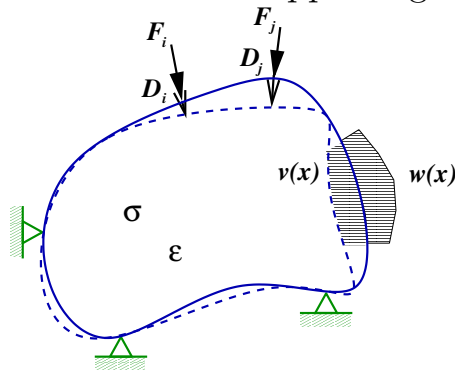


## Minimum Total Potential Energy, Quadratic Programming and Lagrange Multipliers

### 1 Minimum Total Potential Energy

Consider a linear elastic solid supporting external distributed loads  $w(x)$  and external point loads  $\{F\}$ , creating internal stresses  $\{\sigma\}$  and strains  $\{\epsilon\}$  throughout the volume of the solid, and displacements  $v(x)$  and  $\{D\}$  on the surface of the solid, consistent with its supporting reactions.



Here are four definitions. The internal strain energy is given by

$$U = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} dV , \quad (1)$$

the work of external forces  $w(x)$  and  $F$  moving through displacements  $v(x)$  and  $D$  is given by

$$W = \frac{1}{2} \int_l w(x) v(x) dl + \frac{1}{2} \{F\}^T \{D\} , \quad (2)$$

the potential energy function of the external loads is given by

$$V = \int_l w(x) v(x) dl + \{F\}^T \{D\} , \quad (3)$$

and the total potential energy function is given by

$$\Pi = U - V. \quad (4)$$

For an elastic solid in equilibrium, the expression  $U = W$  is a statement of the principle of real work: the work of external forces on an elastic solid is completely stored as strain energy within the solid.

The expression  $\delta\Pi = 0$  is a statement of the principle of minimum total potential energy.

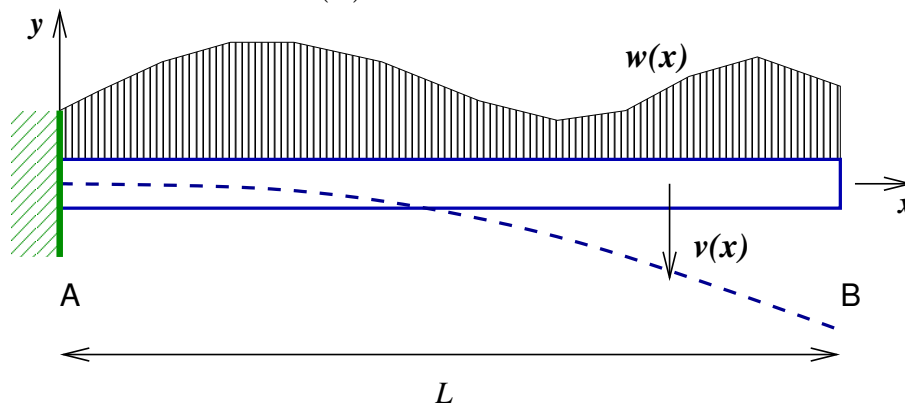
*Among all the possible displacements consistent with the reactions, the correct state of displacement is that which minimizes the total potential energy.*

If the displacements can be expressed in terms of a set of coefficients,  $\mathbf{a} = \{a_1, a_2, \dots\}$  then the coefficients become the unknown variables, and the correct values of the coefficients are those which minimize the total potential energy. Minimizing the total potential energy with respect to the coefficients is equivalent to setting the variation in the total potential energy with respect to the coefficients equal to zero, and solving for the coefficients. The variation in the total potential energy with respect to coefficient  $a_i$  is

$$\delta\Pi_i = \frac{\partial}{\partial a_i}\Pi(\mathbf{a}) \delta a_i = \frac{\partial}{\partial a_i}U(\mathbf{a}) \delta a_i - \frac{\partial}{\partial a_i}V(\mathbf{a}) \delta a_i . \quad (5)$$

## 2 Application to Beams in Bending

To understand that the equilibrium of an elastic solid may be analyzed as a minimization of the total potential energy, consider a cantilever beam carrying a distributed load  $w(x)$ .



