VIBRATIONS OF
SINGLE DEGREE OF FREEDOM SYSTEMS
CE 130L. Uncertainty, Design, and Optimization
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This document describes the free- and forced- response of single degree of freedom (SDOF) systems. A single degree of freedom system is a spring-mass-damper system in which the spring has no damping or mass, the mass has no stiffness or damping, the damper has no stiffness or mass. Furthermore, the mass is allowed to move in only one direction. The horizontal vibrations of a single-story building can be conveniently modeled as a single degree of freedom system. In part 1 of this document we examine some useful trigonometric identities. In part 2 of this document we determine how lightly-damped SDOF systems vibrate freely after being released from an initial displacement with some initial velocity. In part 3 of this document we determine how lightly-damped SDOF systems respond to a persistent sinusoidal forcing. Lastly, in part 4, we determine how damped SDOF systems respond to persistent periodic forcing.

The general form of the differential equations describing a SDOF oscillator are

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t), \qquad x(0) = d_o, \quad \dot{x}(0) = v_o \tag{1}$$

where $x(t)$ is the position of the mass, m is the mass, c is the damping rate, k is the stiffness, and $f(t)$ is the external dynamic load. The initial displacement is d_o, and the initial velocity is v_o.

Figure 1. The proto-typical single degree of freedom oscillator.

1 Trigonometric and Exponential Forms for Oscillations

We expect the free vibrational response of lightly damped systems to decay over time. Note that damping may be introduced into a structure through diverse mechanisms, including linear viscous damping, nonlinear viscous damping, visco-elastic damping, friction damping, and plastic deformation. All but linear viscous damping are somewhat complicated to analyze with closed-form expressions, so we will restrict our attention to linear viscous damping, in which the damping force f_D is proportional to the velocity, $f_D = c\dot{x}$.

1.1 Constant Amplitude

In general a constant-amplitude oscillation, $x(t)$, of frequency ω can be described by sinusoidal functions. These sinusoidal functions may be equivalently written in terms of complex exponentials $e^{\pm i\omega t}$ with complex coefficients, $X = A + iB$ and $X^* = A - iB$. The complex constant X^* is the complex conjugate of X.

$$
\begin{aligned}
x(t) &= a\cos(\omega t) + b\sin(\omega t) && (2)\\
&= X\ e^{i\omega t} + X^*\ e^{-i\omega t} && (3)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
X\ e^{i\omega t} + X^*\ e^{-i\omega t} &= (A + iB)\ (\cos(\omega t) + i\sin(\omega t)) + \\
&\quad\ (A - iB)\ (\cos(\omega t) - i\sin(\omega t)) && (4)\\
&= A\cos(\omega t) + iA\sin(\omega t) + iB\cos(\omega t) - B\sin(\omega t) + \\
&\quad\ A\cos(\omega t) - iA\sin(\omega t) - iB\cos(\omega t) - B\sin(\omega t) && (5)\\
&= 2A\cos(\omega t) - 2B\sin(\omega t) && (6)\\
&= a\cos(\omega t)\ + b\sin(\omega t) && (7)
\end{aligned}
$$

Comparing these forms, we see that $a = 2A$ and $b = -2B$. Note that *all* of the above expressions are *exactly* equivalent. Equation (2), is *exactly* the same as equation (3). Equation (2) is easier to interpret as describing a sinusoidal oscillation, however equation (3) is much easier to work with, mathematically. We will endeavor to use both forms in this document, just to emphasize how the two forms are one and the same.

Equations (2) and (3) describe a sinusoidal oscillation with a constant amplitude. The amplitude, $|X|$, of the oscillation $x(t)$ can be found by adding the magnitudes of the complex amplitudes X and X^*, or by solving $\dot{x}(\hat{t}) = 0$ for $\hat{t}$, and substituting into equation (2). Either way, the amplitude of the oscillation is

$$|X| = \sqrt{a^2 + b^2} \ , \tag{8}$$

The oscillation attains it's maximum and minimum values at times $\hat{t}$, where $\tan(\omega\hat{t}) = b/a$. The oscillation attains a value of zero at times $\check{t}$ where $\tan(\omega\check{t}) = -b/a$. The oscillation can be expressed with a single sine term with a phase-shift.

$$x(t) = |X| \sin(\omega t \pm \theta_s) \tag{9}$$

$$= |X| \sin(\omega t) \cos(\theta_s) \pm |X| \cos(\omega t) \sin(\theta_s) \tag{10}$$

