Duke University
Department of Civil and Environmental Engineering
CE 130L. Uncertainty, Design, and Optimization
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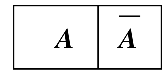
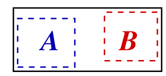
Summary of some Rules of Probability with Examples



- Complementary Events

  If the probability of event A occurring is $P[A]$ then the probability of event A not occurring, $P[\bar{A}]$, is given by $P[\bar{A}] = 1 - P[A]$.

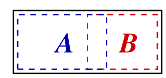
  > Example: This and following examples pertain to traffic and accidents on a certain stretch of highway from 8am to 9am on weekdays.
  > If the probability of no accident is 0.789 then the probability of an accident is 0.211.



- Mutually Exclusive (Non-Intersecting) Events (ME)

  Mutually exclusive events can not occur together; one precludes the other. If A and B are mutually exclusive events, then $P[A \text{ and } B] = 0$. All complementary events are mutually exclusive, but not vice versa.

  > Example: If X represents the uncertain flow of traffic in cars/minute, the following sets of events are mutually exclusive: $0 < X \leq 5$, $5 < X \leq 10$, and $20 < X \leq 50$.
  > The pair of events $X = 0$ and $20 < X \leq 50$ are mutually exclusive but not complementary.

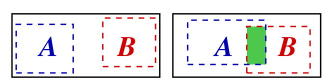


- Collectively Exhaustive Events (CE)

  If A and B are collectively exhaustive events then the set $[A \text{ and } B]$ represents all possible events, and $P[A \text{ or } B] = 1$, The sum of the probabilities of "ME" and "CE" events must be 1.00.

  > Example: Let Y represent the uncertain number of cars in an accident.
  > If $P[Y = 0] = 0.789$ (no accident) and $P[Y = 1] = 0.100$ then $P[Y > 1] = 0.111$.

  > Example: The following sets are mutually exclusive and collectively exhaustive:
  > $X = 0$, $0 < X \leq 5$, $5 < X \leq 10$, $10 < X \leq 20$, $20 < X \leq 50$, and $50 < X$.
  > In the above if the "$<$" were replaced by "$\leq$" the sets would be "CE" but not "ME".



- Conditionally Dependent Events

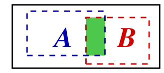
  If the occurrence (or non-occurrence) of an event A affects the probability of event B, then A and B are conditionally dependent; $P[B|A| = P[A \text{ and } B]/P[A]$, and $P[B|\bar{A}| = P[B \text{ and } \bar{A}]/P[\bar{A}]$.

  In the equation $P[B|A] = P[A \text{ and } B]/P[B]$, the set B can be thought of as a more restrictive set of all possibilities.

  > Example: The term $P[Y > 1|10 < X \leq 20]$ denotes the probability that a multi-car accident will occur given the traffic flow is between 10 and 21 cars per minute (non inclusive).

  If A and B are mutually exclusive, the occurrence of one implies the non-occurrence of the other, and therefore a dependence between A and B. So all mutually exclusive events are dependent, but not vice versa.

  > Example: The events $[Y > 1]$ and $[10 < X \leq 20]$ are dependent but not mutually exclusive.



- Statistically Independent Events

  If the occurrence (or nonoccurence) of event A has no bearing on the probability of B occurring, then A and B are statistically independent events; $P[A|B] = P[A]$.

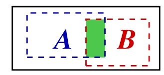
  > Example: If Z represents the uncertain number of migrating geese above above a certain stretch of highway from 8am-9am on a weekday, $P[Y > 1|Z > 20] = P[Y > 1]$.

  Intersecting sets of events can be dependent or independent.

  > Example: The events $[Y > 1]$ and $[Z > 20]$ can occur together (are not mutually exclusive; are intersecting) but are independnet.

  A pair of events cannot be both mutually exclusive and independent.

  > Example: The events $[Z \leq 10]$ and $[Z > 20]$ are mutually exclusive. The occurance of one implies the non-occurance of the other, and therefore a dependence.



- Intersection of Events

  If A and B are dependent events, $P[A \text{ and } B] = P[A \cap B] = P[AB] = P[A|B]P[B]$.

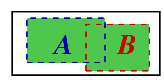
  > Example: If $P[Y > 1|10 < X \leq 20] = 0.07$ and $P[10 < X \leq 20] = 0.30$ then
  > $P[Y > 1 \cap 10 < X \leq 20] = (0.07)(0.30) = 0.021$.

  If A and B are independent events, $P[A|B] = P[A]$; and $P[A \cap B] = P[AB] = P[A]P[B]$

  > Example: $P[Y > 1|Z \leq 20] = P[Y > 1]$.
  > If $P[Z > 20] = 0.04$ then $P[Y > 1 \cap Z \leq 20] = (0.111)(0.96) = 0.107$.

  If A, B, and C, are dependent events, $P[A \cap B \cap C] = P[ABC] = P[A|BC]P[BC] = P[A|BC]P[B|C]P[C]$.

  If A, B, and C are independent events, $P[A|BC] = P[A]$, $P[B|C] = P[B]$, and
  $P[A \cap B \cap C] = P[ABC] = P[A]P[B]P[C]$



- Union of Events

  If A and B are not mutually exclusive events, $P[A \text{ or } B] = P[A \cup B] = P[A] + P[B] - P[AB]$.

  > Example: $P[Y > 1 \cup 10 < X \leq 20] = 0.111 + 0.30 - 0.021 = 0.390$

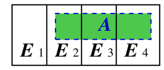
  If A and B are mutually exclusive events, $P[A \cap B] = 0$ and $P[A \cup B] = P[A] + P[B]$.