In order to find the displacements  $v(x)$  corresponding to the minimum total potential energy, it is *essential* to approximate the displacement function in a way that is consistent with the boundary conditions. If the beam

is clamped at  $x = 0$ , then  $v(0)$  must be zero and  $v'(0)$  must be zero. A displacement function  $v(x)$  consistent with these boundary conditions can be expressed in terms of a simple polynomial, with coefficients  $\mathbf{a} = \{a_2, a_3, \dots\}$ .

$$v(x; \mathbf{a}) = a_2x^2 + a_3x^3 + a_4x^4, \quad (6)$$

from which

$$v''(x; \mathbf{a}) = 2a_2 + 6a_3x + 12a_4x^2. \quad (7)$$

Additional terms may be added for more complicated problems.

In beams, stresses are related to the bending moments,  $\sigma = -My/I$ , moments are related to the curvature,  $M = EI(v''(x))$ , the bending strain energy is

$$U = \frac{1}{2} \int_l \frac{M(x)^2}{EI} dl = \frac{1}{2} \int_l EI(v''(x; \mathbf{a}))^2 dl, \quad (8)$$

and the potential energy function of the external loads is

$$V = \int_l w(x) v(x; \mathbf{a}) dl. \quad (9)$$

Inserting equations (8) and (9) into equation (4),

$$\Pi = \frac{1}{2} \int_l EI(v''(x; \mathbf{a}))^2 dl - \int_l w(x) v(x; \mathbf{a}) dl. \quad (10)$$

The variation in the total potential energy with respect to coefficient  $a_i$  is

$$\begin{aligned} \delta\Pi_i &= \frac{\partial}{\partial a_i} \left( \frac{1}{2} \int_l EI(v''(x; \mathbf{a}))^2 dl \right) \delta a_i - \frac{\partial}{\partial a_i} \left( \int_l w(x) v(x; \mathbf{a}) dl \right) \delta a_i \\ &= \frac{1}{2} \int_l \frac{\partial}{\partial a_i} EI(v''(x; \mathbf{a}))^2 dl \delta a_i - \int_l \frac{\partial}{\partial a_i} w(x) v(x; \mathbf{a}) dl \delta a_i \\ &= \int_l EI v''(x; \mathbf{a}) \frac{\partial v''(x; \mathbf{a})}{\partial a_i} \delta a_i dl - \int_l w(x) \frac{\partial v(x; \mathbf{a})}{\partial a_i} \delta a_i dl. \end{aligned} \quad (11)$$

According to the principle of minimum total potential energy, the optimum coefficients,  $\mathbf{a}$ , correspond to  $\delta\Pi_i = 0$ . Inserting equations (6) and (7) into equation (11), setting  $\Pi_i$  equal to zero for each coefficient,  $a_2$ ,  $a_3$ , and  $a_4$ , and factoring out the  $\delta a_i$  terms, results in three equations for the three unknown

coefficients required to minimize the total potential energy  $\Pi$ .

$$0 = \delta\Pi_2 = \int_0^L EIv''(x) \cdot 2 \cdot dx - \int_0^L w(x) \cdot x^2 \cdot dx \quad (12)$$

$$0 = \delta\Pi_3 = \int_0^L EIv''(x) \cdot 6x \cdot dx - \int_0^L w(x) \cdot x^3 \cdot dx \quad (13)$$

$$0 = \delta\Pi_4 = \int_0^L EIv''(x) \cdot 12x^2 \cdot dx - \int_0^L w(x) \cdot x^4 \cdot dx \quad (14)$$

For constant values of  $EI$  and  $w$ , substituting equation (7) results in

$$0 = \delta\Pi_2 = EI \int_0^L (4a_2 + 12a_3x + 24a_4x^2)dx - w \int_0^L x^2 \cdot dx \quad (15)$$

$$0 = \delta\Pi_3 = EI \int_0^L (12a_2x + 36a_3x^2 + 72a_4x^3)dx - w \int_0^L x^3 \cdot dx \quad (16)$$

$$0 = \delta\Pi_4 = EI \int_0^L (24a_2x^2 + 72a_3x^3 + 144a_4x^4)dx - w \int_0^L x^4 \cdot dx \quad (17)$$

and integrating results in

$$0 = \delta\Pi_2 = EI(4a_2L + 6a_3L^2 + 8a_4L^3) - wL^3/3 \quad (18)$$

$$0 = \delta\Pi_3 = EI(6a_2L^2 + 12a_3L^3 + 18a_4L^4) - wL^4/4 \quad (19)$$

$$0 = \delta\Pi_4 = EI(8a_2L^3 + 18a_3L^4 + 28.8a_4L^5) - wL^5/5 \quad (20)$$

In matrix form, these three equations with three unknowns are

$$EI \begin{bmatrix} 4L & 6L^2 & 8L^3 \\ 6L^2 & 12L^3 & 18L^4 \\ 8L^3 & 18L^4 & 28.8L^5 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ a_4 \end{bmatrix} = w \begin{bmatrix} L^3/3 \\ L^4/4 \\ L^5/5 \end{bmatrix} \quad (21)$$

