

Four Strain Energy Concepts in Pictures

CE 130L. Uncertainty, Design and Optimization

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1 The Principle of Real Work

External distributed external loads w distributed over the surface S of an elastic object give rise to displacements distributed over the surface v (consistent with the support conditions of the object), internal stresses $\{\sigma\}$ and internal strains $\{\epsilon\}$.

For any general (possibly nonlinear) elastic stress-strain relationship $\{\sigma(\{\epsilon\})\}$ (where $\{\sigma\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}\}$ and $\{\epsilon\} = \{\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\}$), the work of the external loads W is stored completely as strain energy U within the object.

The internal strain energy is the integral of $\sigma_{ij}d\epsilon_{ij}$ from $\epsilon_{ij} = 0$ to $\epsilon_{ij} = \epsilon_{ij}$, integrated throughout the volume of the solid,

$$U = \int_V \int_{\{\epsilon\}=\{0\}}^{\{\epsilon\}=\{\epsilon\}} \{\sigma(\{\epsilon\})\}^T \{d\epsilon\} dV , \quad (1)$$

and may be thought of as the area under the stress-strain diagram up to a certain value of strain ϵ .

The work of the external forces w moving through displacements v is the integral of $(w dv)$, integrated over the surface of the object

$$W = \int_S \int_{\nu=0}^{\nu=v} w(\nu) d\nu dS \quad (2)$$

and may be thought of as the area under the load-displacement diagram, up to a certain value of displacement, v .

The principle of real work may be thought of as the equivalence of the areas under the $\sigma_{ij} - \epsilon_{ij}$ diagrams with the area under the $w - v$ diagram.

For linear elastic solids, the areas under the $\sigma - \epsilon$ and the $w - v$ diagrams are triangular, and the expressions for the internal strain energy and the external work simplify to

$$U = \int_V \frac{1}{2} \{\sigma\}^T \{\epsilon\} dV \quad (3)$$

and

$$W = \int_S \frac{1}{2} w v dS . \quad (4)$$

The work of the external loads $w(v)$, increasing as they pass through displacements v , on an elastic solid is stored completely as strain energy within the solid. By itself, the principle of real work has limited potential for solving engineering problems. Nevertheless, a number of extremely useful methods may be developed by expanding upon this fundamental concept.

2 Castigliano's Theorems

Using the fact that external work W is conserved as internal strain energy U , the area under the load-displacement displacement diagram may be represented exactly as the strain energy U .

Consider a solid carrying n point forces F_i resulting in n *colocated* displacements D_i ($i = 1, \dots, n$). Each force F_i acts in the same location and in the same direction as each corresponding displacement D_i .

Because $U = W$, the strain energy may be expressed as the sum of the areas below the individual force-displacement diagrams,

$$U = \sum_{i=1}^n \int F_i dD_i . \quad (5)$$

Differentiating both sides of this expression with respect to a particular displacement D_i results in Castigliano's First Theorem,

$$\frac{\partial U}{\partial D_i} = F_i . \quad (6)$$

Now, define the *complementary strain energy* U^* as the sum of the areas to the left of the force-displacement diagrams,

$$U^* = \sum_{i=1}^n \int D_i dF_i . \quad (7)$$

Differentiating both sides of this expression with respect to a particular force F_i results in Castigliano's Second Theorem.

$$\frac{\partial U^*}{\partial F_i} = D_i . \quad (8)$$

The gradient of the strain energy with respect to a displacement is the value of the force colocated with that displacement. The gradient of the complementary strain energy with respect to a force is the displacement colocated with that force.

Internal forces (axial forces, bending moments, etc), stresses, and strains may be related to the external loads on the solid using methods of structural analysis. So the strain energy may also be expressed in terms of the external loads.

If the solid is linear elastic then the areas below the stress-strain diagrams and the areas below the force-displacement diagrams are triangular, the value of the strain energy is equal to the value of the complementary strain energy, $U = U^*$, and Castigliano's second theorem may be written

$$\frac{\partial U}{\partial F_i} = D_i \quad (\text{for linear elastic solids only}). \quad (9)$$

3 The Principle of Virtual Work

Consider a linear elastic solid carrying loads w distributed over the surface. Suppose the load w is separated into two parts, $w = w_o + \delta w$. Note that if the loads w_o are applied first, then these loads w_o need to be held constant as the loads δw are subsequently applied in order for the final load to reach a value of $w + \delta w$. The load w_o gives rise to displacements v_o , stresses $\{\sigma_o\}$ and strains $\{\epsilon_o\}$. Likewise, the load δw gives rise to displacements δv , stresses $\{\delta\sigma\}$ and strains $\{\delta\epsilon\}$. The principle of real work corresponding to the loads w_o alone is

$$\frac{1}{2} \int_V \{\sigma_o\}^T \{\epsilon_o\} dV = \frac{1}{2} \int_S w_o v_o dS . \quad (10)$$

