THE THEOREMS OF BETTI, MAXWELL, AND CASTIGLIANO

CE 130 — Structural Design and Optimization

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Betti's Theorem:

Consider a structure with a set of coordinates, $i = 1, 2, \cdots, n, n + 1, \cdots, N$.

Consider two systems of forces, $\{P_i\}$, $(i = 1, \cdots, n)$ and $\{Q_i\}$, $(i = n+1, \cdots, N)$.

Suppose that the $\{P\}$ system is applied to the structure alone, producing displacements $\{D_P\}$, stresses $\{\sigma_P\}$, and strains $\{\epsilon_P\}$. The internal work is equal to the external work:

$$\frac{1}{2} \int_V \{\sigma_P\}^T \{\epsilon_P\} \, dV = \frac{1}{2} \sum_{i=1}^{n} P_i D_{P_i} \tag{1}$$

Instead, suppose that the $\{Q\}$ system is applied to the structure alone, producing displacements $\{D_Q\}$, stresses $\{\sigma_Q\}$, and strains $\{\epsilon_Q\}$. Equating internal work and internal work,

$$\frac{1}{2}\int_V \{\sigma_Q\}^T\{\epsilon_Q\}\, dV = \frac{1}{2}\sum_{i=n+1}^{N} Q_i D_{Q_i} \tag{2}$$

Now, with the $\{P\}$ system of forces acting, apply the $\{Q\}$ system of forces. The internal and external work done by the $\{Q\}$ forces must be equal.

$$\frac{1}{2}\int_V \{\sigma_Q\}^T\{\epsilon_Q\}\, dV + \int_V \{\sigma_P\}^T\{\epsilon_Q\}\, dV = \frac{1}{2}\sum_{i=n+1}^{N} Q_i D_{Q_i} + \sum_{i=1}^{n} P_i D_{Q_i} \tag{3}$$

Substituting equation (2) into equation (3) gives

$$\int_V \{\sigma_P\}^T\{\epsilon_Q\}\, dV = \sum_{i=1}^{n} P_i D_{Q_i} \tag{4}$$

Instead, with the $\{Q\}$ system of forces acting, apply the $\{P\}$ system of forces. The internal and external work done by the $\{P\}$ forces must be equal.

$$\frac{1}{2}\int_V \{\sigma_P\}^T\{\epsilon_P\}\, dV + \int_V \{\sigma_Q\}^T\{\epsilon_P\}\, dV = \frac{1}{2}\sum_{i=1}^{n} P_i D_{P_i} + \sum_{i=n+1}^{N} Q_i D_{P_i} \tag{5}$$

Substituting equation (1) into equation (5) gives

$$\int_V \{\sigma_Q\}^T\{\epsilon_P\}\, dV = \sum_{i=n+1}^{N} Q_i D_{P_i} \tag{6}$$

Betti's Theorem[1] (1872)

For a linear elastic structure, equations (6) and (4) are equivalent.

Proof: In linear elastic structures stress is proportional to strain. We can write this fact for all six of our stress and strain components as

$$\{\sigma_P\}_{6\times1} = [S]_{6\times6}\{\epsilon_P\}_{6\times1}, \qquad \{\sigma_Q\}_{6\times1} = [S]_{6\times6}\{\epsilon_Q\}_{6\times1},$$

where the material stiffness matrix $[S]$ is symmetric. Substituting,

$$\{\sigma_P\}^T\{\epsilon_Q\} = \{\sigma_P\}^T[S]^{-1}\{\sigma_Q\}, \qquad \{\sigma_Q\}^T\{\epsilon_P\} = \{\sigma_Q\}^T[S]^{-1}\{\sigma_P\}.$$

[1] Betti, E., *Il Nuovo Cimento.* Series 2, Vol's 7 and 8, 1872.

The right hand sides of these two equations are equal to each other.

$$\{\sigma_P\}^T [S]^{-1} \{\sigma_Q\} = \{\sigma_Q\}^T [S]^{-T} \{\sigma_P\} = \{\sigma_Q\}^T [S]^{-1} \{\sigma_P\}$$

Therefore,

$$\int_V \{\sigma_Q\}^T \{\epsilon_P\} \; dV = \int_V \{\sigma_P\}^T \{\epsilon_Q\} \; dV,$$

and

$$\sum_{i=1}^{n} P_i D_{Q_i} = \sum_{i=n+1}^{N} Q_i D_{P_i}$$

QED

Maxwell's Theorem[2] (1864) ... a special case of Betti's theorem.
The flexibility matrix is symmetric.

Proof: Assume that there is only one force, $P_i = 1$ acting at coordinate i in the $\{P\}$ system, and one force $Q_j = 1$ acting at coordinate j in the $\{Q\}$ system.