This matrix is always symmetric and invertible, and depends only on the properties of the beam alone ( $EI$  and  $L$ ). It can be thought of as a kind of *stiffness matrix*. Solution to this matrix equation gives the three coefficients

$$a_2 = wL^2/(4EI) \quad (22)$$

$$a_3 = -wL/(6EI) \quad (23)$$

$$a_4 = w/(24EI) \quad (24)$$

so,

$$v(x) = \frac{w}{EI} \left( \frac{1}{4}L^2x^2 - \frac{1}{6}Lx^3 + \frac{1}{24}x^4 \right), \quad (25)$$

which is *exactly* the deflection of a cantilever beam carrying a uniformly distributed load.

## 2.1 Virtual Work and the Principle of Minimum Potential Energy

Note that variations in  $v$  and  $v''$  may be expressed in terms of variations in the individual polynomial coefficients,  $\delta a_i$ .

$$\delta v = \frac{\partial v(x; \mathbf{a})}{\partial a_i} \delta a_i . \quad (26)$$

$$\delta v'' = \frac{\partial v''(x; \mathbf{a})}{\partial a_i} \delta a_i . \quad (27)$$

In the case of beams in bending, the variation of the strain energy is the work of the moments arising from the loads  $w$  rotating through curvatures  $\delta v'' dl$ ,

$$\delta U = \int_l EI v''(x) \delta v''(x) dl , \quad (28)$$

and the variation of the potential function of external loads is the work of the loads  $w$  moving through displacements,  $\delta v$ .

$$\delta V = \int_l w(x) \delta v(x) dl . \quad (29)$$

The variation in strain energy can be interpreted as internal virtual work, the variation in the potential energy function can be interpreted as external virtual work, and the principle of minimum potential energy is the same as the principle of virtual work, for example,

$$\int_l EI v''(x) \delta v''(x) dl = \int_l w(x) \delta v(x) dl . \quad (30)$$

### 3 Quadratic Programming

The previous example shows that the solution to the static equilibrium problem may be found by solving a matrix equation of the form

$$[\mathbf{K}]\{\mathbf{a}\} = \{\mathbf{f}\} . \quad (31)$$

This solution was derived from, and corresponds to, the minimization of the total potential energy function  $\Pi = U - V$ . It may now be seen that the total potential energy function may be expressed in terms of  $[\mathbf{K}]$ ,  $\{\mathbf{a}\}$ , and  $\{\mathbf{f}\}$  as follows:

$$\Pi = \frac{1}{2}\{\mathbf{a}\}^T[\mathbf{K}]\{\mathbf{a}\} - \{\mathbf{f}\}^T\{\mathbf{a}\} . \quad (32)$$

The variation in  $\Pi$  is then,

$$\delta\Pi = \frac{\partial\Pi(\mathbf{a})}{\partial\mathbf{a}}\delta\mathbf{a} = \{\mathbf{a}\}^T[\mathbf{K}]\{\delta\mathbf{a}\} - \{\mathbf{f}\}^T\{\delta\mathbf{a}\} . \quad (33)$$

Minimizing the total potential energy with respect to the unknown coefficients is equivalent to setting the variation of the total potential energy function with respect to the coefficients equal to zero,

$$(\{\mathbf{a}\}^T[\mathbf{K}] - \{\mathbf{f}\}^T)\{\delta\mathbf{a}\} = \{\mathbf{0}\} , \quad (34)$$

or

$$[\mathbf{K}]\{\mathbf{a}\} = \{\mathbf{f}\} , \quad (35)$$

so, equations (10) and (32) are equivalent expressions for the total potential energy function,

$$\Pi = U - V = \frac{1}{2}\{\mathbf{a}\}^T[\mathbf{K}]\{\mathbf{a}\} - \{\mathbf{f}\}^T\{\mathbf{a}\} , \quad (36)$$

in which the matrix  $[\mathbf{K}]$  is the Hessian of the strain energy

$$K_{ij} = \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} U = \frac{1}{2} \int_l EI \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} (v''(x; \mathbf{a}))^2 dl . \quad (37)$$

and the vector  $\{\mathbf{f}\}$  is the gradient of the potential energy function of external forces

$$f_i = \frac{\partial}{\partial a_i} V = \int_l w \frac{\partial}{\partial a_i} v(x; \mathbf{a}) dl . \quad (38)$$

