

**THE THEOREMS OF CASTIGLIANO**  
**CE 130L. — Uncertainty, Design, and Optimization**  
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The flexibility relationship between a colocated force,  $F$ , and displacement,  $D$ , in statically determinate systems can be determined using the *principle of real work*,

$$\frac{1}{2} F D = U = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} dV . \quad (1)$$

The flexibility relationships for statically indeterminate systems, or systems with multiple external forces or distributed loads, can not be found with the principle of real work.

Instead, a new method must be developed.

### Castigliano's Theorem - Part I (Force Theorem)

$$U = \int F \, dD \quad \dots \quad \text{strain energy}$$

$$F_i = \frac{\Delta U}{\Delta D_i} = \frac{\partial U}{\partial D_i} \quad (2)$$

A force,  $F_i$ , on an elastic solid is equal to the derivative of the *strain energy* with respect to the displacement,  $D_i$ , in the direction and location of the force  $F_i$ .

### Castigliano's Theorem - Part II (Deflection Theorem)

$$U^* = \int D \, dF \quad \dots \quad \text{complementary strain energy}$$

$$D_i = \frac{\Delta U^*}{\Delta F_i} = \frac{\partial U^*}{\partial F_i} \quad (3)$$

A displacement,  $D_i$ , in an elastic solid is equal to the derivative of the *complementary strain energy* with respect to the force,  $F_i$ , in the direction and location of the displacement  $D_i$ .

If the solid is linear elastic, then  $U^* = U$ .

**Castigliano's Deflection Theorem:** (1873)

*The partial derivative of the complementary strain energy of a linearly elastic system with respect to a selected force acting on the system gives the displacement of that force along its direction.*

$$\frac{\partial U^*}{\partial F_i} = D_i$$

For linear elastic solids

$$U^* = U = \frac{1}{2} \int_l \frac{N^2}{EA} dl + \frac{1}{2} \int_l \frac{M_z^2}{EI_z} dl + \frac{1}{2} \int_l \frac{M_y^2}{EI_y} dl + \frac{1}{2} \int_l \frac{V_z^2}{G(A/\alpha_z)} dl + \frac{1}{2} \int_l \frac{V_y^2}{G(A/\alpha_y)} dl + \frac{1}{2} \int_l \frac{T^2}{GJ} dl . \quad (4)$$

So,

$$\frac{\partial U^*}{\partial F_i} = \frac{\partial U}{\partial F_i} = \int_l \frac{N \frac{\partial N}{\partial F_i}}{EA} dl + \int_l \frac{M_z \frac{\partial M_z}{\partial F_i}}{EI_z} dl + \int_l \frac{M_y \frac{\partial M_y}{\partial F_i}}{EI_y} dl + \int_l \frac{V_z \frac{\partial V_z}{\partial F_i}}{G(A/\alpha_z)} dl + \int_l \frac{V_y \frac{\partial V_y}{\partial F_i}}{G(A/\alpha_y)} dl + \int_l \frac{T \frac{\partial T}{\partial F_i}}{GJ} dl . \quad (5)$$

## Superposition

Superposition is an extremely powerful method for reducing complicated problems down to a set of more simple problems. To use the principle of superposition, the system must behave in a *linear* elastic fashion.

The principle of superposition states:

*Any response of a linear system to multiple inputs can be represented as the sum of the responses to the inputs taken individually.*

By “response” we can mean a strain, a stress, a deflection, an internal force, a rotation, an internal moment, etc.

By “input” we can mean an externally applied load, a temperature change, a support settlement, etc.

## Strain Energy and Temperature Changes

Consider a statically determinate system subject to only temperature change.

- What are the reactions?
- What are the internal forces and moments?
- What are the internal stresses?
- What is the strain energy?
- What are the strains?
- Are the deflections zero?

How can we find a deflection of interest?

1. *Before* temperature changes are applied, apply a “virtual force”,  $F$ , collocated with the desired deflection. Hold  $F$  constant.
2. Apply the change in temperature, and find the strain energy,  $U$ , of that *constant* force moving through the temperature-induced displacements.
3. Use Castigliano’s Second Theorem ( $D = \partial U^* / \partial F$ ) to find an expression for  $D$ . Evaluate  $D$  for  $F = 0$ .

$$U = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} dV + \int_V \{\sigma\}^T \{\alpha \Delta T\} dV \quad (6)$$

**Axial Forces**

$$\sigma = N/A$$

$$dV = A dl$$

$$\sigma = E\epsilon$$

$$U = \frac{1}{2} \int_l \frac{N^2}{EA} dl + \int_l N \alpha \Delta T dl \quad (7)$$

$$D = \frac{\partial U}{\partial F} = \int_l \frac{N \frac{\partial N}{\partial F}}{EA} dl + \int_l \frac{\partial N}{\partial F} \alpha \Delta T dl \quad (8)$$

