear stress and shear rate are discussed in sections 2.7.4 and 2.11. The apparent viscosity is a function of temperature, pressure, and the shear rate. Several general classes of fluids have been defined based upon the relationship between the shear stress and the shear rate (Figure 2.17) or the behavior of η_{app} as a function of shear rate (Figure 2.19). Note that η_{app} is determined at each shear rate from measured values of the shear rate and shear stress.

Fluid viscosity and other steady rheological properties are measured in devices known as viscometers. Another class of devices known as rheometers can measure dynamic properties of fluids and solids. Several common designs are used which require fluid volumes ranging from 0.5 to 3 ml. These include the cone and plate viscometer, Couette viscometer, rotating disk viscometer and capillary viscometer. For smaller sample volumes and rheological measurements with cells involve the use of magnetic beads which oscillate or rotate in the fluid.

A general feature of viscometers and rheometers is the independent determination of shear stress and shear rate. For Couette, cone and plate and rotating disk viscometers, one surface is rotating at a constant speed and the torque needed to maintain constant speed is measured. The shear rate is proportional to the angular velocity and the shear stress is proportional to the torque. For capillary tube viscometers, the shear rate is proportional to the flow rate and the shear stress is proportional to the pressure drop per unit length. These devices will be discussed in more detail later in this chapter once we have discussed momentum balances and conservation relations.

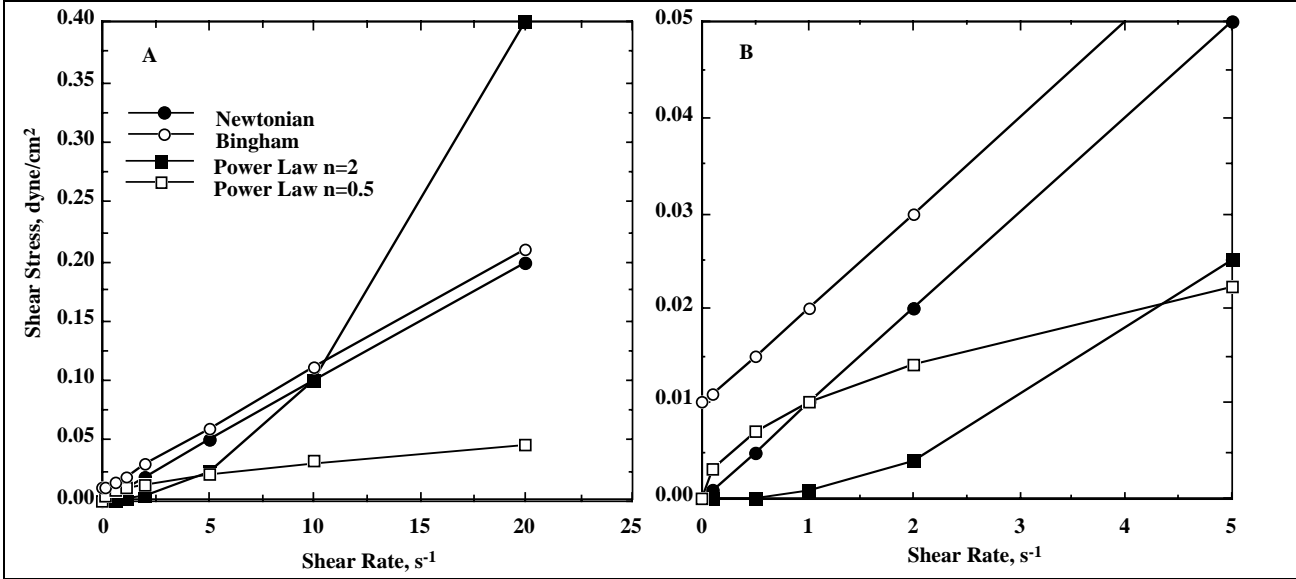


Figure 2.17. Relationship between wall shear stress and shear rate for Newtonian and non-Newtonian fluids. Fluid properties are: Newtonian fluid: $\mu = 0.01 \text{ g cm}^{-1}\text{s}^{-1}$; Bingham plastic: $\tau_0 = 0.01 \text{ dyne/cm}^2$, $\mu_0 = 0.01 \text{ g cm}^{-1}\text{s}^{-1}$; shear-thickening fluid (dilatant): $m = 0.001 \text{ g cm}^{-1}\text{s}^2$, $n = 2.0$; and shear thinning fluid (pseudoplastic): $m = 0.01 \text{ g cm}^{-1}\text{s}^{0.5}$, $n = 0.5$.