### 3.1 Constraints and Lagrange Multipliers

Returning to the example of the cantilever beam, if the beam is supported at additional locations  $x = x_p$  and  $x = x_q$  so that  $v(x_p) = 0$  and  $v(x_q) = 0$ , the displacements must minimize  $\Pi$  subject to the constraints that  $v(x_p) = 0$  and  $v(x_q) = 0$ .

With these additional conditions, the problem is now more complicated, and the beam displacement should be approximated by a higher order polynomial, such as,

$$v(x) = a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 . \quad (39)$$

The conditions  $v(x_p) = 0$  and  $v(x_q) = 0$  in matrix form are now

$$\begin{bmatrix} x_p^2 & x_p^3 & x_p^4 & x_p^5 & x_p^6 & x_p^7 \\ x_q^2 & x_q^3 & x_q^4 & x_q^5 & x_q^6 & x_q^7 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} , \quad (40)$$

or  $[\mathbf{A}]\{\mathbf{a}\} = \{\mathbf{0}\}$ . So, the problem at hand is to minimize  $\Pi$  such that  $[\mathbf{A}]\{\mathbf{a}\} = \{\mathbf{0}\}$ . This may be done by augmenting the cost function with Lagrange multipliers,

$$\tilde{\Pi} = \frac{1}{2}\{\mathbf{a}\}^T[\mathbf{K}]\{\mathbf{a}\} - \{\mathbf{f}\}^T\{\mathbf{a}\} + \{\boldsymbol{\lambda}\}^T[\mathbf{A}]\{\mathbf{a}\} . \quad (41)$$

Recognizing that

$$[\mathbf{A}]\{\mathbf{a}\} = \begin{Bmatrix} v(x_p) \\ v(x_q) \end{Bmatrix} , \quad (42)$$

and examining the form of equation (41) the values of  $\boldsymbol{\lambda}$  can be seen to correspond to forces acting at  $x = x_p$  and  $x = x_q$ . The term  $\{\boldsymbol{\lambda}\}^T[\mathbf{A}]\{\mathbf{a}\}$  represents the potential energy function of forces  $\{\boldsymbol{\lambda}\}$  at a displacement of  $[\mathbf{A}]\{\mathbf{a}\}$ . Because the displacements  $[\mathbf{A}]\{\mathbf{a}\}$  will be enforced to be zero, the Lagrange multiplier can be viewed as the reaction forces required to enforce the zero displacement condition at the reaction locations. The Lagrange multipliers are, in fact, equal to the external forces at the constraining reactions.

As in the previous unconstrained example, the minimization of  $\tilde{\Pi}$  with respect to the vector of polynomial coefficients,  $\mathbf{a}$ , corresponds to

$$\{\mathbf{0}\} = \frac{\partial}{\partial \mathbf{a}} \tilde{\Pi} = \{\mathbf{a}\}^T [\mathbf{K}] - \{\mathbf{f}\}^T + \{\boldsymbol{\lambda}\}^T [\mathbf{A}] . \quad (43)$$

As long as  $[\mathbf{A}]\{\mathbf{a}\} = \{\mathbf{0}\}$ ,  $\tilde{\Pi} = \Pi$ . Moreover, as long as  $[\mathbf{A}]\{\mathbf{a}\} = \{\mathbf{0}\}$ ,  $\tilde{\Pi}$  does not depend on the values of  $\boldsymbol{\lambda}$ . Therefore, setting  $\frac{\partial}{\partial \boldsymbol{\lambda}} \tilde{\Pi}$  equal to zero is equivalent to enforcing the constraint  $[\mathbf{A}]\{\mathbf{a}\} = \{\mathbf{0}\}$ ,

$$\{\mathbf{0}\} = \frac{\partial}{\partial \boldsymbol{\lambda}} \tilde{\Pi} = \{\mathbf{a}\}^T [\mathbf{A}]^T . \quad (44)$$

The conditions arising from equations (43) and (44) may be transposed and combined into a single matrix equation

$$\begin{bmatrix} \mathbf{K} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix} . \quad (45)$$