Setting $x(\check{t}) = 0$ results in $\tan(\omega\check{t}) = \mp \tan(\theta_s)$, and,

$$\tan(\theta_s) = \pm\frac{b}{a} \tag{11}$$

The oscillation can also be expressed as a single cosine term, with a different phase-shift.

$$x(t) = |X| \cos(\omega t \pm \theta_c) \tag{12}$$

$$= |X| \cos(\omega t) \cos(\theta_c) \mp |X| \sin(\omega t) \sin(\theta_c) \tag{13}$$

Setting $x(\check{t}) = 0$ results in $\tan(\omega\check{t}) = \pm \cot(\theta_c)$, and,

$$\tan(\theta_c) = \mp\frac{a}{b} \tag{14}$$

Note, again, that equations (2) and (3) are equivalent to one-another and are also equivalent to equations (9) and (12) using the definitions for $|X|$, θ_s, and θ_c given above.

Figure 2. A constant-amplitude oscillation.

1.2 Decaying Amplitude

To describe an oscillation which decays with time, we can multiply the expression for a constant amplitude oscillation by a positive-valued function which decays with time. Here we will use a real exponential, $e^{\sigma t}$, where $\sigma < 0$. Multiplying equations (2) through (3) by $e^{\sigma t}$,

$$\begin{aligned} x(t) &= e^{\sigma t}(a\cos(\omega t) + b\sin(\omega t)) && (15)\\ &= e^{\sigma t}(Xe^{i\omega t} + X^* e^{-i\omega t}) && (16)\\ &= Xe^{(\sigma+i\omega)t} + X^* e^{(\sigma-i\omega)t} && (17)\\ &= Xe^{\lambda t} + X^* e^{\lambda^* t} && (18) \end{aligned}$$

Again, note that *all* of the above equations are *exactly* equivalent. The exponent λ is complex, $\lambda = \sigma + i\omega$ and $\lambda^* = \sigma - i\omega$. If σ is negative, then these equations describe an oscillation with exponentially decreasing amplitudes. Note that in equation (15) the unknown constants are σ, ω, a, and b. Angular frequencies, ω, have units of radians per second. Circular frequencies, $f = \omega/(2\pi)$ have units of cycles per second, or Hertz. Periods, $T = 2\pi/\omega$, have unites of seconds.

In the next section we will find that for an un-forced vibration, σ and ω are determined from the mass, damping, and stiffness of the system. We will see that the constant a equals the initial displacement d_o, but that the constant b depends on the initial displacement and velocity, as well mass, damping, and stiffness.

Figure 3. A decaying oscillation.

2 Free response of systems with mass, stiffness and damping

Using equation (18) to describe the free response of a single degree of freedom system, we will set $f(t) = 0$ and will substitute $x(t) = Xe^{\lambda t}$ into equation (1).

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 , \quad x(0) = d_o , \quad \dot{x}(0) = v_o , \tag{19}$$

$$m\lambda^2 Xe^{\lambda t} + c\lambda Xe^{\lambda t} + kXe^{\lambda t} = 0 , \tag{20}$$

$$(m\lambda^2 + c\lambda + k)Xe^{\lambda t} = 0 , \tag{21}$$

Note that m, c, k, λ and X do *not* depend on time. For equation (21) to be true for all time,

$$(m\lambda^2 + c\lambda + k)X = 0 . \tag{22}$$

Equation (22) is trivially satisfied if $X = 0$. The *non-trivial solution* is $m\lambda^2 + c\lambda + k = 0$. This is a quadratic equation in λ which has the roots,

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}. \tag{23}$$

The solution to a homogeneous second order ordinary differential equation requires two independent initial conditions, an initial displacement and an initial velocity. These two initial conditions are used to determine the coefficients, X and X^* (or A and B) of the two linearly independent solutions corresponding to λ_1 and λ_2.

The level of damping, c, qualitatively affects the quadratic roots, $\lambda_{1,2}$, and the free response solutions.

- **Case 1:** $c = 0$ "undamped"
  If the system has no damping, $c = 0$, and
  $$\lambda_{1,2} = \pm i\sqrt{k/m} = \pm i\omega_{\rm n} \ . \tag{24}$$
  This is called the *natural frequency* of the system. Undamped systems oscillate freely at their natural frequency, $\omega_{\rm n}$. The solution in this case is
  $$x(t) = Xe^{i\omega_{\rm n}t} + X^* e^{-i\omega_{\rm n}t} \ , \tag{25}$$
  which is a *real-valued* function. The amplitudes depend on the initial displacement, d_o, and the initial velocity, v_o.