2.5.2.A. Newtonian Fluid. As noted above, the shear stress is proportional to the shear rate and the apparent viscosity equals the true viscosity, $\eta_{app} = \mu$, which is a function of temperature and pressure:

$$\tau_{yx} = \mu \dot{\gamma}_x \quad (2.5.8)$$

2.5.2.B. Bingham Plastic A Bingham plastic does not flow until the applied stress exceeds the yield stress τ_0 for the fluid. Below the yield stress, the shear rate and velocity gradient are zero:

$$\tau_{yx} < \tau_0 \quad \dot{\gamma}_x = 0 \quad (2.5.9a)$$

In rectangular coordinates, a zero shear rate is equivalent to a constant velocity. Above the yield stress, the relationship between shear stress and shear rate is:

$$\tau_{yx} > \tau_0 \quad \tau_{yx} = \pm \tau_0 + \mu_0 \dot{\gamma}_x \quad (2.5.9b)$$

where μ_0 depends upon temperature and pressure and is independent of shear rate. Equation 2.5.9b is similar to Newton's law of viscosity, except that the fluid exhibits a yield stress.

2.5.2.C. Power Law Fluids In general, a power law fluid is one the apparent viscosity is a function of the shear rate raised to a power. For power law fluids, the apparent viscosity can be written as:

$$\eta_{app} = m |\dot{\gamma}_x|^{n-1} \quad (2.5.10)$$

The quantities m and n depend upon the particular fluid. Note that n is dimensionless, but the units of m depend upon the value of n ($\text{g cm}^{-1}\text{s}^n$). Power law fluids are classified as follows:

$$n = 1 \quad \text{Newtonian fluid (} m = \mu \text{)}$$

- $n < 1$ Pseudoplastic (shear thinning) fluid
- $n > 1$ Dilatant (shear thickening) fluid

Examination of Figure 2.18 shows that for $n < 1$, the slope of the shear stress vs. shear rate curve decreases as the value of the shear rate increases. Alternatively, if $n > 1$ slope of the shear stress vs. shear rate curve increases as the value of the shear rate increases.

For a pseudoplastic fluid the apparent viscosity (i.e. the slope at a particular value of $\dot{\gamma}_x$) decreases as shear rate increases. Examples of pseudoplastic or shear thinning fluids are high molecular weight polymers as well as synovial fluid and mucus. Blood exhibits shear thinning behavior and a yield stress, but its rheology is more complex (see Section 2.11). For a dilatant fluid, the apparent viscosity increases as the shear rate increases. Examples of dilatant or shear thickening fluids are slurries and suspensions.

One way to characterize a fluid is to plot the logarithm of the apparent viscosity versus the logarithm of the shear rate (Figure 2.19). For a power law fluid, equation 2.5.10 becomes:

$$\log(\eta_{app}) = \log(m) + (n-1)\log(\dot{\gamma}_x) \tag{2.5.11}$$

For a Newtonian fluid, the viscosity is independent of the shear rate ($n=1$). For $n > 1$ the slope will be positive and for $n < 1$ the slope will be negative. Deviations from a power law fluid can be detected as nonlinearities in the plot (Figure 2.18). A log-log plot for Bingham plastics produces a nonlinear dependence of apparent viscosity which asymptotically approaches a constant value. This nonlinear behavior is due to the existence of a yield stress.