#### 4 Application to Beams in Bending

The deflections of a cantilever beams with extra supports is more complicated than the deflections of a simple cantilever beam, and should be described by a more complicated expression. Using a seventh-order polynomial to approximate the deflected shape of the beam,

$$v(x; \mathbf{a}) = a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 , \quad (46)$$

$$v''(x; \mathbf{a}) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 , \quad (47)$$

adjoining the potential energy function  $\Pi$  with  $\{\boldsymbol{\lambda}\}^T [\mathbf{A}]\{\mathbf{a}\}$ , substituting into  $\partial \tilde{\Pi} / \partial a_i = 0$  for  $i = 2, \dots, 7$ , and  $\partial \tilde{\Pi} / \partial \lambda_j = 0$ , for  $j = 1, \dots, 2$ , and integrating, the system of linear equations for  $\mathbf{a}$  and  $\boldsymbol{\lambda}$  are

$$\begin{aligned}
EI & \begin{bmatrix} 4L & 6L^2 & 8L^3 & 10L^4 & 12L^5 & 14L^6 \\ 6L^2 & 12L^3 & 18L^4 & 24L^5 & 30L^6 & 36L^7 \\ 8L^3 & 18L^4 & 28.8L^5 & 40L^6 & 51.43L^7 & 63L^8 \\ 10L^4 & 24L^5 & 40L^6 & 57.14L^7 & 75L^8 & 93.33L^9 \\ 12L^5 & 30L^6 & 51.43L^7 & 75L^8 & 100L^9 & 120L^{10} \\ 14L^6 & 36L^7 & 63L^8 & 93.33L^9 & 126L^{10} & 160.36L^{11} \end{bmatrix} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} \\
-w & \begin{pmatrix} L^3/3 \\ L^4/4 \\ L^5/5 \\ L^6/6 \\ L^7/7 \\ L^8/8 \end{pmatrix} + \begin{bmatrix} x_p^2 & x_q^2 \\ x_p^3 & x_q^3 \\ x_p^4 & x_q^4 \\ x_p^5 & x_q^5 \\ x_p^6 & x_q^6 \\ x_p^7 & x_q^7 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (48)
\end{aligned}$$

and

$$\begin{bmatrix} x_p^2 & x_p^3 & x_p^4 & x_p^5 & x_p^6 & x_p^7 \\ x_q^2 & x_q^3 & x_q^4 & x_q^5 & x_q^6 & x_q^7 \end{bmatrix} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (49)$$

Figures 1, 2, and 3 provide numerical results for this example. The approximate 7-th order polynomial solution is compared to a solution computed derived from an equilibrium moment equation such that displacements are zero at the extra supports. Note that the approximated deflections are exactly zero at the extra supports. Also note that the internal moment computed from the approximate deflections is not zero at the free end. *The approximation does not satisfy equilibrium!* A zero-moment condition was not included in this approximate formulation, although it could be by adding an additional constraint equation,  $v''(L) = 0$ . This extra constraint equation would be

$$2a_2 + 6a_3L + 12a_4L^2 + 20a_5L^3 + 30a_6L^4 + 42a_7L^5 = 0, \quad (50)$$

corresponding to a third row  $[ 2 \quad 6L \quad 12L^2 \quad 20L^3 \quad 30L^4 \quad 42L^5 ]$  in the  $[\mathbf{A}]$  matrix.

A scalar-valued error indicator provides a quantitative measure of the difference between the equilibrium solution and the approximate solution based on our assumption for  $v(x)$ . For the difference in the displacements,

$$\epsilon_v = \frac{\|v_{\text{equil}}(x) - v_{\text{approx}}(x)\|}{\|v_{\text{equil}}(x)\|} \quad (51)$$

And similarly for the difference in the moments,

$$\epsilon_M = \frac{\|M_{\text{equil}}(x) - M_{\text{approx}}(x)\|}{\|M_{\text{equil}}(x)\|} \quad (52)$$

For the seventh-order polynomial approximation,  $\epsilon_v \approx 0.10$  and  $\epsilon_M \approx 0.20$ . Constraining  $v''(L)$  to be zero does force  $M(L)$  to be zero, but does not improve the overall accuracy of  $v(x)$  or  $M(x)$ , and in some cases can measurably reduce the overall accuracy of the approximation.