- **Case 2:** $c = c_{\rm c}$ "critically damped"
  If $(c/(2m))^2 = k/m$, or, equivalently, if $c = 2\sqrt{mk}$, then the discriminant of equation (23) is zero, This special value of damping is called the *critical damping* rate, $c_{\rm c}$,
  $$c_{\rm c} = 2\sqrt{mk} \ . \tag{26}$$
  The ratio of the actual damping rate to the critical damping rate is called the *damping ratio*, ζ.
  $$\zeta = \frac{c}{c_{\rm c}} \ . \tag{27}$$
  The two roots of the quadratic equation are real and are repeated at
  $$\lambda_1 = \lambda_2 = -c/(2m) \ , \tag{28}$$
  and the two basic solutions are equal to each other, $e^{\lambda_1 t} = e^{\lambda_2 t}$. In order to admit solutions for arbitrary initial displacements and velocities, the solution in this case is
  $$x(t) = x_1 \ e^{(-c_{\rm c}/2m)t} + x_2 \ t \ e^{(-c_{\rm c}/2m)t} \ . \tag{29}$$
  where the real constants x_1 and x_2 are determined from the initial displacement, d_o, and the initial velocity, v_o. Details regarding this special case are at the end of this document.

- **Case 3:** $c > c_{\rm c}$ "over-damped"
  If the damping is greater than the critical damping, then the roots, λ_1 and λ_2 are distinct and real. If the system is over-damped it will not oscillate freely. The solution is
$$x(t) = x_1 \ e^{\lambda_1 t} + x_2 \ e^{\lambda_2 t} \ , \tag{30}$$
  which can also be expressed using hyperbolic sine and hyperbolic cosine functions. The real constants x_1 and x_2 are determined from the initial displacement, d_o, and the initial velocity, v_o.

- **Case 4:** $0 < c < c_{\rm c}$ "under-damped"
  If the damping rate is positive, but less than the critical damping rate, the system will oscillate freely from some initial displacement and velocity. The roots are complex conjugates, $\lambda_1 = \lambda_2^*$, and the solution is
$$x(t) = X \ e^{\lambda_1 t} + X^* \ e^{\lambda_2 t} \ . \tag{31}$$
  where the complex amplitudes depend on the initial displacement, d_o, and the initial velocity, v_o.

We can re-write the dynamic equations of motion using the new dynamic variables for natural frequency, $\omega_{\rm n}$, and damping ratio, ζ. Note that
$$\frac{c}{m} = c\frac{\sqrt{k}}{\sqrt{k}}\frac{1}{\sqrt{m}\sqrt{m}} = \frac{c}{\sqrt{k}\sqrt{m}}\frac{\sqrt{k}}{\sqrt{m}} = 2\frac{c}{2\sqrt{km}}\sqrt{\frac{k}{m}} = 2\zeta\omega_{\rm n}. \tag{32}$$
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t), \tag{33}$$
$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}f(t), \tag{34}$$
$$\ddot{x}(t) + 2\zeta\omega_{\rm n} \ \dot{x}(t) + {\omega_{\rm n}}^2 \ x(t) = \frac{1}{m}f(t), \tag{35}$$
The expression for the roots $\lambda_{1,2}$, can also be written in terms of $\omega_{\rm n}$ and ζ.
$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \ , \tag{36}$$
$$= -\zeta\omega_{\rm n} \pm \sqrt{(\zeta\omega_{\rm n})^2 - {\omega_{\rm n}}^2} \ , \tag{37}$$
$$= -\zeta\omega_{\rm n} \pm \omega_{\rm n}\sqrt{\zeta^2 - 1} \ . \tag{38}$$

Some useful facts about the roots λ_1 and λ_2 are:

- $\lambda_1 + \lambda_2 = -2\zeta\omega_\mathrm{n}$
- $\lambda_1 - \lambda_2 = 2\sqrt{\zeta^2 - 1}\ \omega_\mathrm{n}$
- ${\omega_\mathrm{n}}^2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 - \frac{1}{4}(\lambda_1 - \lambda_2)^2$
- $\omega_\mathrm{n} = \sqrt{\lambda_1 \lambda_2}$
- $\zeta = -(\lambda_1 + \lambda_2)/(2\omega_\mathrm{n})$

