PLASTIC ANALYSIS OF BEAMS
CE 130 — Structural Design and Optimization
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Consider a prismatic beam with a rectangular cross section ($b \times d$) carrying a bending moment, M,

We will assume that:

1. Plane sections remain plane;
2. The stress-strain behavior is ideally elastic-plastic; and
3. The deformations are small.

In the first part of this analysis, we will examine the moment-curvature relationship of this beam in three different cases:

1. There is no material yielding in the beam cross section;
2. There is some material yielding in the beam cross section; and
3. The material is yielding everywhere.

1 Elastic Cross-Section

When the beam is elastic, recall that the moment, M is proportional to the curvature, $M = EIy'' = EI\phi$ and that the curvature is given by $\phi = \epsilon/(d/2)$. When the beam first starts to yield, the stress at the top and bottom of the beam equals the yield stress. We will define the *yield moment*, M_y, to be the moment that causes initial yielding in the cross section, $\sigma_y = M_y(d/2)/I$, or $M_y = \sigma_y I/(d/2)$. The *yield curvature*, ϕ_y, is the corresponding curvature, $\phi_y = \epsilon_y/(d/2)$. Now, dividing both sides of the elastic moment-curvature relationship by M_y we get

$$M = EI\phi \tag{1}$$

$$\frac{M}{M_y} = \frac{EI\phi}{EI\phi_y} \tag{2}$$

$$\frac{M}{M_y} = \frac{\phi}{\phi_y} \tag{3}$$

2 Partially Plastic Cross-Section

Even after the cross section starts to yield it can still carry additional moment. The curvature is now given by $\phi = \epsilon_y/y_o$. The moment that the beam carries

when the cross section is partially plastic can be found by integrating σy over the cross section.

$$
\begin{aligned}
M &= 2\left[\frac{1}{2}y_o\sigma_y b\left(\frac{2}{3}y_o\right)+\left(\frac{1}{2}d-y_o\right)\sigma_y b\left(y_o+\frac{1}{2}\left(\frac{1}{2}d-y_o\right)\right)\right] && (4)\\
&= \left[\frac{2}{3}y_o^2+2\left(\frac{1}{2}d-y_o\right)\left(y_o+\frac{1}{4}d-\frac{1}{2}y_o\right)\right]\sigma_y b && (5)\\
&= \left[\frac{2}{3}y_o^2+\left(\frac{1}{2}d-y_o\right)\left(\frac{1}{2}d+y_o\right)\right]\sigma_y b && (6)\\
&= \left[\frac{2}{3}y_o^2+\left(\frac{1}{2}d-y_o\right)\left(\frac{1}{2}d+y_o\right)\right]\sigma_y b && (7)\\
&= \left[\frac{2}{3}y_o^2+\left(\frac{1}{4}d^2-y_o^2\right)\right]\sigma_y b && (8)
\end{aligned}
$$

Now, substituting $y_o = \epsilon_y/\phi$, we can obtain the moment, M as a function of curvature, ϕ.

$$
M = \left[\frac{2}{3}\left(\frac{\epsilon_y}{\phi}\right)^2+\left(\frac{1}{4}d^2-\left(\frac{\epsilon_y}{\phi}\right)^2\right)\right]\sigma_y b \qquad (9)
$$

Dividing both sides by $M_y = \sigma_y I/(d/2)$,

$$
\begin{aligned}
\frac{M}{M_y} &= \left[\frac{2}{3}\left(\frac{\epsilon_y}{\phi}\right)^2+\left(\frac{1}{4}d^2-\left(\frac{\epsilon_y}{\phi}\right)^2\right)\right]\frac{\sigma_y bd}{\sigma_y I 2} && (10)\\
&= \left[\frac{2}{3}\left(\frac{\epsilon_y}{\phi}\right)^2+\left(\frac{1}{4}d^2-\left(\frac{\epsilon_y}{\phi}\right)^2\right)\right]\frac{6}{d^2} && (11)
\end{aligned}
$$

Now, taking the limit as $\phi \to \infty$,

$$
\begin{aligned}
\frac{M}{M_y} &= \left[0+\left(\frac{1}{4}d^2-0\right)\right]\frac{6}{d^2} && (12)\\
&= \frac{6}{4} && (13)\\
&= 1.5 && (14)
\end{aligned}
$$

The moment, M asymptotically approaches a limiting moment, which is called

the *plastic moment*, M_p. The plastic moment is always greater than the yielding moment. Beams therefore have an ability to carry loads beyond the initial yielding of the beam. For all rectangular cross sections, $M_p/M_y = 1.5$; for circular cross sections, $M_p/M_y = 1.7$; and for I-beams $M_p/M_y \approx 1.12$. Thus, for I-beams in particular, the moment curvature relationship can be conveniently idealized as being bi-linear without much loss of accuracy.

3 Fully Plastic Cross Section

The plastic moment, M_p can be found directly, without resorting to the more difficult analysis of a partially plastic cross section. When the cross section is fully plastic, *all* of the material is yielding ($y_o \rightarrow 0$). The plastic bending

stresses are in compression above the *plastic neutral axis* and are in tension below the plastic neutral axis. The total compressive force must equal the total tensile force, and the *plastic neutral axis* is defined as the location in the cross section for which the area above the axis equals the area below. For symmetric cross sections, the plastic neutral axis is the same as the elastic neutral axis (which is the same as the centroid). However, for a-symmetric cross sections, the plastic neutral axis is not the same as the centroid.

For any rectangular cross section,

$$M_p = 2\left[\sigma_y b \frac{d}{2}\left(\frac{d}{4}\right)\right] \tag{15}$$

$$= \frac{bd^2}{4}\sigma_y \tag{16}$$

As an example of an a-symmetric cross section, let's consider a triangular cross section. We must first find the location of the plastic neutral axis, i.e., the

location in the cross section for which the area above, A_c equals the area below, A_t.

$$A_c = A_t \tag{17}$$

$$\frac{1}{2}yb' = \frac{1}{2}(b+b')(d-y) \tag{18}$$
$$\frac{1}{2}y\frac{y}{d}b = \frac{1}{2}(b+\frac{y}{d}b)(d-y) \tag{19}$$
$$\frac{1}{2}y^2\frac{b}{d} = \frac{1}{2}(bd+yb-by-y^2\frac{b}{d} \tag{20}$$
$$y^2 = \frac{1}{2}d^2 \tag{21}$$
$$y = \sqrt{\frac{1}{2}}d \tag{22}$$

With the location of the plastic neutral axis found, we can find the plastic moment by finding the distance between the centroids of A_c and A_t. For this triangular shape, this distance is

$$L = \frac{y}{3} + \frac{d-y}{2} = \left(\frac{1}{2} - \frac{1}{6}\sqrt{\frac{1}{2}}\right) d \tag{23}$$

and the plastic moment is given by

$$M_p = A_t L \sigma_y \tag{24}$$
$$= \frac{1}{2}\frac{1}{2}d^2\frac{d}{b}\left(\frac{1}{2} - \frac{1}{6}\sqrt{\frac{1}{2}}\right) d\sigma_y \tag{25}$$

In general, to find the plastic moment of a rectangular cross section, follow these three steps:

- Find the location of the plastic neutral surface:
  Divide the cross sectional area into two areas, A_t and A_c, such that $A_t = A_c$.
- Find the distance, L, between the centroids of the two areas.
- The plastic moment is $M_p = \sigma_y \cdot L \cdot A_c$.