Betti's Theorem states that

$$1 \cdot D_{Q_i} = 1 \cdot D_{P_j} \qquad \text{or} \qquad D_{Q_i} = D_{P_j} \qquad \text{or} \qquad f_{ij} = f_{ji}$$

where f_{ij} is the displacement at i due to a unit force at j, and f_{ji} is the displacement at j due to a unit force at i. That is, f_{ij} is a *flexibility coefficient.* In other words, the flexibility matrix is symmetric. QED

[2]Maxwell, J.C., "On the Caclulation of the Equilibrium nad Stiffness of Frames," *Philosophical Magazine,* vol. 27, pp. 294–299, 1864.

Castigliano's Deflection Theorem: (1873)

The partial derivative of the strain energy of a linearly elastic system with respect to a selected force acting on the system gives the displacement of that force along its direction.

$$\frac{\partial U}{\partial F_i} = D_i$$

Recall that

$$\begin{aligned} U &= \frac{1}{2}\int_l \frac{N^2}{EA}\, dl + \frac{1}{2}\int_l \frac{M_z^2}{EI_z}\, dl + \frac{1}{2}\int_l \frac{M_y^2}{EI_y}\, dl + \\ & \frac{1}{2}\int_l \frac{V_z^2}{G(A/\alpha_z)}\, dl + \frac{1}{2}\int_l \frac{V_y^2}{G(A/\alpha_y)}\, dl + \frac{1}{2}\int_l \frac{T^2}{GJ}\, dl. \end{aligned}$$

So

$$\begin{aligned} \frac{\partial U}{\partial F_i} &= \int_l \frac{N \frac{\partial N}{\partial F_i}}{EA}\, dl + \int_l \frac{M_z \frac{\partial M_z}{\partial F_i}}{EI_z}\, dl + \int_l \frac{M_y \frac{\partial M_y}{\partial F_i}}{EI_y}\, dl + \\ & \int_l \frac{V_z \frac{\partial V_z}{\partial F_i}}{G(A/\alpha_z)}\, dl + \int_l \frac{V_y \frac{\partial V_y}{\partial F_i}}{G(A/\alpha_y)}\, dl + \int_l \frac{T \frac{\partial T}{\partial F_i}}{GJ}\, dl. \end{aligned}$$

Notes:

1. The terms $\frac{\partial M}{\partial F_i}$ correspond to the virtual force terms in the principle of virtual work.
2. Castigliano's Theorems as presented here require a linear elastic system. The principle of virtual work does not. Therefore the principle of virtual work is more general.
3. Generalizations of Castigliano's Theorems for nonlinear elastic systems exist, and are described by T. Au, *Elementary Structural Mechanics,* (Prentice Hall, 1963).

Proof of Castigliano's Theorem

Consider a structure subjected to a set of forces F_i, $i = 1, \cdots, N$. The internal strain energy created by these forces is a (positive) function of these forces.

$$U = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}\, dV = U(F_1, F_2, \cdots, F_k, \cdots F_N).$$

If all of these forces increase by a small amount, ΔF_i, then the change in the internal strain energy, ΔU, will be

$$\Delta U = \frac{\partial U}{\partial F_1}\Delta F_1 + \frac{\partial U}{\partial F_2}\Delta F_2 + \cdots + \frac{\partial U}{\partial F_k}\Delta F_k + \cdots + \frac{\partial U}{\partial F_N}\Delta F_N,$$

or if only force k were changed by an amount ΔF_k, then the change in the internal strain energy would be

$$\Delta U = \frac{\partial U}{\partial F_k}\Delta F_k.$$

So the total internal strain energy from all forces F_i ($i = 1, \cdots, N$), plus an extra force ΔF_k is

$$U + \Delta U = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}\, dV + \frac{\partial U}{\partial F_k}\Delta F_k.$$

Now if ΔF_k is applied first, it does a little bit of external work, $\frac{1}{2}\,\Delta F_k\,\Delta D_k$ (assuming linear elastic properties). This is so small that it can be neglected. Next, when the rest of the forces F_i, ($i = 1, \cdots, N$) are applied, the small force, ΔF_k, moves through a displacement D_k, and the total external work from all the forces (ΔF_k plus all the F_i's) equals (again assuming linear elasticity),

$$W_{\mathrm{E}} + \Delta W_{\mathrm{E}} = \frac{1}{2}\sum_{i=1}^{N} F_i D_i + \Delta F_k\, D_k.$$

Setting $U + \Delta U$ equal to $W_{\mathrm{E}} + \Delta W_{\mathrm{E}}$, and noting that $U = W_{\mathrm{E}}$, we get $\Delta U = \Delta W_{\mathrm{E}}$, or, $\frac{\partial U}{\partial F_k}\Delta F_k = \Delta F_k\, D_k$, or,

$$\frac{\partial U}{\partial F_k} = D_k.$$

QED