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Strain Energy in Linear Elastic Solids

Consider a force, F_i, applied gradually to a structure. Let D_i be the resulting displacement at the location and in the direction of the force F_i. If the structure is elastic, the force-displacement curve follows the same path on loading and unloading.

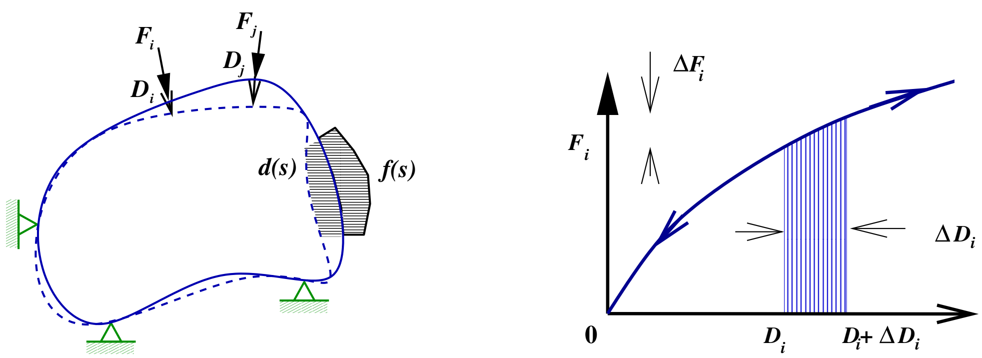

Figure 1. Forces and displacements on the surface of an elastic solid.

If F_i is increased by ΔF_i and the corresponding increase in the displacement is ΔD_i, then as $\Delta F_i \to 0$, the incremental work, ΔW, done by the load increment ΔF_i is

$$\Delta W \approx F_i \, \Delta D_i,$$

or, more precisely,

$$\Delta W = \int_{D_i}^{D_i + \Delta D_i} F_i(D_i) \, dD_i.$$

When the structure is *elastic* **and** *linear*, that is $F_i(D_i) = k_i D_i$, the total work corresponding to a displacement D_i is

$$W = \int_0^{D_i} F_i \, dD_i = \int_0^{D_i} k_i D_i \, dD_i = \frac{1}{2} k_i D_i^2 = \frac{1}{2}\frac{1}{k_i} F_i^2 = \frac{1}{2} F_i D_i.$$

If a linear elastic structure is subjected to a system of point forces, $F_1, F_2, \ldots, F_n$

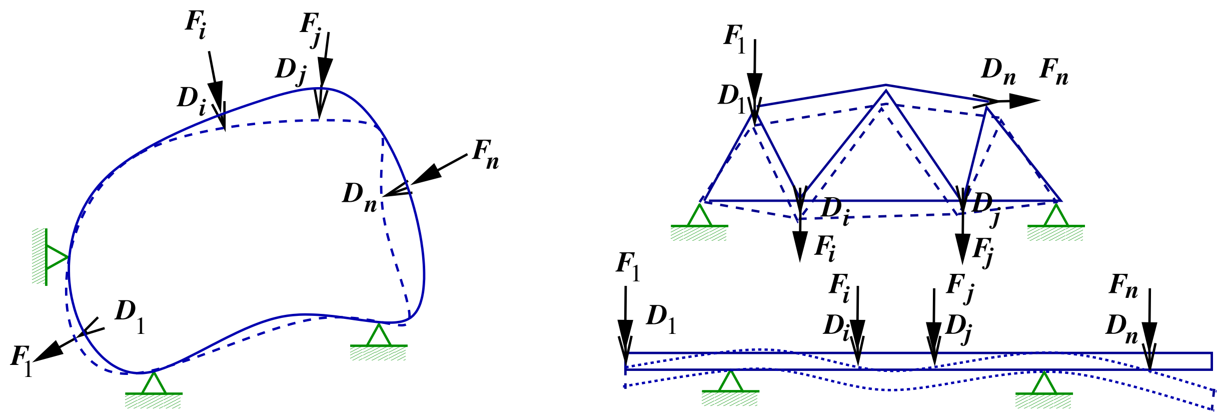

Figure 2. Point forces and collocated displacements on linear elastic solids and structures.

causing displacements $D_1, D_2, \ldots, D_n$, in the direction of those forces, then the total *external work* is

$$W = \frac{1}{2}\left\{F_1 D_1 + F_2 D_2 + \cdots + F_n D_n\right\} = \frac{1}{2}\left\{F\right\}^T\left\{D\right\}.$$

This work will be completely stored in the structure in the form of *strain energy*. Therefore, the *external work* and *strain energy* are equal to one another.