2.1 Critical Damping

The solution to a homogeneous second order ordinary differential equation requires two initial conditions, an initial displacement and an initial velocity. These two initial conditions are used to determine the coefficients of the two linearly independent solutions corresponding to λ_1 and λ_2. If $\lambda_1 = \lambda_2$, then the solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are not independent. In fact they are identical. In such a case, a new trial solution can be determined as follows. Assume a new solution of the form

$$x(t) = u(t) x_1 e^{\lambda_1 t} \ , \tag{39}$$

$$\dot{x}(t) = \dot{u}(t) x_1 e^{\lambda_1 t} + u(t)\lambda_1 x_1 e^{\lambda_1 t} \ , \tag{40}$$

$$\ddot{x}(t) = \ddot{u}(t) x_1 e^{\lambda_1 t} + 2\dot{u}(t)\lambda_1 x_1 e^{\lambda_1 t} \ , + u(t)\lambda_1^2 x_1 e^{\lambda_1 t} \tag{41}$$

substitute these expressions into

$$\ddot{x}(t) + 2\zeta\omega_\mathrm{n}\dot{x}(t) + {\omega_\mathrm{n}}^2 x(t) = 0 \ ,$$

collect terms, and divide by $x_1 e^{\lambda_1 t}$, to get

$$\ddot{u}(t) + (2\zeta\omega_\mathrm{n} + 2(-\omega_\mathrm{n}))\dot{u}(t) = 0$$

which is a first order ordinary differential equation for $\dot{u}(t)$. The solution of this ordinary differential equation is

$$\dot{u}(t) = C \ ,$$

from which the new trial solution is found.

$$u(t) = x_2 \ t$$

and

$$x(t) = x_1 \ e^{(-c_c/2m)t} + x_2 \ t \ e^{(-c_c/2m)t} \ . \tag{42}$$

2.2 Free response of underdamped systems

If the system is under-damped, then $\zeta < 1$, $\sqrt{\zeta^2 - 1}$ is imaginary, and

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{|\zeta^2 - 1|} = \sigma \pm i\omega. \tag{43}$$

The frequency $\omega_n\sqrt{|\zeta^2 - 1|}$ is called the *damped natural frequency*, ω_d,

$$\omega_d = \omega_n\sqrt{|\zeta^2 - 1|} \ . \tag{44}$$

It is the frequency at which under-damped SDOF systems oscillate freely, With these new dynamic variables (ζ, ω_n, and ω_d) we can re-write the solution to the damped free response,

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t}(a\cos\omega_d t + b\sin\omega_d t), & (45) \\ &= Xe^{\lambda t} + X^* e^{\lambda^* t}. & (46) \end{aligned}$$

Now we can solve for X, (or, equivalently, A and B) in terms of the initial conditions. At the initial point in time, $t = 0$, the position of the mass is $x(0) = d_o$ and the velocity of the mass is $\dot{x}(0) = v_o$.

$$\begin{aligned} x(0) = d_o &= Xe^{\lambda \cdot 0} + X^* e^{\lambda^* \cdot 0} & (47) \\ &= X + X^* & (48) \\ &= (A + iB) + (A - iB) = 2A = a. & (49) \\ \dot{x}(0) = v_o &= \lambda X e^{\lambda \cdot 0} + \lambda^* X^* e^{\lambda^* \cdot 0}, & (50) \\ &= \lambda X + \lambda^* X^*, & (51) \\ &= (\sigma + i\omega_d)(A + iB) + (\sigma - i\omega_d)(A - iB), & (52) \\ &= \sigma A + i\omega_d A + i\sigma B - \omega_d B + \\ &\quad \sigma A - i\omega_d A - i\sigma B - \omega_d B, & (53) \\ &= 2\sigma A - 2\omega B & (54) \\ &= -\zeta\omega_n \ d_o - 2 \ \omega_d \ B, & (55) \end{aligned}$$

from which we can solve for B or b.

$$B = -\frac{v_o + \zeta\omega_\mathrm{n} d_o}{2\omega_\mathrm{d}} \tag{56}$$

$$b = \frac{v_o + \zeta\omega_\mathrm{n} d_o}{\omega_\mathrm{d}} \tag{57}$$

Putting this all together, the free response of a lightly-damped system to an arbitrary initial condition, $x(0) = d_o, \dot{x}(0) = v_o$ is

$$x(t) = e^{-\zeta\omega_\mathrm{n} t}\left(d_o \cos\omega_\mathrm{d} t + \frac{v_o + \zeta\omega_\mathrm{n} d_o}{\omega_\mathrm{d}} \sin\omega_\mathrm{d} t\right). \tag{58}$$

Figure 4. Under-damped response.