*Example*

**Bending Moments**     $\sigma = -My/I$      $dV = A dl = b(y) dy dl$      $\sigma = E\epsilon$

Assume a *linear* variation in temperature through the cross section.

$$\Delta T(y) = \frac{1}{2} (\Delta T_{\text{top}} + \Delta T_{\text{bottom}}) + \frac{1}{h} (\Delta T_{\text{top}} - \Delta T_{\text{bottom}}) y \quad (9)$$

This is a *big* assumption.

$$\begin{aligned} U &= \frac{1}{2} \int_V \frac{\sigma^2}{E} dV + \int_V \sigma \alpha \Delta T(y) dV \\ &= \frac{1}{2} \int_V \frac{M^2 y^2}{EI^2} dA dl + \int_V -\frac{My}{I} \alpha \Delta T(y) b(y) dy dl \\ &= \frac{1}{2} \int_l \frac{M^2 y^2}{EI^2} dA dl \\ &\quad + \int_V -\frac{My}{I} \alpha \frac{1}{2} (\Delta T_{\text{top}} + \Delta T_{\text{bottom}}) b(y) dy dl \\ &\quad + \int_V -\frac{My}{I} \alpha \frac{1}{h} (\Delta T_{\text{top}} - \Delta T_{\text{bottom}}) y b(y) dy dl \end{aligned}$$

Recall:  $I = \int_A y^2 dA = \int_y y^2 b(y) dy \dots$  and  $\dots 0 = \int_A y dA = \int_y y b(y) dy \dots$

$$U = \frac{1}{2} \int_l \frac{M^2}{EI} dl - \int_l M \alpha \frac{1}{h} (\Delta T_{\text{top}} - \Delta T_{\text{bottom}}) dl \quad (10)$$

$$D = \frac{\partial U}{\partial F} = \int_l \frac{M \frac{\partial M}{\partial F}}{EI} dl - \int_l \frac{\partial M}{\partial F} \alpha \frac{1}{h} (\Delta T_{\text{top}} - \Delta T_{\text{bottom}}) dl \quad (11)$$

## Castigliano's Theorem and Superposition for Statically Indeterminate Systems

Recall the strain energy method formulation for a statically indeterminate system with three redundant forces: The redundant forces are:  $R_B$ ,  $R_C$ ,

and  $R_D$ . Note that in this problem the reaction forces do not move (or settle) in the direction of the reaction forces. The reactions are fixed.

From the principle of superposition:

$$M(x) = M_o(x) + m_1(x)R_B + m_2(x)R_C + m_3(x)R_D \quad (12)$$

$$N = N_o + n_1R_B + n_2R_C + n_3R_D \quad (13)$$

The total strain energy,  $U$ , in systems with bending strain energy and axial strain energy is,

$$U = \frac{1}{2} \int_0^L \frac{M(x)^2}{EI} dx + \frac{1}{2} \sum \frac{N^2 H}{EA} \quad (14)$$

We are told that the displacements at points B, C, and D are all zero and we will assume the structure behaves linear elastically, therefore, from Castigliano's Second Theorem,

$$D_i = \frac{\partial U^*}{\partial F_i} = \frac{\partial U}{\partial F_i},$$

we obtain three expressions for the facts that  $D_B = 0$ ,  $D_C = 0$ , and  $D_D = 0$ .

$$D_B = 0 = \frac{\partial U}{\partial R_B}$$

$$D_C = 0 = \frac{\partial U}{\partial R_C}$$

$$D_D = 0 = \frac{\partial U}{\partial R_D}$$

Inserting equation (14) into the three expressions for zero displacement at the fixed reactions, noting that  $EI$  and  $EA$  are constants in this problem, and noting that the strain energy,  $U$ , depends on the reactions  $R$ , only through the internal forces,  $M$  and  $N$ , we obtain

$$D_B = 0 = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial R_B} dx + \frac{H}{EA} N \frac{\partial N}{\partial R_B}$$

$$D_C = 0 = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial R_C} dx + \frac{H}{EA} N \frac{\partial N}{\partial R_C}$$

$$D_D = 0 = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial R_D} dx + \frac{H}{EA} N \frac{\partial N}{\partial R_D}$$