## 5 Higher Order Polynomials

This section investigates the use of a higher order approximating function for the displacements  $v(x)$ . In general, a polynomial approximating the displacements of a cantilever beam can be written as a power series,

$$v(x; \mathbf{a}) = \sum_{n=2}^N a_n x^n . \quad (53)$$

The curvature written as a power series is

$$v''(x; \mathbf{a}) = \sum_{n=2}^N (n-1)(n) a_n x^{n-2} = \sum_{n=2}^N (n^2 - n) a_n x^{n-2} . \quad (54)$$

The anticipation that a matrix equation such as equation (48) will result from the substitutions of equations (53) and (54) into

$$0 = EI \int_l v''(x; \mathbf{a}) \frac{\partial v''(x; \mathbf{a})}{\partial a_i} dl - w \int_l \frac{\partial v(x; \mathbf{a})}{\partial a_i} dl \quad (55)$$

for  $i = 2, \dots, N$ , motivates us to find general expressions for the elements of  $[\mathbf{K}]$  and  $\{\mathbf{f}\}$  directly.

Note that in the matrix equation (48), the matrix  $[\mathbf{K}]$  has terms

$$\begin{aligned}
K_{ij} &= \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} U = \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} \frac{1}{2} \int_l EI (v''(x, \mathbf{a}))^2 dl \\
&= \frac{1}{2} \int_0^L EI \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} \left( \sum_{n=2}^N (n^2 - n) a_n x^{n-2} \right)^2 dx \\
&= \int_0^L EI ((i^2 - i)x^{i-2}) ((j^2 - j)x^{j-2}) dx \\
&= EI \frac{(i^2 - i)(j^2 - j)}{i + j - 3} L^{i+j-3} .
\end{aligned} \tag{56}$$

Also note that in the matrix equation (48), the vector  $\{\mathbf{f}\}$  has terms

$$\begin{aligned}
f_i &= \frac{\partial}{\partial a_i} V = \int_l w \frac{\partial}{\partial a_i} v(x, \mathbf{a}) dl \\
&= \int_0^L w(x) \frac{\partial}{\partial a_i} \left( \sum_{n=2}^N a_n x^n \right) dx \\
&= \int_0^L w(x) x^i dx \\
&= w \frac{1}{i + 1} L^{i+1} .
\end{aligned} \tag{57}$$

The columns of the constraint matrix  $[\mathbf{A}]$  for  $v(x_p) = 0$  and  $v(x_q) = 0$  are simply

$$\{A_j\} = \left\{ \begin{array}{c} x_p^{j+1} \\ x_q^{j+1} \end{array} \right\} , \tag{58}$$

and the columns of  $[\mathbf{A}]$  for  $v(x_p) = 0$ ,  $v(x_q) = 0$  and  $M(L) = 0$  are simply

$$\{A_j\} = \left\{ \begin{array}{c} x_p^{j+1} \\ x_q^{j+1} \\ ((j + 1)^2 - (j + 1))L^{j-1} \end{array} \right\} . \tag{59}$$

Equations (45), (56), (57), and (58), or (59) completely define the matrix equation associated with the constrained minimization of the augmented total potential energy,  $\tilde{\Pi}$ , associated with the bending of a cantilever beam supported at two additional points and carrying a uniform load<sup>1</sup>.

For  $N = 11$ ,  $\epsilon_v \approx 0.02$  and  $\epsilon_M \approx 0.11$ . Increasing the polynomial order much beyond 11 quickly leads to a point of diminishing returns. Even

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<sup>1</sup>That's a run-on sentence.

well beyond the point of diminishing returns, a polynomial approximation is unable to capture the sharp discontinuities in the slope of the moment. For beams supported by a “pad” instead of a “knife-edge” support, the moment diagram is smooth and the shear diagram is continuous.

## 6 Constrained Optimization and Minimum Total Potential Energy

An analogy between constrained optimization and minimum total potential energy of structures with indeterminate reactions may be tabulated as follows.

Table 1. An analogy between constrained optimization and minimum total potential energy.