2.3 Free response of over-damped systems

If the system is over-damped, then $\zeta > 1$, and $\sqrt{\zeta^2 - 1}$ is real, and the roots are both real and negative

$$\lambda_{1,2} = -\zeta\omega_\mathrm{n} \pm \omega_\mathrm{n}\sqrt{\zeta^2 - 1} = \sigma \pm \omega_\mathrm{d}. \tag{59}$$

Substituting the initial conditions $x(0) = d_o$ and $\dot{x}(0) = v_o$ into the solution (equation (30)), and solving for the coefficients results in

$$x_1 = \frac{v_o + d_o(\zeta\omega_\mathrm{n} + \omega_\mathrm{d})}{2\omega_\mathrm{d}} , \tag{60}$$

$$x_2 = d_o - x_1 . \tag{61}$$

Substituting the hyperbolic sine and hyperbolic cosine expressions for the exponentials results in

$$x(t) = e^{-\zeta\omega_{\rm n} t}\left(d_o\ \cosh\omega_{\rm d} t + \frac{v_o + \zeta\omega_{\rm n} d_o}{\omega_{\rm d}}\ \sinh\omega_{\rm d} t\right)\ . \tag{62}$$

The undamped free response can be found as a special case of the under-damped free response. While special solutions exist for the critically damped response, this response can also be found as limiting cases of the under-damped or over-damped responses.

Figure 5. Critically-damped and over-damped response.

To review, some of the important equations of this section are:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$

$$\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + {\omega_{\rm n}}^2 x(t) = 0$$

$$\omega_{\rm n} = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{c_{\rm c}} = \frac{c}{2\sqrt{mk}}$$

$$\omega_{\rm d} = \omega_{\rm n}\sqrt{|\zeta^2 - 1|}$$

3 Response of systems with mass, stiffness, and damping to sinusoidal forcing

If the SDOF system is dynamically forced with a sinusoidal forcing function, then $f(t) = |F|\cos(\omega t)$, where ω is the frequency of the forcing, in radians per second. If $f(t)$ is persistent, then after several cycles the system will respond only at the frequency of the external forcing, ω. Let's suppose that this *steady-state response* is described by the function

$$x(t) = a\cos\omega t + b\sin\omega t, \tag{63}$$

then

$$\dot{x}(t) = \omega(-a\sin\omega t + b\cos\omega t), \tag{64}$$

and

$$\ddot{x}(t) = \omega^2(-a\cos\omega t - b\sin\omega t). \tag{65}$$

Substituting this trial solution into equation (1), we obtain

$$\begin{aligned} m\omega^2 \quad & (-a\cos\omega t - b\sin\omega t) + \\ c\omega \quad & (-a\sin\omega t + b\cos\omega t) + \\ k \quad & (a\cos\omega t + b\sin\omega t) = |F|\cos\omega t. \end{aligned} \tag{66}$$

Equating the sine terms and the cosine terms

$$(-m\omega^2 a + c\omega b + ka)\cos\omega t = |F|\cos\omega t \tag{67}$$

$$(-m\omega^2 b - c\omega a + kb)\sin\omega t = 0, \tag{68}$$

which is a set of two equations for the two unknown constants, a and b,

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} |F| \\ 0 \end{bmatrix}, \tag{69}$$

for which the solution is

$$a(\omega) = \frac{c\omega}{(k - m\omega^2)^2 + (c\omega)^2}\,|F| \tag{70}$$

$$b(\omega) = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2}\,|F|. \tag{71}$$

Figure 6. The amplitude of the sum of two oscillations in quadrature.

The forced vibration solution (equation (63)) may be written

$$x(t) = a(\omega)\cos\omega t + b(\omega)\sin\omega t = |X(\omega)|\cos(\omega t + \theta_c). \tag{72}$$

The amplitude of this oscillation is $|X|$, $|X| = \sqrt{a^2 + b^2}$. The angle θ_c is the phase between the force $f(t)$ and the response $x(t)$, and

$$\tan\theta_c = -\frac{a}{b} = -\frac{cw}{k - m\omega^2} \tag{73}$$

Note that θ_c is negative, regardless of frequency, meaning that the response always lags the force. The ratio of the response amplitude $|X|$ to the forcing amplitude $|F|$ is

$$\frac{|X|}{|F|} = \frac{1}{\sqrt{(k - m\omega^2) + (c\omega)^2}} \ . \tag{74}$$

This equation shows how the response amplitude $|X|$ depends on the amplitude of the forcing $|F|$ and the frequency of the forcing ω, and has units of flexibility.