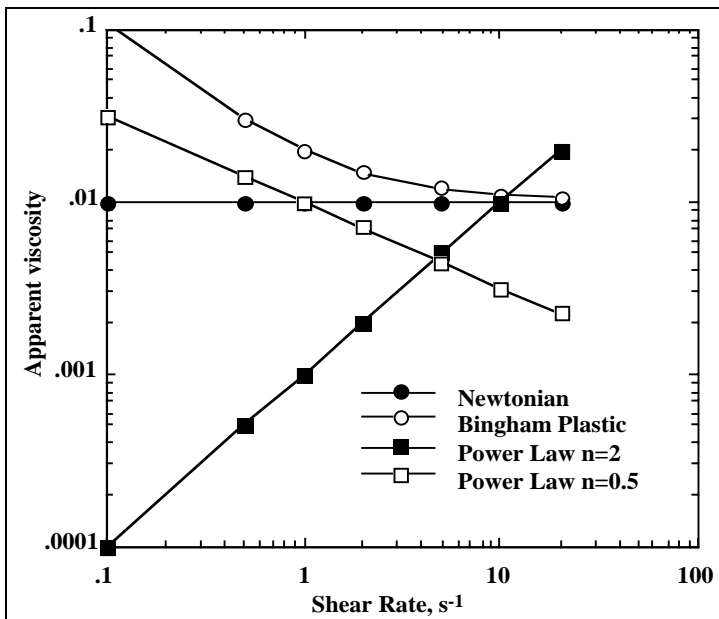


Figure 2.18. Apparent viscosity vs. shear rate for fluids shown in Figure 2.17.

2.5.3. Time Dependent Viscoelastic Behavior Some fluids exhibit both viscous and elastic properties. As a result, the constitutive equation is a function of both the strain and the time rate of change of strain or the shear rate. For example, consider flow of a fluid between two flat plates separated by a distance h . At time zero the upper plate is suddenly set into motion. For a Newtonian fluid, the fluid velocity distribution asymptotically approaches the steady state value with a characteristic time of $\rho h^2/\mu$. A viscoelastic fluid may undergo a sudden almost instantaneous displacement followed by a gradual decay to the steady value or the fluid might overshoot the steady velocity before declining to the steady state. In some cases, a recoil may occur. Other transient phenomena associated with viscoelastic fluids include stress relaxation and creep (Bird et al., 1987).

For a viscous Newtonian fluid, the shear stress is proportional to the shear rate ($\dot{\gamma}_x$) as given by equation 2.5.8. For a linear elastic (Hookean) material, the shear stress is proportional to the strain (γ_x):

$$\tau_{yx} = G\gamma_x \tag{2.5.12}$$

where G is the elastic modulus and the shear rate (or strain rate) and the strain are related by:

$$\gamma_x(t_0, t) = \int_{t_0}^t \dot{\gamma}_x(t') dt' \quad \text{or} \quad \frac{d\gamma_x(t)}{dt} = \dot{\gamma}_x(t) \tag{2.5.13}$$

One constitutive equation which has been used is the Maxwell model:

$$\tau_{yx} + \frac{\mu}{G} \frac{\partial \tau_{yx}}{\partial t} = \mu \dot{\gamma}_x \tag{2.5.14a}$$

where G is the elastic modulus. The ratio μ/G has units of s^{-1} and is often represented as the "relaxation time" λ . Assuming that the stress in the fluid is finite at $t = -\infty$, then the solution of equation 2.5.14 can be represented as (Bird et al., 1987):

$$\tau_{yx} = - \int_{-\infty}^t \left[\frac{\mu}{\lambda} \exp(-(t-t')/\lambda) \right] \dot{\gamma}_x(t') dt' \tag{2.5.14b}$$

The number of fluids which behave similar to a Maxwell fluid is limited. Although a number of other expressions have been proposed (Bird et al., 1987), the most useful is a generalized expression for a linear viscoelastic fluid:

$$\tau_{yx} = \int_{-\infty}^t G(t-t') \dot{\gamma}_x(t') dt' \quad (2.5.15)$$

where $G(t-t')$ is known as the relaxation modulus. The relaxation modulus is a property of the fluid whereas the shear rate is a property of the flow. For a Maxwell fluid $G(t-t') = \frac{\mu}{\lambda} \exp(-(t-t')/\lambda)$.