<b>Constrained Optimization</b>	<b>Minimum Total Potential Energy</b>
cost function, $J$	total potential energy function, $\Pi$
parameters, $p_i$	displacement coefficients, $a_i$
active constraints, $g_j(\mathbf{p}) = 0$	specified displacements $v(x_p; \mathbf{a}) = 0$
Lagrange multipliers, $\lambda$	reaction forces at specified displacements, $\lambda$
Hessian matrix of the cost function, $[\mathbf{H}]$	stiffness matrix, $[\mathbf{K}]$

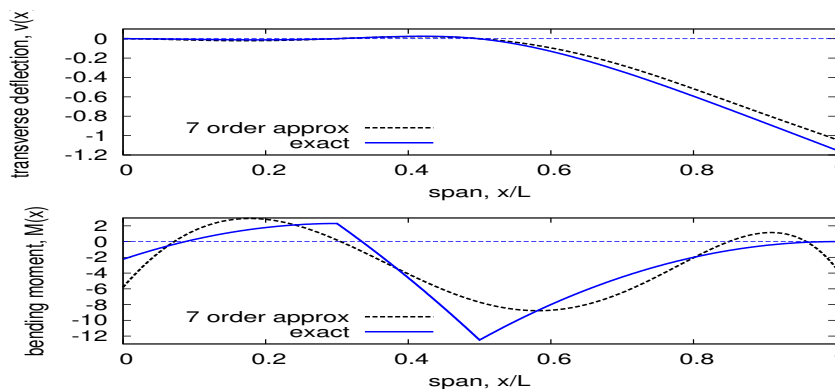


Figure 1.  $N = 7$ ; Roller supports at  $x_p = 0.3L$  and  $x_q = 0.5L$ ; Approximate Reactions:  $R_p = +78.80$ ,  $R_q = -140.66$ ; Exact Reactions:  $R_p = +64.18$ ,  $R_q = -134.00$ ; Approximation Errors:  $\epsilon_v = 0.12$ ;  $\epsilon_M = 0.34$

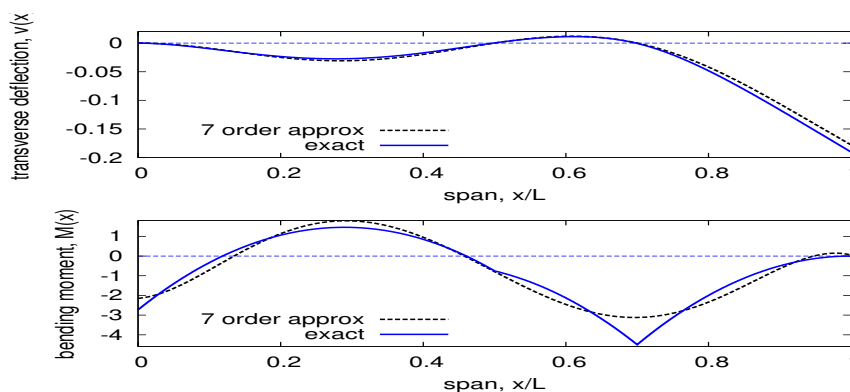


Figure 2.  $N = 7$ ; Roller supports at  $x_p = 0.5L$  and  $x_q = 0.7L$ ; Approximate Reactions:  $R_p = -10.98$ ,  $R_q = -59.51$ ; Exact Reactions:  $R_p = -12.34$ ,  $R_q = -58.71$ ; Approximation Errors:  $\epsilon_v = 0.09$ ;  $\epsilon_M = 0.20$

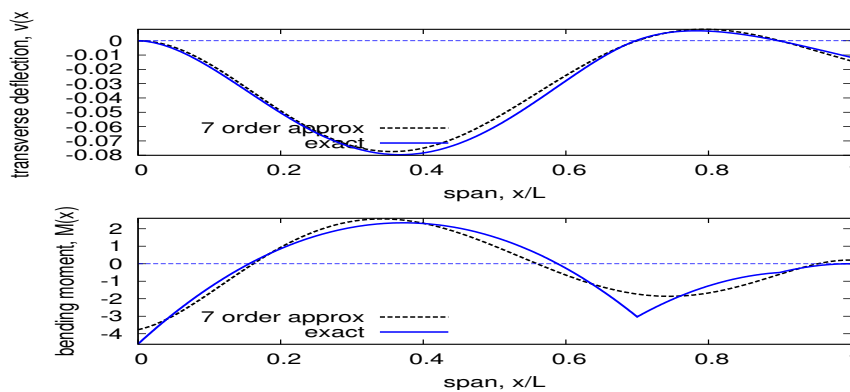


Figure 3.  $N = 7$ ; Roller supports at  $x_p = 0.7L$  and  $x_q = 0.9L$ ; Approximate Reactions:  $R_p = -56.08$ ,  $R_q = -6.88$ ; Exact Reactions:  $R_p = -55.45$ ,  $R_q = -7.32$ ; Approximation Errors:  $\epsilon_v = 0.06$ ;  $\epsilon_M = 0.22$