Let's re-derive this expression using complex exponential notation. The equations of motion are

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = |F|\cos\omega t = \frac{1}{2}|F|e^{i\omega t} + \frac{1}{2}|F|e^{-i\omega t} \ . \tag{75}$$

In a solution of the form, $x(t) = Xe^{i\omega t} + X^* e^{-i\omega t}$, the coefficient X corresponds to the positive exponents (positive frequencies), and X^* corresponds to negative exponents (negative frequencies). Positive exponent coefficients and negative exponent coefficients may be found separately. Considering the positive exponent solution, the forcing is expressed as $Fe^{i\omega t}$ and the partial solution $Xe^{i\omega t}$ is substituted into the forced equations of motion, resulting in

$$(-m\omega^2 + ci\omega + k)\ X\ e^{i\omega t} = F\ e^{i\omega t}\ , \tag{76}$$

from which

$$\frac{X}{F} = \frac{1}{(k - m\omega^2) + i(c\omega)}\ , \tag{77}$$

which is complex-valued. This complex function has a magnitude

$$\frac{|X|}{|F|} = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\ , \tag{78}$$

which is the same as equation (74).

Equation (74) may be written in terms of the dynamic variables, $\omega_{\rm n}$ and ζ. Dividing the numerator and the denominator of equation (74) by k, we obtain

$$\frac{|X|}{|F|} = \frac{1/k}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}, \tag{79}$$

$$= \frac{1/k}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_{\rm n}}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_{\rm n}}\right)^2}}, \tag{80}$$

$$|X| = \frac{|F|/k}{\sqrt{\left(1 - \Omega^2\right)^2 + \left(2\zeta\Omega\right)^2}}, \tag{81}$$

$$\frac{|X|}{x_{\rm st}} = \frac{1}{\sqrt{\left(1 - \Omega^2\right)^2 + \left(2\zeta\Omega\right)^2}}, \tag{82}$$

where the frequency ratio Ω is the ratio of the forcing frequency to the natural frequency, $\Omega = \omega/\omega_{\rm n}$, and the static deflection $x_{\rm st}$ is the response to a static load F, $F = kx_{\rm st}$. This equation is called the *dynamic amplification factor.* It is the factor by which displacement responses are amplified due to the fact that the external forcing is dynamic, not static. See figure 7.

3.1 Ground Motion Excitation

When the dynamic loads are caused by motion of the supports (or the ground) the forcing on the structure equals the mass of the structure times

Figure 7. The dynamic amplification factor for external forcing $|X|/x_{\rm st}$, equation (82).

the ground acceleration, $f(t) = -m\ddot{z}(t)$.

Figure 8. The proto-typical SDOF oscillator subjected to base motions, $z(t)$

$$m(\ \ddot{x}(t) + \ddot{z}(t)\ ) + c\dot{x}(t) + kx(t) \ = \ 0 \qquad (83)$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) \ = \ -m\ddot{z}(t) \qquad (84)$$

$$\ddot{x}(t) + 2\zeta\omega_{\rm n}\dot{x}(t) + {\omega_{\rm n}}^2 x(t) \ = \ -\ddot{z}(t) \qquad (85)$$

If the ground displacements are sinusoidal $z(t) = |Z| \cos \omega t$, then the ground accelerations are $\ddot{z}(t) = -|Z|\omega^2 \cos \omega t$, and $f(t) = m|Z|\omega^2 \cos \omega t$.

Using the complex exponential formulation, we can find the dynamic amplification factor as a function of the frequency of the ground motion, ω.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = mZ\omega^2 \cos\omega t = \frac{1}{2}mZ\omega^2 e^{i\omega t} + \frac{1}{2}mZ^*\omega^2 e^{-i\omega t} \tag{86}$$

Again assuming a solution of the form $x(t) = Xe^{i\omega t}$ the dynamic amplification factor is

$$\frac{X}{Z} = \frac{m\omega^2}{(k - m\omega^2) + i(c\omega)}, \tag{87}$$

and

$$\begin{aligned} \frac{|X|}{|Z|} &= \frac{m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} && (88) \\ &= \frac{\Omega^2}{\sqrt{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}} && (89) \end{aligned}$$

See figure 9.

Figure 9. The dynamic amplification factor for base-excitation $|X|/|Z|$, equation (89).