To see why $G(t-t')$ is called the relaxation modulus, consider a situation where a strain of magnitude γ_0 is applied to a viscoelastic fluid at time 0 (Figure 2.19).

$$t \leq 0 \quad \gamma_x = 0 \quad (2.5.16a)$$

$$t > 0 \quad \gamma_x = \gamma_0 \quad (2.5.16b)$$

An alternate way to represent this strain function is $\gamma_x = \gamma_0 H(t-t_0)$ where $H(t-t_0)$ is the unit step function. For this particular case $t_0 = 0$. From the relation between the strain and shear rate (equation 2.5.13), we have:

$$\dot{\gamma}_x(t) = \gamma_0 \frac{dH(t)}{dt} = \gamma_0 \delta(t) \quad (2.5.17) \quad w$$

where $\delta(t)$ is the Dirac delta function which equals 1 at $t = 0$ and is equal to 0 at all other times. Thus, equation 2.5.15 becomes:

$$\tau_{yx} = \int_{-\infty}^t G(t-t') \dot{\gamma}_x(t') dt' \quad (2.5.18a)$$

$$\tau_{yx} = \gamma_0 \int_{-\infty}^t G(t-t') dt' \quad (2.5.18b)$$

By means of Laplace Transforms one can show that:

$$\tau_{yx} = \gamma_0 \int_{-\infty}^0 G(t-t') \delta(t') dt' = \gamma_0 G(t) \quad (2.5.19)$$

(See problem 10 for an alternate derivation of equation of 2.5.19.) If the strain is applied at some

time t_0 , then $\tau_{yx} = \gamma_0 G(t - t_0)$. Thus, equation 2.5.19 shows that a viscoelastic fluid responds transiently to an applied strain. For a Newtonian fluid, the relaxation function equals zero when motion stops.

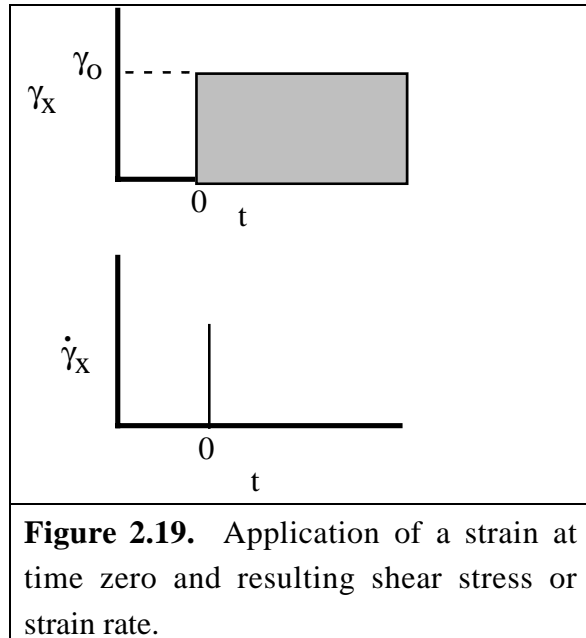


Figure 2.19. Application of a strain at time zero and resulting shear stress or strain rate.

(See problem 10 for an alternate derivation of equation of 2.5.19.) If the strain is applied at some time t_0 then $\tau_{yx} = \gamma_0 G(t - t_0)$. Thus, equation 2.5.19 shows that a viscoelastic fluid responds transiently to an applied strain. For a Newtonian fluid, the relaxation function equals zero when motion stops.