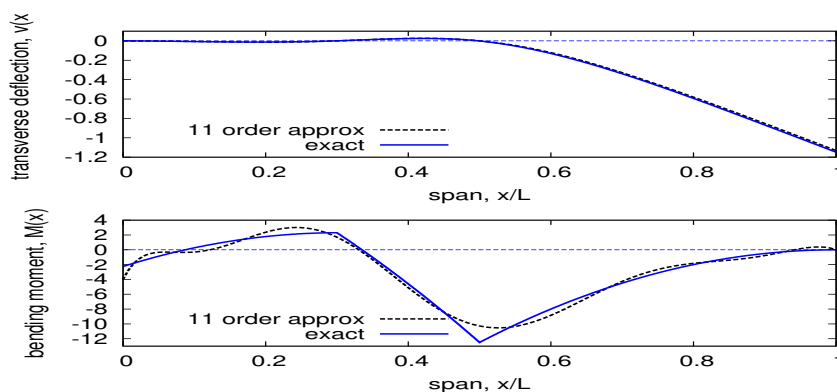


Figure 4.  $N = 11$ ; Roller supports at  $x_p = 0.3L$  and  $x_q = 0.5L$ ; Approximate Reactions:  $R_p = +65.18$  and  $R_q = -134.63$ ; Exact Reactions:  $R_p = +64.18$  and  $R_q = -134.00$ ; Approximation Errors:  $\epsilon_v = 0.02$  and  $\epsilon_M = 0.11$

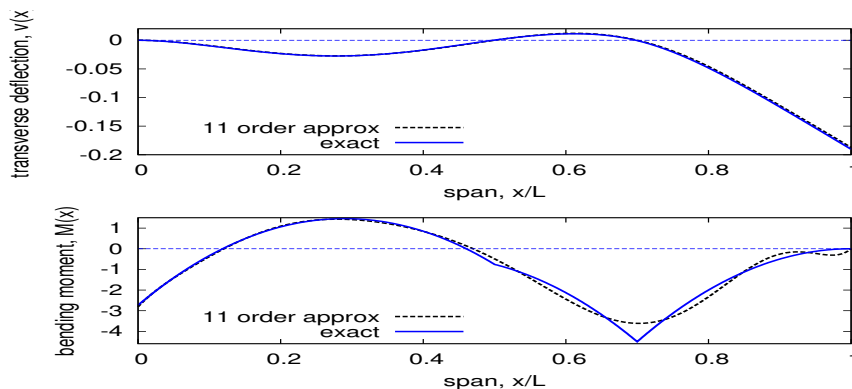


Figure 5.  $N = 11$ ; Roller supports at  $x_p = 0.5L$  and  $x_q = 0.7L$ ; Approximate Reactions:  $R_p = -12.18$  and  $R_q = -58.80$ ; Exact Reactions:  $R_p = -12.34$  and  $R_q = -58.71$ ; Approximation Errors:  $\epsilon_v = 0.02$  and  $\epsilon_M = 0.11$

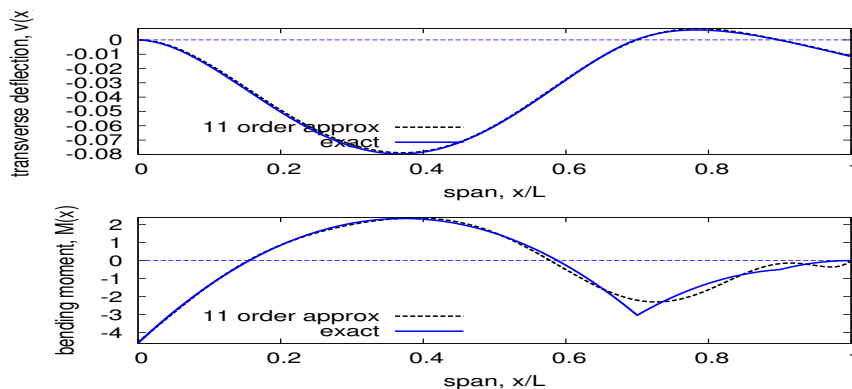


Figure 6.  $N = 11$ ; Roller supports at  $x_p = 0.7L$  and  $x_q = 0.9L$ ; Approximate Reactions:  $R_p = -55.47$  and  $R_q = -7.30$ ; Exact Reactions:  $R_p = -55.45$  and  $R_q = -7.32$ ; Approximation Errors:  $\epsilon_v = 0.02$  and  $\epsilon_M = 0.11$