Finally, let's consider the total motion of the mass $x(t) + z(t)$.

$$\frac{X}{Z} = \frac{m\omega^2}{(k - m\omega^2) + i(c\omega)},$$

$$\frac{X+Z}{Z}=\frac{X}{Z}+1 = \frac{m\omega^2+(k-m\omega^2)+i(c\omega)}{(k-m\omega^2)+i(c\omega)}, \tag{90}$$

$$= \frac{k+i(c\omega)}{(k-m\omega^2)+i(c\omega)}, \tag{91}$$

$$\frac{|X+Z|}{|Z|} = \frac{\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}, \tag{92}$$

$$= \frac{\sqrt{1+(2\zeta\Omega)^2}}{\sqrt{(1-\Omega^2)^2+(2\zeta\Omega)^2}} = \mathrm{Tr}(\Omega). \tag{93}$$

This function is called the *transmissibility ratio*, $\mathrm{Tr}(\Omega)$. It determines the ratio between the total response amplitude $|X+Z|$ and the base motion $|Z|$. See figure 10.

Figure 10. The transmissibility ratio $|X+Z|/|Z| = \mathrm{Tr}(\Omega)$, equation (93).

4 Response of systems with mass, stiffness, and damping to periodic forcing

Suppose the external forcing, $f(t)$, is persistent and periodic with period T.

$$\cdots = f(t-2T) = f(t-T) = f(t) = f(t+T) = f(f+2T) = \cdots \quad (94)$$

Any periodic function may be represented as a series expansion of sines and cosines, as a Fourier series,

$$f(t) = \frac{1}{2}a_0 + \sum_{q=1}^{\infty} a_q \cos\frac{2\pi qt}{T} + \sum_{q=1}^{\infty} b_q \sin\frac{2\pi qt}{T} \ , \quad (95)$$

where the Fourier coefficients, a_q and b_q are given by the Fourier integrals,

$$a_q = \frac{2}{T}\int_{t_o}^{t_o+T} f(t) \ \cos\frac{2\pi qt}{T} \ dt \ , \quad q = 0,1,2,\ldots \quad (96)$$

$$b_q = \frac{2}{T}\int_{t_o}^{t_o+T} f(t) \ \sin\frac{2\pi qt}{T} \ dt \ , \quad q = 0,1,2,\ldots \quad (97)$$

and the time t_o is arbitrary. Section 1 of this document emphasizes the connection between various expressions for an oscillation. The Fourier series (95) may also be represented using complex exponential notation.

$$f(t) = \frac{1}{2}a_0 + \sum_{q=1}^{\infty}\left[a_q \cos\frac{2\pi qt}{T} + b_q \sin\frac{2\pi qt}{T}\right] \quad (98)$$

$$= \sum_{q=-\infty}^{q=\infty} F_q \exp\left[i\frac{2\pi q}{T}t\right] \quad (99)$$

$$= \sum_{q=-\infty}^{q=\infty} F_q \ e^{i\omega_q t} \quad (100)$$

as long as $F_q = F^*_{-q}$.
Proof:

$$f(t) = \sum_{q=-\infty}^{q=\infty} F_q \ e^{i\omega_q t} \quad (101)$$

$$= \sum_{q=-\infty}^{q=0} F_q \ e^{i\omega_q t} + \sum_{q=1}^{q=\infty} F_q \ e^{i\omega_q t} \quad (102)$$

$$= \sum_{q=0}^{q=\infty} F_{-q} e^{-i\omega_q t} + \sum_{q=1}^{n=\infty} F_q e^{i\omega_q t} \quad (103)$$

$$= F_0 + \sum_{q=1}^{\infty} \left[F_q \, e^{i\omega_q t} + F_{-\mathrm{q}} \, e^{-i\omega_q t} \right] \tag{104}$$

$$= f_0 + \sum_{q=1}^{\infty} \left[(F_q^r + iF_q^i)\,(\cos\omega_q t + i\sin\omega_q t) + (F_q^r - iF_q^i)\,(\cos\omega_q t - i\sin\omega_q t) \right] \tag{105}$$

$$= F_0 + \sum_{q=1}^{\infty} \left[2F_q^r \cos\omega_q t - 2F_q^i \sin\omega_q t \right] \tag{106}$$

So, the real part of F_q, F_q^r, is half of a_q, the imaginary part of F_q, F_q^i, is half of $-b_q$, and $F_q = F_{-q}^*$. The complex coefficients may be found directly from

$$F_q = \frac{1}{T} \int_{t_o}^{t_o+T} f(t) \;\exp\left[-\frac{2\pi i q t}{T} \right] \; dt \; . \tag{107}$$

Complex exponential notation allows us to directly determine the steady-state periodic response to general periodic forcing, in terms of both the magnitude of the response and the phase of the response. Recall the relationship between the complex magnitudes X and F for a sinusoidally-driven spring-mass-damper oscillator, from equation (77),

$$X_q = \frac{1}{(k - m\omega_q^2) + i(c\omega_q)} \; F_q \tag{108}$$

$$X_q = H(\omega_q) \; F_q \tag{109}$$

The function $H(\omega)$ is called the frequency response function for the dynamic system relating the input $f(t)$ to the output $x(t)$.