  > Example: $P[Y = 0 \cup Y > 1] = 0.789 + 0.111 = 0.90$

  If $E_1, E_2, \cdots, E_n$ are n events $P[E_1 \cup E_2 \cup \cdots \cup E_n] = 1 - P[\bar{E}_1 \cup \bar{E}_2 \cup \cdots \cup \bar{E}_n]$

  If $E_1, E_2, \cdots, E_n$ are n mutually exclusive events,
  $P[E_1 \cup E_2 \cup \cdots \cup E_n] = P[E_1] + P[E_2] + \cdots + P[E_n]$

- Theorem of Total Probability



If $E_1, E_2, \cdots, E_n$ are n mutually exclusive (M.E.) and collectively exhaustive (C.E.) events, and if A is an event that shares the same space as the events E_i, ($P[A|E_i] > 0$ for at least some events E_i) then

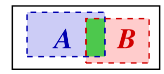
$$P[A] = P[A|E_1]P[E_1] + P[A|E_2]P[E_2] + \cdots + P[A|E_n]P[E_n]$$

Example: The table below shows the probabilities of a number of events.

$P[X = 0]$	$P[0 < X \leq 10]$	$P[10 < X \leq 20]$	$P[20 < X \leq 50]$	$P[50 < X]$
0.00	0.20	0.30	0.35	0.15
$P[Y > 1\|X = 0]$	$P[Y > 1\|0 < X \leq 10]$	$P[Y > 1\|10 < X \leq 20]$	$P[Y > 1\|20 < X \leq 50]$	$P[Y > 1\|50 < X]$
0.00	0.00	0.07	0.15	0.25

Then, $P[Y > 1] = (0.00)(0.00) + (0.00)(0.20) + (0.07)(0.30) + (0.15)(0.35) + (0.25)(0.15) = 0.111$

- Bayes' Theorem



For two dependent events A and B, $P[A \text{ and } B] = P[A|B]\ P[B] = P[B|A]\ P[A]$, so

$$P[A|B] = \frac{P[B|A]\ P[A]}{P[B]}.$$

If event A depends on n ME and CE events, $E_1, \ldots, E_n$

$$P[E_i|A] = \frac{P[A|E_i]\ P[E_i]}{P[A|E_1]P[E_1] + P[A|E_2]P[E_2] + \cdots + P[A|E_n]P[E_n]}.$$

Example: Using the probabilities from the table above, given the observation that a multi-car accident occured, find the probability that the the traffic was very heavy ($50 < X$).
$P[50 < X|Y > 1] = P[Y > 1|50 < X]P[50 < X]/P[Y > 1] = (0.25)(0.15)/(0.111) = 0.338.$

- Bernoulli sequence with probability p of independent events (mean return period $\tilde{T} = 1/p$).

Example: Given the probability of a multi-car accident on any given day is 0.111, and assuming the probability of an accident today in no way depends on the occurance of an accident on any other day, what is the probability of two days with multi-car accidents this work-week?
Given:
$P[\text{accident on any given day}] = p = 0.111.$
From the intersection of independent events:
$P[\text{accident on any two days}] = (p)(p) = (0.111)(0.111) = (0.111)^2 = 0.0123.$
From complementary events:
$P[\text{no accident on any given day}] = 1 - p = 1 - 0.111 = 0.889.$
From the intersection of independent events:
$P[\text{no accident on any three days}] = (1 - p)(1 - p)(1 - p) = (1 - 0.111)^3 = 0.7026.$
From the intersection of independent events:
$P[\text{accident on any two days} \cap \text{no accident on any three days}] = (p)^2(1 - p)^{5-2} = (0.0123)(0.7026)$
From the union of independent events:
number of ways to pick two days out of five $= 4 + 3 + 2 + 1 = 5!/(2!\ 3!) = 120/((2)(6)) = 10$
$P[\text{two days with accidents out of five}] = (10)(.0123)(.7026) = 0.0864$

This is called the binomial distribution.

$$P[n \text{ events out of } m \text{ attempts}] = \frac{m!}{n!(m - n)!} p^n (1 - p)^{m-n}$$

$$P[0 \text{ events out of } m \text{ attempts}] = (1 - p)^m$$

- Poisson process with mean occurrence rate ν (mean return period $\tilde{T} = 1/\nu$).

  For a Bernoulli sequence with a small event probability p and a large number of trials, the binomial distribution approaches the Poisson distribution, in which the event probability p is replaced by a mean occurance rate, ν, or a return period, $\tilde{T}$.

$$P[n \text{ events during time } t] = ((t/\tilde{T})^n/n!)\ e^{-t/\tilde{T}}$$

$$P[\text{time between two events} > t] = P[0 \text{ events during time } t] = e^{-t/\tilde{T}}$$

$$P[\text{time between two events} \leq t] = 1 - P[0 \text{ events during time } t] = 1 - e^{-t/\tilde{T}}$$

Example: Given the probability of a multi-car accident on any given work-day is 0.111, the return period of a multi-car accident is 1/0.111 work-days or 9 work-days. On average drivers would expect a multi-car accident every nine work-days.
Assuming multi-car accidents are a Poisson process, find $P[\text{two accidents in five days}]$
$P[2 \text{ accidents in 5 days}] = (5/9)^2/2! \exp(-5/9) = 0.0884$
...only slightly more than what one would get assuming a Bernoulli sequence.
$P[\text{time between accidents} > \tilde{T}] = P[0 \text{ accidents in time } \tilde{T}] = e^{-\tilde{T}/\tilde{T}} = e^{-1} = 0.368$
$P[\text{one or more accident in } \tilde{T} \text{ work-days}] = 1 - ((\tilde{T}/\tilde{T})^0/0!)\ e^{-\tilde{T}/\tilde{T}} = 1 - e^{-1} = 0.632$