A variety of different transients are used to study viscoelastic behavior. The response to an imposed oscillatory shear rate (or velocity for flow of a viscous fluid through narrow channels) is the most common input for the quantitative characterization of biological and nobiological fluids. The most general statement is to represent an imposed oscillation using the complex representation:

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t) \tag{2.5.20}$$

Thus,

$$\dot{\gamma}_x(t) = \dot{\gamma}_x^0 \exp(i\omega t) \tag{2.5.20a}$$

and

$$\gamma(t) = \frac{\gamma_0 \exp(i\omega t)}{i\omega} \tag{2.5.20b}$$

The shear stress can also be written in complex notation:

$$\tau_{yx} = (\eta' + i\eta'') \dot{\gamma}_x^0 \tag{2.5.21a}$$

$$\tau_{yx} = (G' + iG'')\gamma_0 \tag{2.5.21b}$$

where η' and η'' are the real and imaginary components of the complex viscosity and G' and G'' are the real and imaginary components of the complex modulus of elasticity. Since $\dot{\gamma}_x^0$ and γ_0 are related (equation 2.5.20a,b), equation 2.5.21b can also be written as:

$$\tau_{yx} = i\omega(\eta' + i\eta'')\gamma_0 \tag{2.5.22}$$

and

$$G' = -\omega\eta'' \qquad G'' = \omega\eta' \tag{2.5.23a,b}$$

G' and G'' are also known as the storage and loss moduli, respectively. For a perfectly elastic material, G' equals the elastic modulus and G'' equals zero. G' also represents the storage of elastic energy, hence the term elastic modulus. Likewise, for a viscous fluid η' equals the viscosity and η'' equals zero. As a result, G'' is nonzero. Since viscous forces dissipate energy, G'' is known as the loss modulus. The relative ratio of G' and G'' (or η'' and η') provides an indication of the relative importance of viscous and elastic contributions to material behavior.

Synovial fluid is present between joints. The fluid consists of electrolytes and proteins. The major protein consists of a protein core linked to hyaluronic acid. Joints contain about 0.2-0.5 ml of fluid. This fluid serves to lubricate joints and absorb compressive loads. Rheological data are presented for synovial fluid in Figure 2.20. Data were obtained in an oscillating Couette rheometer. This data show clearly that synovial fluid exhibits viscoelastic behavior. At low frequencies, The loss modulus is much larger than the storage modulus ($G'' \gg G'$) and the fluid behavior is largely viscous. As the frequency increases, both moduli increase in magnitude. For frequencies greater than 2 rad/s the storage modulus exceeds the loss

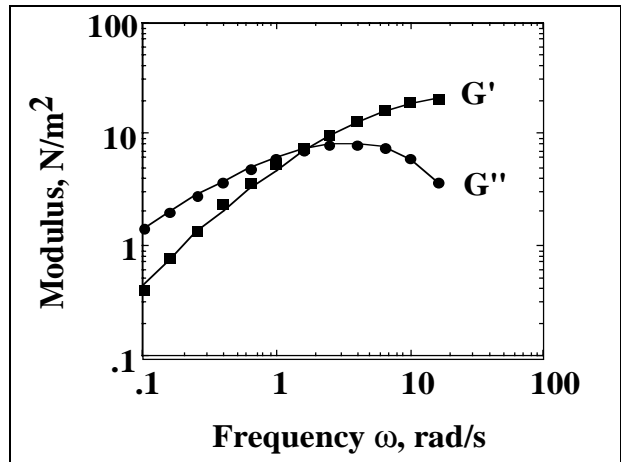


Figure 2.20. Storage and loss moduli for human synovial fluid. From (Balazs and Gibbs, 1970).

modulus and elastic behavior dominates. Consistent with this behavior, steady flow measurements in a viscometer indicate that synovial fluid exhibits shear thinning behavior. Frequencies associated with walking (3 rad/s) and running (15.7 rad/s) lie in a range where elastic forces dominate. Elastic behavior may be beneficial at these frequencies whereas viscous behavior may be helpful when bearing loads without motion. With age, both the storage and loss moduli decrease and G'' contributes more significantly at all frequencies.