Because the oscillator is linear, if the response to $f_1(t)$ is $x_1(t)$, and the response to $f_2(t)$ is $x_2(t)$, then the response to $c_1 f_1(t) + c_2 f_2(t)$ is $c_1 x_1(t) + c_2 x_2(t)$. More generally, then,

$$x(t) = \sum_{q=-\infty}^{q=\infty} \frac{1}{(k - m\omega_q^2) + i(c\omega_q)} \; F_q \; e^{i\omega_q t} \tag{110}$$

where $\omega_q = 2\pi q / T$.

Note that ω_q is *not* the same symbol as ω_{n}; $\omega_{\mathrm{n}} = \sqrt{k/m}$, $\omega_q = 2\pi q/T$.

The series expansion for the response $x(t)$ converges with fewer terms than the Fourier series for the external forcing, because $|H(\omega_q)|$ decreases as $1/\omega_q^2$.

5 Fourier Transforms

Recall for periodic functions of period, T, the Fourier series expansion is

$$f(t) = \sum_{q=-\infty}^{q=\infty} F_q \, e^{i\omega_q t} \ , \tag{111}$$

where the Fourier coefficients, F_q, have the same units as $f(t)$, and are given by the Fourier integral,

$$F_q = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-i\omega_q t} \, dt \ , \tag{112}$$

in which the interval of integration arbitrarily starts at $-T/2$.

Now, consider a change of variables, by defining a frequency increment, $\Delta\omega$, and a scaled amplitude, $F(\omega_q)$.

$$\Delta\omega \ \triangleq \ \omega_1 = \frac{2\pi}{T} \qquad (\omega_q = q \ \Delta\omega) \tag{113}$$

$$F(\omega_q) \ \triangleq \ T F_q = \frac{2\pi}{\Delta\omega} F_q \tag{114}$$

Where the scaled amplitude, $F(\omega_q)$, has units of $f(t) \cdot t$ or $f(t)/\omega$.

Using these new variables,

$$f(t) \ = \ \frac{1}{2\pi} \sum_{q=-\infty}^{q=\infty} F(\omega_q) \, e^{i\omega_q t} \ \Delta\omega \ , \tag{115}$$

$$F(\omega_q) \ = \ \int_{-T/2}^{T/2} f(t) \, e^{-i\omega_q t} \ dt \ . \tag{116}$$

Finally, taking the limit as $T \rightarrow \infty$, implies $\Delta\omega \rightarrow d\omega$ and $\sum \rightarrow \int$

$$f(t) \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \, e^{i\omega t} \ d\omega \ , \tag{117}$$

$$F(\omega) \ = \ \int_{-\infty}^{\infty} f(t) \ \ e^{-i\omega t} \ dt \ . \tag{118}$$

These expressions are the famous Fourier transform pair.

Table 1. Peak coordinates for various frequency response functions.

$\lvert H\rvert^2(\Omega,\zeta)$	Ω_{peak}	$\lvert H\rvert^2_{\text{peak}}$
$\frac{1}{(1-\Omega^2)^2+(2\zeta\Omega)^2}$	$\sqrt{1-2\zeta^2}$	$\frac{1}{4\zeta^2(1-\zeta^2)}$
$\frac{\Omega^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}$	1	$\frac{1}{4\zeta^2}$
$\frac{\Omega^4}{(1-\Omega^2)^2+(2\zeta\Omega)^2}$	$\frac{1}{\sqrt{1-2\zeta^2}}$	$\frac{1}{4\zeta^2(1-\zeta^2)}$
$\frac{1+(2\zeta\Omega)^2}{(1-\Omega^2)^2+(2\zeta\Omega)^2}$	$\frac{\sqrt{\sqrt{1+8\zeta^2}-1}}{2\zeta}$	$\frac{8\zeta^4}{8\zeta^4-4\zeta^2-1+\sqrt{1+8\zeta^2}}$