The principle of real work corresponding to the loads δw alone is

$$\frac{1}{2} \int_V \{\delta\sigma\}^T \{\delta\epsilon\} dV = \frac{1}{2} \int_S \delta w \delta v dS . \quad (11)$$

Now, with the external forces w_o applied to the system and held constant, additional forces δw may be applied, increasing the total load to $w_o + \delta w$, the total stresses to $\{\sigma_o\} + \{\delta\sigma\}$, the total strain to $\{\epsilon_o\} + \{\delta\epsilon\}$ and the total displacements to $v_o + \delta v$. With this additional load, the strain energy is now

$$\frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} dV + \frac{1}{2} \int_V \{\delta\sigma\}^T \{\delta\epsilon\} dV + \int_V \{\sigma\}^T \{\delta\epsilon\} dV , \quad (12)$$

and the external work is now

$$\frac{1}{2} \int_l w(x) v(x) dl + \frac{1}{2} \int_l \delta w(x) \delta v(x) dl + \int_l w(x) \delta v(x) dl . \quad (13)$$

Corresponding terms of the strain energy and the external work are equal to one another. Canceling equal terms (equations (10) and (11)) results in

$$\int_V \{\sigma_o\}^T \{\delta\epsilon\} dV = \int_S w_o \delta v dS . \quad (14)$$

The work of constant loads w_o passing through displacements δv is equal to the strain energy of the stresses $\{\sigma_o\}$ corresponding to the loads w_o passing through the strains $\delta\epsilon$ corresponding to the displacements δv .

This is a statement of the *principle of virtual work*. The principle of virtual work holds for linear elastic solids as well as for nonlinear elastic solids.

4 The Principle of Minimum Potential Energy

Define the *potential energy function of external forces* (F_i and w) acting through displacements (D_i and v) on the surface S of an elastic solid as

$$V = \sum_{i=1}^n F_i D_i + \int_S w v dS . \quad (15)$$

Note that the *potential energy of the external forces* is not the same as the work of external loads, increasing as they pass through the displacements. The potential energy of external forces is the work of *constant* loads w and F_i passing through displacements v and D_i .

Next define the *total potential energy* as the difference between the strain energy U and the potential energy of external forces V ,

$$\Pi = U - V . \quad (16)$$

As an introductory example, consider the strain energy U and the potential energy function V of a linear elastic spring with stiffness k carrying a tensile load F resulting in a displacement D .

$$\Pi = U - V = \frac{1}{2}kD^2 - FD . \quad (17)$$

Unlike the difference between strain energy and external work ($U - W$), which is identically zero in a state of equilibrium, the total potential energy is *minimized* in a state of equilibrium. This may be seen by plotting the parabola $U = \frac{1}{2}kD^2$, the line $V = FD$, and the parabola $\Pi = U - V$ with respect to D .

The *principle of minimum potential energy* states that the total potential energy $\Pi = U - V$ is minimized in a state of stable static equilibrium,

$$\delta\Pi = 0 . \quad (18)$$

To examine this principle further, consider the following three force – displacement diagrams for a linear elastic solid. In the first diagram the displacements are greater than those required for equilibrium. In the second diagram the displacements are less than those required for equilibrium. In the third diagram the displacements satisfy equilibrium. By comparing the areas corresponding to U , V , and Π , it can be seen that the area beneath Π is minimized (is most negative) in the condition of static equilibrium.

excessive displacements

insufficient displacements

equilibrium

Similar graphical arguments for the principle of minimum potential energy can be made for nonlinear elastic solids.

excessive displacements

insufficient displacements

equilibrium

For linear or nonlinear elastic solids, the minimum value of Π is the complementary strain energy in the solid. For the special case of linear elastic solids, $V = 2W$, so $\Pi = U - V = U - 2W$. In a condition of static equilibrium, $U = W$, and the minimum value of Π is $-U$ (or $-W$), which is the negative of the strain energy (and the work done) on the solid.

The external virtual work δW can be thought of as a variation of the potential function of external loads δV . Likewise, the internal virtual work δU can be thought of as a variation of the strain energy δU .

5 Application of the Principle of Minimum Potential Energy

If the actual displacement at any point of the structure is expressed by an “assumed” displacement in terms of a set of unknown coefficients, then the principle of minimum potential energy results in a system of equations for the unknowns.

5.1 Example 1: A linear elastic bar in tension

Assume that the axial displacement along the bar is $u(x) = \frac{x}{L}u_L$ where u_L is the unknown displacement at the end of the bar. This assumed displacement satisfies the boundary condition that $u(0) = 0$.

$$\Pi(u_L) = \frac{1}{2} \int_V \sigma_{xx} \epsilon_{xx} dV - Fu_L \quad (19)$$

$$= \frac{1}{2} \int_A \int_{x=0}^L E \left(\frac{\partial u(x)}{\partial x} \right)^2 dA dx - Fu_L \quad (20)$$

$$= \frac{1}{2} EA \int_{x=0}^L \left(\frac{u_L}{L} \right)^2 dx - Fu_L \quad (21)$$

$$= \frac{1}{2} \frac{EA}{L} u_L^2 - Fu_L \quad (22)$$

$$\delta \Pi = \frac{\partial \Pi}{\partial u_L} du_L = \frac{EA}{L} u_L - F = 0 \quad (23)$$

$$u_L = \frac{FL}{EA} . \quad (24)$$

Note that $\frac{d^2 \Pi}{du_L^2} = EA/L > 0$, so the stationary point is a minimum and the equilibrium point is stable. The minimum value of the potential energy is $-\frac{1}{2} \frac{EA}{L} u_L^2 = -\frac{1}{2} \frac{L}{EA} F^2$, which is the negative of the elastic strain energy stored in the bar. This is the correct solution for the axial displacement of a bar, and our original assumption was correct.

5.2 Example 2: A linear elastic bar in tension, wrong assumption

What if we assumed that $u(x)$ increased quadratically with x ? $u(x) = (x/L)^2 u_L$. This assumption still meets the criterion that $u(x=0) = 0$. In this case,

$$\Pi(u_L) = \frac{2EA}{3L} u_L^2 - F u_L \quad (25)$$

$$\delta\Pi = \frac{\partial\Pi}{\partial u_L} du_L = \frac{4EA}{3L} u_L - F = 0 \quad (26)$$

$$u_L = \frac{3FL}{4EA} . \quad (27)$$

The minimum value of the potential energy is now $-\frac{1}{8}\frac{EA}{L}u_L^2 = -\frac{1}{8}\frac{L}{EA}F^2$, which is not as negative as the value of Π found using the correct assumption for $u(x)$. In this way two assumptions for $u(x)$ may be compared, and the better assumption may be retained.

5.3 Example 3: A cantilever beam

The assumed transverse displacement of the beam satisfies the boundary conditions specified by the supports ($v(0) = 0$, $v'(0) = 0$).

$$v(x) = \frac{1}{2L^3}(3Lx^2 - x^3)v_B \quad (28)$$

$$v(x=L) = v_B \quad (29)$$

$$v(x=0) = 0 \quad (30)$$

$$v'(x=0) = 0 \quad (31)$$

The total potential energy is

$$\Pi(v_B) = \frac{1}{2} \int_V \sigma_{xx} \epsilon_{xx} dV - P v_B \quad (32)$$

$$= \frac{1}{2} \int_{x=0}^L EI \left(\frac{\partial^2 v(x)}{\partial x^2} \right)^2 dx - P v_B \quad (33)$$

$$= \frac{1}{2} \int_{x=0}^L EI \left(\frac{6L - 6x}{2L^3} \right)^2 v_B^2 dx - P v_B \quad (34)$$

$$\delta \Pi = \frac{\partial \Pi}{\partial v_B} dv_B = 0 \implies \quad (35)$$

$$v_B = \frac{PL^3}{3EI}, \quad (36)$$

which is the correct displacement of a tip-loaded cantilever beam. The minimum value of the potential energy is $-\frac{1}{6} \frac{P^2 L^3}{EI}$.

5.4 Example 3: A cantilever beam, wrong assumption

Now assume that the beam takes on the following displaced shape $v(x) = v_B(1 - \cos(\frac{\pi x}{2L}))$. This assumed shape still meets the conditions imposed by the fixed end, $v(0) = 0$ and $v'(0) = 0$. This displacement function results in an end displacement of

$$v_B = \frac{32 PL^3}{\pi^4 EI} \approx 0.3285 \frac{PL^3}{EI}. \quad (37)$$

This answer is within two percent of the correct value. The minimum value of the potential energy is now $-\frac{16}{\pi^4} \frac{P^2 L^3}{EI}$ which is not as negative as $-\frac{1}{6} \frac{P^2 L^3}{EI}$. Moreover, the incorrect displacement function does not satisfy equilibrium, since $M(x) = P(L - x) \neq EI \frac{d^2 v(x)}{dx^2}$.

6 References

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