External Work = Strain Energy = Internal Work

$$W_{\mathrm{E}} = U = W_{\mathrm{I}}$$

Example: Small element subjected to normal stress σ_{xx}

Strain Energy in a general state of stress and strain

A three dimensional linear elastic solid with loads supplied by external forces, $F_1, \ldots, F_n$, and through support reactions, can be considered to be made up of small cubic elements as shown below:

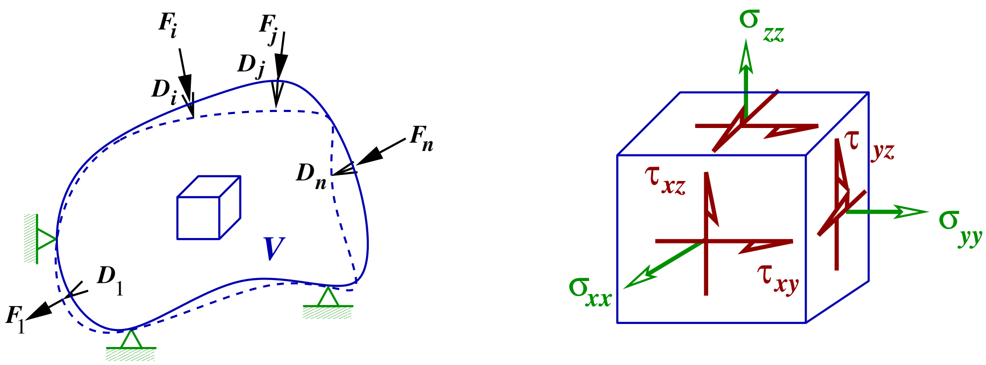

Figure 3. Stresses within a linear elastic solid.

The incremental strain energy, dU, for this elemental cube of volume dV can be written:

$$dU = \frac{1}{2} \{\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\} \, dV.$$

Integrating the incremental strain energy, dU, over an entire volume, V, the total strain energy, U, is

$$U = \frac{1}{2} \int_V \{\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\} \, dV.$$

If the stresses and strains are re-written as vectors,

$$\begin{aligned} \{\sigma\}^T &= \{\sigma_{xx} \; \sigma_{yy} \; \sigma_{zz} \; \tau_{xy} \; \tau_{xz} \; \tau_{yz}\} \\ \{\epsilon\}^T &= \{\epsilon_{xx} \; \epsilon_{yy} \; \epsilon_{zz} \; \gamma_{xy} \; \gamma_{xz} \; \gamma_{yz}\} \, , \end{aligned}$$

then the total strain energy can be written compactly as

$$U = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} \, dV.$$

This equation is a general expression for the strain energy of a linear elastic structure of any type. It can be simplified significantly for structures made of prismatic members, such as trusses and frames.

Axial Strain Energy, $\sigma_{xx} = N_x/A$, $\epsilon_{xx} = u'(x)$

Consider a rod subjected to a normal force, N_x:

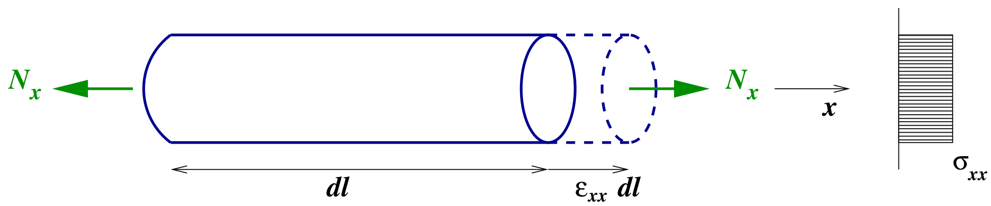

Figure 4. Internal axial forces, deformation, and stresses in an axially-loaded prismatic bar.

The normal stress on an element dA is

$$\sigma_{xx} = E\epsilon_{xx} = \frac{N_x}{A} \; .$$

The corresponding strain is

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = u'(x) \; .$$

The incremental internal strain energy, dU, in an incremental volume element, dV, in terms of axial forces, N_x, or axial displacements, $u(x)$, is

$$\begin{aligned} dU = \frac{1}{2}\sigma_{xx}\epsilon_{xx}\, dV &= \frac{1}{2}\frac{\sigma_{xx}^2}{E}\, dV = \frac{1}{2}\frac{N_x^2}{EA^2}\, dA\; dl \\ &= \frac{1}{2}E\epsilon_{xx}^2\, dV = \frac{1}{2}E(u'(x))^2\, dA\; dl \end{aligned}$$

and the total strain energy in a bar in tension or compression is

$$U = \frac{1}{2}\int_l \frac{N_x^2}{EA^2}\iint_A dA\; dl \qquad \text{or} \qquad U = \frac{1}{2}\int_l E(u'(x))^2 \iint_A dA\; dl.$$

Since $A = \iint_A dA$,

$$U = \frac{1}{2}\int_l \frac{N_x^2}{EA}\, dl \qquad \text{or} \qquad U = \frac{1}{2}\int_l EA\left(u'(x)\right)^2\, dl.$$

A prismatic bar with a constant axial force, N_x, and a constant strain $\epsilon_{xx} = \Delta_x/L$, along its length, is like a truss element, and the strain energy can be expressed as

$$U = \frac{1}{2}\frac{N_x^2 L}{EA} \qquad \text{or} \qquad U = \frac{1}{2}\frac{EA}{L}\Delta_x^2.$$

Bending Strain Energy, $\sigma_{xx} = -M_z y / I_z,\ \epsilon_{xx} = -v''(x)\ y$

Consider a beam subjected to a pure bending moment about the z-axis, M_z:

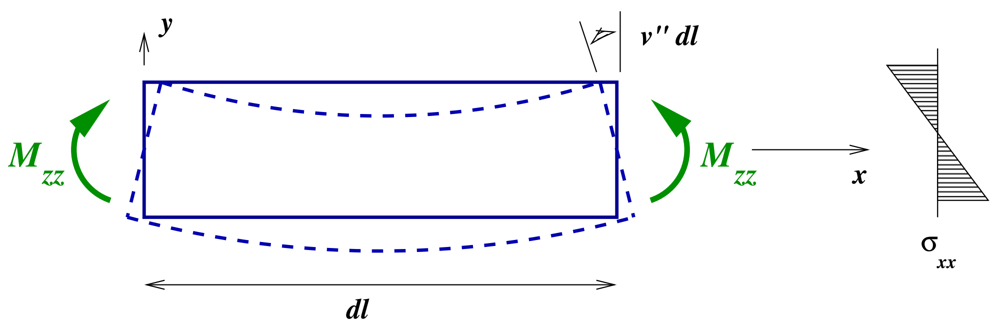

Figure 5. Internal bending moments, deformation, and stresses in a prismatic beam.

The normal stress on an element dA at a distance y from the neutral axis is

$$\sigma_{xx}(y) = E\epsilon_{xx}(y) = -\frac{M_z y}{I_z}.$$

The corresponding strain is

$$\epsilon_{xx}(y) = \frac{\sigma_{xx}}{E} = -\kappa y \approx -v''(x)\ y.$$

The incremental internal strain energy, dU, in a volume element, dV, in terms of bending moments, $M_z(x)$, or transverse displacement, $v(x)$, is

$$\begin{aligned} dU = \frac{1}{2}\sigma_{xx}\epsilon_{xx}\ dV &= \frac{1}{2}\frac{\sigma_{xx}^2}{E}\ dV = \frac{1}{2}\frac{M_z^2 y^2}{E I_z^2}\ dA\ dl \\ &= \frac{1}{2}E\epsilon_{xx}^2\ dV = \frac{1}{2}E\left(v''(x)\ y\right)^2\ dA\ dl\ , \end{aligned}$$

and the total strain energy in a beam under pure bending moments is

$$U = \frac{1}{2}\int_l \frac{M_z^2}{E I_z^2} \iint_A y^2\ dA\ dl \qquad \text{or} \qquad U = \frac{1}{2}\int_l E(v''(x))^2 \iint_A y^2\ dA\ dl.$$

Since the bending moment of inertia, I, is $\iint_A y^2\ dA$, provided that the origin of the coordinate system lies on the neutral axis of the beam ($\iint_A\ yz\ \ dy\ \ dz\ =\ 0$),

$$U = \frac{1}{2}\int_l \frac{M_z^2}{E I_z}\ dl \qquad \text{or} \qquad U = \frac{1}{2}\int_l E I_z (v''(x))^2\ dl.$$

Shear Strain Energy, $\tau_{xy} = V_y Q(y)/I_z t(y),\ \gamma_{xy} = v_s'(x)$

Consider a beam subjected to a shear force, V, (and bending moment):

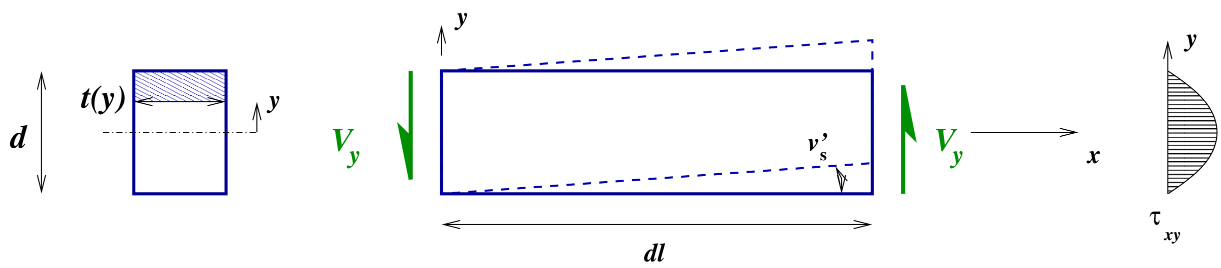

Figure 6. Internal shear forces, deformation, and stresses, of a prismatic beam.

$$\tau_{xy}(y) = G\gamma_{xy}(y) = \frac{V_y Q(y)}{I_z t(y)}$$

$$Q(y) = \text{Moment of Area of Cross Section} = \int_y^{d/2} t(y) y \, dy$$

$$dU = \frac{1}{2}\tau_{xy}\gamma_{xy}\, dV = \frac{1}{2}\frac{\tau_{xy}^2}{G}\, dA\, dl = \frac{1}{2}\frac{V_y^2 Q(y)^2}{I_z^2 G t(y)^2}\, dA\, dl$$

$$U = \frac{1}{2}\int_l \frac{V_y^2}{I_z^2 G}\iint_A \frac{Q(y)^2}{t(y)^2}\, dA\, dl = \frac{1}{2}\int_l \frac{V_y^2}{GA}\left[\frac{A}{I_z^2}\iint_A \frac{Q(y)^2}{t(y)^2}\, dA\right] dl$$

This last integral reduces to a constant that depends only upon the shape of the cross-section. This constant is given the variable name α.

$$\alpha = \frac{A}{I_z^2}\iint_A \frac{Q(y)^2}{t(y)^2}\, dA$$

Values of α for some common cross-section shapes are given below ($\alpha > 1$).

solid circular sections:	$\alpha \approx 1.08$
solid rectangular sections:	$\alpha \approx 1.15$
thin-walled circular tubes:	$\alpha \approx 1.95$
thin-walled square tubes:	$\alpha \approx 2.35$
I-sections in strong-axis shear:	$\alpha \approx A/(td)$

With this simplification, the internal strain energy due to shear forces is

$$U = \frac{1}{2}\int_l \frac{\alpha V_y^2}{GA}\, dl = \frac{1}{2}\int_l \frac{V_y^2}{G(A/\alpha)}\, dl.$$

The term (A/α) is called the *effective shear area.*

As a review of shear stresses in beams, consider the shear stress in a rectangular section (with section $d \times b$).

$$\tau_{xy} = \frac{V_y Q(y)}{I_z t(y)}$$

$$Q(y) = \int_y^{d/2} t(y) y \, dy = b \int_y^{d/2} y \, dy = b \left[\frac{y^2}{2} \right]_y^{d/2} = b \left[\frac{d^2}{8} - \frac{y^2}{2} \right]$$

$$\tau_{xy} = \frac{V_y}{2I_z} \left(\frac{d^2}{4} - y^2 \right) .$$

This stress varies parabolically along the direction of the applied shear. It is maximum at the centroid of the section and zero at the ends.