2.6. Laminar and Turbulent Flow A remarkable feature of fluid flow is that fluid motion undergoes a transition in its behavior at higher flow rates. Such a change in fluid motion is evident to anyone who has watched smoke rise from a fire or water stream from a hose or faucet. In the late nineteenth century, Osborne Reynolds documented this transition for flow in cylindrical tubes. Consider a Newtonian fluid flowing in a cylindrical tube as shown in Figure 2.21A. A dye is injected continuously into the fluid. For low flow rates, the dye traces a straight line path. (A more quantitative statement about what constitutes a low flow rate is considered below.) The width of the dye band does broaden due to molecular diffusion as the dye moves farther downstream. Since this is a steady flow, the dye pathline traces a fluid streamline. Flow in which the fluid travels in a steady time-independent manner at each location is known as *laminar* flow. Fluid motion is said to occur in lamina, in this case, concentric cylinders around the centerline.

Now consider that the flow rate is increased. At first, the dye pattern is the same. But at some flow rate, depending on the radius of the cylinder and the fluid properties, the flow pattern undergoes a transition. Dye starts to move radially in a random manner. At first, this behavior occurs intermittently and for a period the flow resumes its laminar behavior. As the flow rate increases, the intermittent random and chaotic behavior of the dye becomes more common until the behavior is completely chaotic (Figure 2.21B). Dye is dispersed throughout the entire radius of the tube. Such fluid flow is known as *turbulence*.

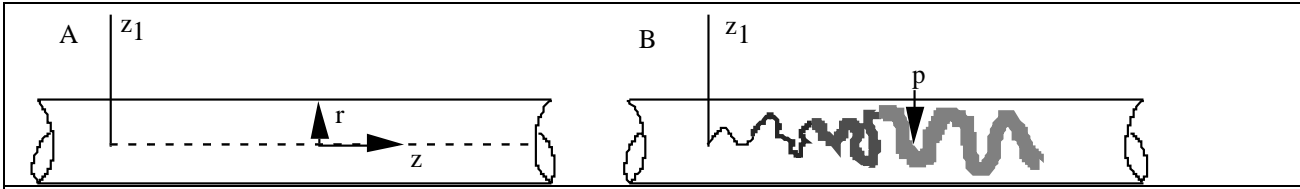


Figure 2.21. Release of dye into a stream of flowing Newtonian liquid and demonstration of transition from laminar (A) to turbulent (B) flow.

In steady laminar flow the velocity at a given location does not change with time unless some external disturbance is applied. A significant feature of turbulent flow is that, even for steady flow, the velocity at a given location changes with time as shown in Figure 2.22 for point p in Figure 2.21B. The time averaged velocity at a given location is, however, a well-defined quantity, as shown by the dotted line in Figure 2.22. Often the velocity in turbulent flow is characterized as the sum of a time averaged component and a fluctuating component with a zero time average.

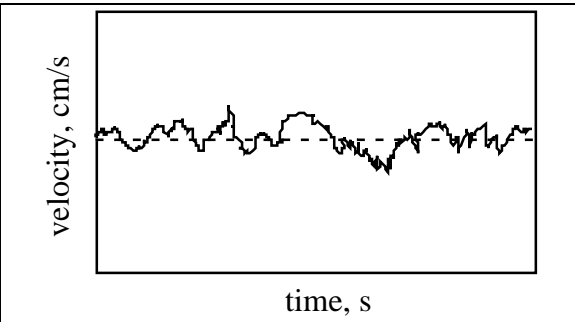


Figure 2.22. Velocity fluctuations in turbulent flow at location p in Figure 2.21B.

Turbulence affects many important transport properties. For example, at low flow rates, the pressure drop is proportional to the average velocity. For turbulent flow in a cylindrical tube, the pressure drop is proportional to the velocity raised to the 1.75 power. For biological organisms, such a change in the functional relationship between pressure drop and velocity will affect the energy needed to create such a pressure drop. Also turbulence affects the transport of molecules, often producing more rapid mixing than observed under laminar flow conditions. Turbulent stresses can, however, damage biological molecules and cells.

Whether a flow is laminar or turbulent can be determined simply by computation of a single ratio of inertial to viscous forces known as the Reynolds number (Re). This quantity is generally defined as: