INTRODUCTION TO
THE PRINCIPLE OF VIRTUAL WORK
CE 130 — Structural Design and Optimization
Spring, 2003

Consider a beam with a single point load, $\bar{F}_*$,

It produces a deflection $\bar{D}_*$ at the loading point, as well as at lots of other locations on the beam. The other deflections we can call $\bar{D}_1$, $\bar{D}_2$, ..., $\bar{D}_N$.

The force $\bar{F}_*$ creates stresses $\bar{\sigma}$ and strains $\bar{\epsilon}$ throughout the beam.

The external work of the force $\bar{F}_*$ passing through the displacement $\bar{D}_*$ is

$$W_{\mathrm{E}} = \frac{1}{2}\bar{F}_*\bar{D}_*, \tag{1}$$

and the total internal work associated with this single point load is

$$W_{\mathrm{I}} = \frac{1}{2}\int_V \{\bar{\sigma}\}^T\{\bar{\epsilon}\}dV \tag{2}$$

Next, consider the same beam with (*real*) forces applied at points 1,2, ..., N. ($\bar{F}_*$ could be at one of these points, but it doesn't have to be.)

All of these forces produce deflections D_1, D_2, ... D_N at the corresponding loading points. Plus a deflection D_* where $\bar{F}_*$ was applied. The sum of these forces produce stresses σ and strains ϵ throughout the beam. With this system of loading the external and internal work are:

$$W_{\rm E} = \frac{1}{2}\sum_{i=1}^{N} F_i D_i = \frac{1}{2}\{F\}^T\{D\} \tag{3}$$

and

$$W_{\rm I} = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\} dV \tag{4}$$

Finally, consider what happens when we apply $\bar{F}_*$ first, then apply all the other (*real*) loads F_1, F_2, ..., F_N.

For the force $\bar{F}_*$

and the external work is

$$W_{\mathrm{E}} = \frac{1}{2}\bar{F}_*\bar{D}_* + \bar{F}_* D_*. \tag{5}$$

Note that $\bar{F}_*$ is held constant as the other loads, F_i, and deflections, D_i, increase. At all the other locations, $i = 1, \ldots, N$,

$$W_{\mathrm{E}} = \frac{1}{2}\sum_{i=1}^{N} F_i D_i \tag{6}$$

The total external work is

$$W_{\mathrm{E}} = \frac{1}{2}\bar{F}_*\bar{D}_* + \bar{F}_* D_* + \frac{1}{2}\sum_{i=1}^{N} F_i D_i. \tag{7}$$

Now, let's find the internal work for the combination of loads $\bar{F}_*$ with all the other loads F_i, $i = 1, \ldots, N$. As before, we will say that the stresses and strains caused by the force $\bar{F}_*$ are $\bar{\sigma}$ and $\bar{\epsilon}$ and that the stresses and strains caused by the forces F_i, $i = 1, \ldots, N$ are σ and ϵ. First we will look at the stresses due to the force $\bar{F}_*$. The stress $\bar{\sigma}$ increases linearly until the strain $\bar{\epsilon}$ is attained. After the force $\bar{F}_*$ is applied the stress $\bar{\sigma}$ remains constant as the strains from the forces F_i, $i = 1, \ldots, N$ are applied.

Next we will look at the stresses σ due to the forces F_1, F_2, ... F_N. As the load $\bar{F}_*$ is applied, (i.e., before the forces F_i are applied), the stress σ are all zero, but the strains increase from 0 to $\bar{\epsilon}$. Once the force $\bar{F}_*$ has been applied, the forces F_i are applied, and the strains increase linearly with the stress.

The total internal work due to the combined actions of all the loads, applied sequentially, is

$$W_{\rm I} = \frac{1}{2} \int_V \{\bar{\sigma}\}^T \{\bar{\epsilon}\} dV + \int_V \{\bar{\sigma}\}^T \{\epsilon\} dV + \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} dV. \tag{8}$$

Equating the external and internal work (equations (7) and (8)), noting from page 1 that

$$\frac{1}{2}\bar{F}_*\bar{D}_* = \frac{1}{2}\int_V \{\bar{\sigma}\}^T\{\bar{\epsilon}\}dV, \tag{9}$$

and from page 2 that

$$\frac{1}{2}\sum_{i=1}^{N} F_i D_i = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}dV, \tag{10}$$

we obtain *the principle of virtual work*:

$$\bar{F}_* D_* = \int_V \{\bar{\sigma}\}^T\{\epsilon\}dV. \tag{11}$$

This expression holds for beams, bars, trusses, frames, plates, shells, bricks, etc. etc. It is customary to call the left hand side of this expression the *external virtual work*

$$\bar{W}_{\mathrm{E}} = \bar{F}_* D_*, \tag{12}$$

and the right hand side of this expression the *internal virtual work.*

$$\bar{W}_{\mathrm{I}} = \int_V \{\bar{\sigma}\}^T\{\epsilon\}dV. \tag{13}$$

Recall the definitions of the various terms in the principle of virtual work equation:

- D_* is the *real deflection* at some point in the structure (point *) caused by the *real forces* $F_1, F_2, ..., F_N$. Often D_* is the unknown in our problem.
- $\bar{F}_*$ is a *virtual force* in the direction and in the location of D_*. Often we set this force equal to 1 unit of force.
- ϵ are the *real strains* associated with the *real forces* $F_1, F_2, ..., F_N$.
- $\bar{\sigma}$ are the *virtual stresses* caused by the *virtual force* $\bar{F}_*$.

In words, the external virtual work of a virtual force ($\bar{F}_*$) moving through a real displacement (D_*) equals the integral of the virtual stresses associated with $\bar{F}_*$ times the real strains associated with D_*, over the volume of the solid.

There are many ways in which the principle of virtual work is applied to problems in many fields of engineering and applied mathematics, including solid mechanics, fluid mechanics, and electro-statics to name a few. The principle of virtual work is a critical component of the finite element method, which is used to solve problems in many disciplines. Here is a typical example of how we can apply the principle of virtual work to find the deflections at some point in an elastic solid. Consider a beam carrying some loads

The problem is to find the displacement Δ at some point on this structure. Note that the principle of real work can not be applied to this problem.

The figure above shows the real displacements of the structure. This system is the one we are "really" interested in and it has internal strains that are "really" there.

To apply the principle of virtual work to this problem, we remove all of the externally applied loads in the figure above and apply a unit force in the direction and location of the unknown displacement Δ. This unit *virtual force* ($\bar{F}_*$) will cause bending and shear in this particular structure, which will have associated *real stresses*. Knowing how to:
— relate the real external loads to internal bending moments $M(x)$ and shear forces $V(x)$;
— Relate these internal moments and shears to strains, ϵ
— Relate the unit virtual load to internal virtual bending moments $\bar{M}(x)$ and shear forces $\bar{V}(x)$; and
— relate those internal virtual moments and shears to virtual stresses, $\bar{\sigma}$,

equation (13) becomes

$$1 \cdot \Delta = \int_V \{\bar{\sigma}\}^T \{\epsilon\} dV. \tag{14}$$

The relationships between internal moments, shears, torques, and axial loads to stresses and strains may be simplified by considering those cases individually. Doing so simplifies the integral in equation (14) for these special cases.

Note that the principle of virtual work applies to non-linear elastic structures also.

Later in the course we will see how to use the principle of virtual work to the analysis of inelastic (i.e., plastic) deformations also.