The corresponding shear strain energy equation in terms of displacements is a bit more subtle

$$U = \frac{1}{2} \int_l G(A/\alpha)(v_{\mathrm{s}}'(x))^2 \, dl \ .$$

where the total transverse displacement arises from bending $v_{\mathrm{b}}(x)$ and shear $v_{\mathrm{s}}(x)$, $v(x) = v_{\mathrm{b}}(x) + v_{\mathrm{s}}(x)$.

Torsional Strain Energy, $\tau_{x\theta} = T_x r/J,\ \gamma_{x\theta} = r\ \theta'$

Consider a circular shaft subjected to a constant torsional moment, T_x:

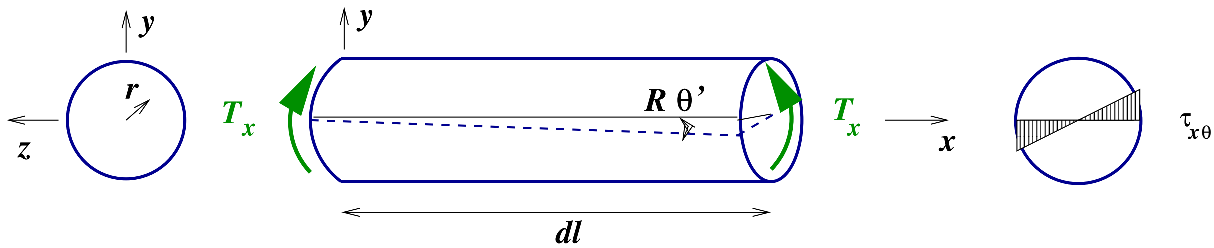

Figure 7. Internal torsional moments, deformation, and stresses in a prismatic rod.

The circumferential shear stress $\tau_{x\theta}(r)$ is

$$\tau_{x\theta}(r) = G\gamma_{x\theta}(r) = \frac{T_x r}{J}$$

and the corresponding shear strain is

$$\gamma_{x\theta}(r) = \frac{\tau_{x\theta}(r)}{G} = r\ \theta'.$$

The incremental internal strain energy, dU, in terms of torsional moments, $T_x(x)$, or torsional rotations, $\theta(x)$, is

$$\begin{aligned} dU = \frac{1}{2}\tau_{x\theta}\gamma_{x\theta}\ dV &= \frac{1}{2}\frac{\tau_{x\theta}^2}{G}\ dV = \frac{1}{2}\frac{T_x^2 r^2}{GJ^2}\ dA\ dl \\ &= \frac{1}{2}G\gamma_{x\theta}^2\ dV = \frac{1}{2}G(r\ \theta')^2\ dA\ dl \end{aligned}$$

and the total strain energy for the shaft is

$$U = \frac{1}{2}\int_l \frac{T_x^2}{J^2 G}\iint_A r^2\ dA\ dl \qquad \text{or} \qquad U = \frac{1}{2}\int_l G(\theta')^2 \iint_A r^2\ dA\ dl.$$

Since the term $\iint_A r^2\ dA$ is the same as the polar moment of inertia, J,

$$U = \frac{1}{2}\int_l \frac{T_x^2}{GJ}\ dl \qquad \text{or} \qquad U = \frac{1}{2}\int_l GJ(\theta')^2.$$

Total Strain Energy arising from Combined Axial Stresses

As a review of the material above, consider a three-dimensional bending problem with a super-imposed normal force, N_x.

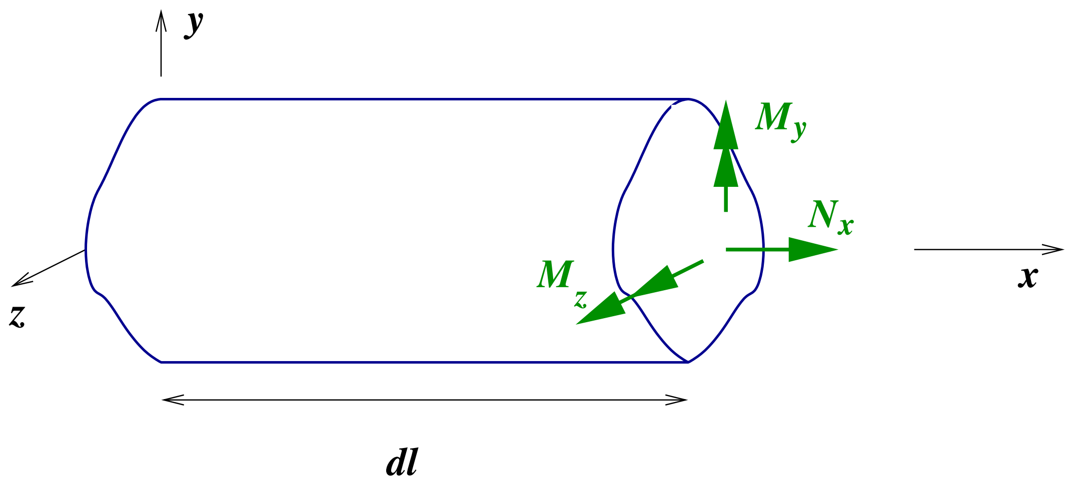

Figure 8. Internal axial force and bending moments in a prismatic beam.

$$\sigma_{xx} = \frac{N_x}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}.$$

The total strain energy arising from axial and pure bending effects is

$$U_n = \frac{1}{2}\int_V \sigma_{xx}\epsilon_{xx}\, dV = \frac{1}{2}\int_V \frac{\sigma_{xx}^2}{E}\, dV = \frac{1}{2}\int_l \frac{1}{E}\iint_A \sigma_{xx}^2\, dA\, dl.$$

The term σ_{xx}^2 in the integral above can be expanded as follows.

$$\iint_A \sigma_{xx}^2\, dA = \iint_A \left\{ \frac{N_x^2}{A^2} + \frac{M_z^2 y^2}{I_z^2} + \frac{M_y^2 z^2}{I_y^2} - 2\frac{N_x M_z y}{A I_z} + 2\frac{N_x M_y z}{A I_y} - 2\frac{M_z M_y z y}{I_z I_y} \right\} dA.$$

But, since the coordinate axes are assumed to pass through the centroid of the cross-sectional area,

$$\iint_A y\, dA = \iint_A z\, dA = \iint_A yz\, dA = 0$$

Therefore, the total potential energy is simply the sum of the potential energies due to axial and bending moments individually.

$$U_n = \frac{1}{2}\left\{ \int_l \frac{N_x^2}{EA}\, dl + \int_l \frac{M_z^2}{EI_z}\, dl + \int_l \frac{M_y^2}{EI_y}\, dl \right\}.$$

Total Strain Energy arising from Combined Shear Stresses

Just as a structural element can be subjected to combined normal and bending stresses, combined shear stresses can also act together.

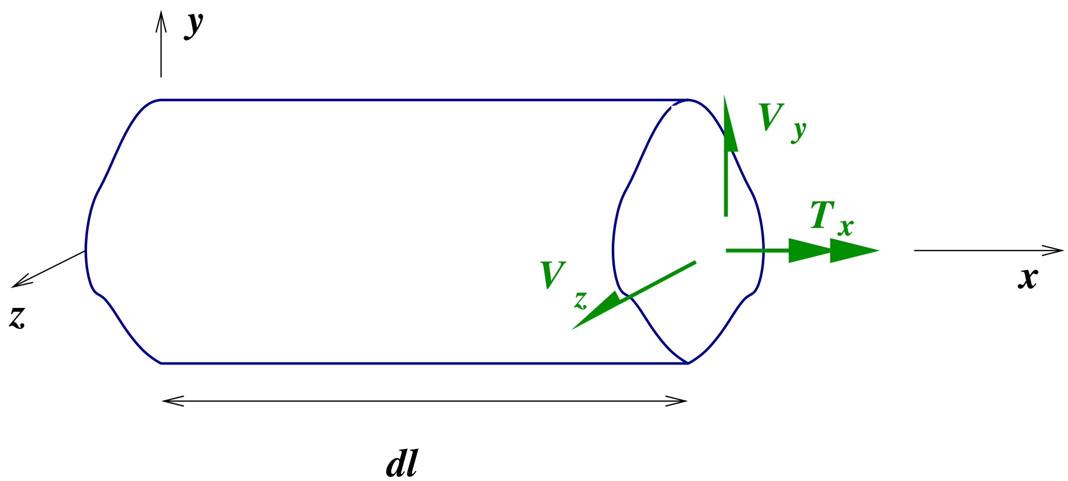

Figure 9. Internal shear forces and torsional moment in a prismatic beam.

$$\tau_{xy} = \frac{V_y Q_y(y)}{I_z t_z(y)} \qquad \tau_{xz} = \frac{V_z Q_z(z)}{I_y t_y(z)} \qquad \tau_{x\theta} = \frac{T_x r}{J}$$

Through mathematical manipulations similar to those above, it can be shown that

$$U_v = \frac{1}{2}\left\{ \int_l \frac{V_y^2}{G(A/\alpha_y)}\, dl + \int_l \frac{V_z^2}{G(A/\alpha_z)}\, dl + \int_l \frac{T_x^2}{GJ}\, dl \right\},$$

where

$$\alpha_y = \frac{A}{I_z^2} \iint_A \left(\frac{Q_y(y)}{t_z(y)} \right)^2 \, dA$$

$$\alpha_z = \frac{A}{I_y^2} \iint_A \left(\frac{Q_z(z)}{t_y(z)} \right)^2 \, dA$$

Total Strain Energy

The total strain energy for solids subjected to axial, bending, shear, and torsional forces is the sum of U_n and U_v above.

Summary

Strain energy is a kind of potential energy arising from the deformation of elastic solids. For structural elements (bars, beams, or shafts) strain energy is expressed in terms of the elasticity of the material (E or G), the dimensions of the element (L, A, I, J, or A/α), and *either* the internal forces (or moments) in the element ($N(x)$, $M(x)$, $V(x)$, or $T(x)$), *or* the deformation of the element ($u'(x)$, $v''(x)$, $v_\mathrm{s}'(x)$, $\theta'(x)$).

	"force"	deformation	force-based strain-energy	deformation-based strain energy
Axial	$N_x(x)$	$u'(x)$	$\int_{x=0}^{L} \frac{N_x(x)^2}{E(x)A(x)} dx$	$\int_{x=0}^{L} E(x)A(x)(u'(x))^2 dx$
Bending	$M_z(x)$	$v''(x)$	$\int_{x=0}^{L} \frac{M_x(x)^2}{E(x)I(x)} dx$	$\int_{x=0}^{L} E(x)I(x)(v''(x))^2 dx$
Shear	$V_y(x)$	$v_\mathrm{s}'(x)$	$\int_{x=0}^{L} \frac{V_x(x)^2}{G(x)(A(x)/\alpha)} dx$	$\int_{x=0}^{L} G(x)(A(x)/\alpha)(v_\mathrm{s}'(x))^2 dx$
Torsion	$T_x(x)$	$\theta'(x)$	$\int_{x=0}^{L} \frac{T_x(x)^2}{G(x)J(x)} dx$	$\int_{x=0}^{L} G(x)J(x)(\theta'(x))^2 dx$

where:

$E(x)$	is Young's modulus
$G(x)$	is the shear modulus
$A(x)$	is the cross sectional area of a bar
$I(x)$	is the bending moment of inertia a beam
$A(x)/\alpha$	is the effective shear area a beam
$J(x)$	is the torsional moment of inertia of a shaft
$N_x(x)$	is the axial force within a bar
$M_z(x)$	is the bending moment within a beam
$V_y(x)$	is the shear force within a beam
$T_x(x)$	is the torque within a shaft
$u'(x)$	is $du(x)/dx$, the axial strain, $u(x)$ is the axial displacement along the bar
$v''(x)$	is $d^2v(x)/dx^2$, the curvature, $v(x)$ is the transverse bending displacement of the beam
$v_\mathrm{s}'(x)$	is $dv_\mathrm{s}(x)/dx$, the shear deformation, $v_\mathrm{s}(x)$ is the transverse shear displacement of the beam
$\theta'(x)$	is $d\theta(x)/dx$, the torsional deformation, $\theta(x)$ is the torsional rotation of the shaft