Now, from the superposition equations (12) and (13),  $\partial M(x)/\partial R_B = m_1(x)$ ,  $\partial M(x)/\partial R_C = m_2(x)$ ,  $\partial M(x)/\partial R_D = m_3(x)$ ,  $\partial N(x)/\partial R_B = n_1$ ,  $\partial N(x)/\partial R_C = n_2$ , and  $\partial N(x)/\partial R_D = n_3$ . Inserting these expressions and the superposition equations (12) and (13) into the above equations for  $D_B$ ,  $D_C$ , and  $D_D$ ,

$$D_B = 0 = \frac{1}{EI} \int_0^L [M_o + m_1 R_B + m_2 R_C + m_3 R_D] m_1 dx + \frac{H}{EA} [N_o + n_1 R_B + n_2 R_C + n_3 R_D] n_1$$

$$D_C = 0 = \frac{1}{EI} \int_0^L [M_o + m_1 R_B + m_2 R_C + m_3 R_D] m_2 dx + \frac{H}{EA} [N_o + n_1 R_B + n_2 R_C + n_3 R_D] n_2$$

$$D_D = 0 = \frac{1}{EI} \int_0^L [M_o + m_1 R_B + m_2 R_C + m_3 R_D] m_3 dx + \frac{H}{EA} [N_o + n_1 R_B + n_2 R_C + n_3 R_D] n_3$$

These three expressions contain the three unknown reactions  $R_B$ ,  $R_C$ , and  $R_D$ . Everything else in these equations ( $m_1(x)$ ,  $m_2(x)$  ...  $n_3$ ) can be found without knowing the unknown reactions. By taking the unknown reactions out of the integrals (they are constants), we can write these three equations in matrix form.

$$\begin{bmatrix} \int_0^L \frac{m_1 m_1}{EI} dx + \frac{n_1 n_1 H}{EA} & \int_0^L \frac{m_1 m_2}{EI} dx + \frac{n_1 n_2 H}{EA} & \int_0^L \frac{m_1 m_3}{EI} dx + \frac{n_1 n_3 H}{EA} \\ \int_0^L \frac{m_2 m_1}{EI} dx + \frac{n_2 n_1 H}{EA} & \int_0^L \frac{m_2 m_2}{EI} dx + \frac{n_2 n_2 H}{EA} & \int_0^L \frac{m_2 m_3}{EI} dx + \frac{n_2 n_3 H}{EA} \\ \int_0^L \frac{m_3 m_1}{EI} dx + \frac{n_3 n_1 H}{EA} & \int_0^L \frac{m_3 m_2}{EI} dx + \frac{n_3 n_2 H}{EA} & \int_0^L \frac{m_3 m_3}{EI} dx + \frac{n_3 n_3 H}{EA} \end{bmatrix} \begin{bmatrix} R_B \\ R_C \\ R_D \end{bmatrix} = - \begin{bmatrix} \int_0^L \frac{M_o m_1}{EI} dx + \frac{N_o n_1 H}{EA} \\ \int_0^L \frac{M_o m_2}{EI} dx + \frac{N_o n_2 H}{EA} \\ \int_0^L \frac{M_o m_3}{EI} dx + \frac{N_o n_3 H}{EA} \end{bmatrix} \quad (15)$$

This 3-by-3 matrix is called a *flexibility matrix*,  $\mathbf{F}$ . The values of the terms in the flexibility matrix depend only on the responses of the structure to unit loads placed at various points in the structure. The flexibility matrix is therefore a property of the structure alone, and does not depend upon the loads on the structure<sup>1</sup>. The vector on the right-hand-side depends on the loads on the structure. Recall that this matrix looks a lot like the matrix from the three-moment equation. All flexibility matrices share several properties:

- All flexibility matrices are symmetric.
- No diagonal terms are negative.
- Flexibility matrices for structures which can not move or rotate without deforming are *positive definite*. This means that all of the eigenvalues of a flexibility matrix describing a fixed structure are positive.
- The unknowns in a flexibility matrix equation are forces (or moments).
- The number of equations (rows of the flexibility matrix) equals the number of unknown forces (or moments).

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<sup>1</sup>There are some fascinating cases in which the behavior does depend upon the loads, but that is a story for another day!

It is instructive to now examine the meaning of the terms in the matrix,  $\mathbf{F}$

$$F_{11} = \int_0^L \frac{m_1 m_1}{EI} dx + \frac{n_1 n_1 H}{EA} = \delta_{11}$$

$$F_{21} = \int_0^L \frac{m_2 m_1}{EI} dx + \frac{n_2 n_1 H}{EA} = \delta_{21}$$

$$F_{12} = \int_0^L \frac{m_1 m_2}{EI} dx + \frac{n_1 n_2 H}{EA} = \delta_{12}$$

$$F_{31} = \int_0^L \frac{m_3 m_1}{EI} dx + \frac{n_3 n_1 H}{EA} = \delta_{31}$$

The fact that  $F_{12} = F_{21}$  is called *Maxwell's Reciprocity Theorem*.