

**INTERNAL STRAIN ENERGY OF  
LINEAR ELASTIC SOLIDS**  
**CE 130 — Structural Design and Optimization**  
**Spring, 2002**

Consider a force,  $F_i$ , applied gradually to a structure. Let  $D_i$  be the resulting displacement at the location and in the direction of the force  $F_i$ .

If the structure is elastic, the force-displacement curve follows the same path on loading and unloading.

If  $F_i$  is increased by  $\Delta F_i$  and the corresponding increase in the displacement is  $\Delta D_i$ , then as  $\Delta F_i \rightarrow 0$ , the incremental work,  $\Delta W$ , done by the load increment  $\Delta F_i$  is

$$\Delta W \approx F_i \Delta D_i,$$

or, more precisely,

$$\Delta W = \int_{D_i}^{D_i + \Delta D_i} F_i dD_i.$$

When the structure is *elastic and linear*, that is  $F_i = k_i D_i$ , the total work corresponding to a displacement  $D_i$  is

$$W = \int_0^{D_i} F_i dD_i = \int_0^{D_i} k_i D_i dD_i = \frac{1}{2} k_i D_i^2 = \frac{1}{2} F_i D_i.$$

If a linear elastic structure is subjected to a system of forces,

$F_1, F_2, \dots, F_n$  causing displacements  $D_1, D_2, \dots, D_n$ , in the direction of those forces, then the total *external work* is

$$W = \frac{1}{2} \{F_1 D_1 + F_2 D_2 + \dots + F_n D_n\} = \frac{1}{2} \{F\}^T \{D\}.$$

This work will be completely stored in the structure in the form of *strain energy*. Therefore, the *external work* and *strain energy* are equal to one another.

**External Work = Strain Energy = Internal Work**

$$W_E = U = W_I$$

**Example:** Small element subjected to normal stress  $\sigma_{xx}$

**STRAIN ENERGY DUE TO:  
AXIAL LOADS, BENDING MOMENTS, SHEAR FORCES, AND  
TORSIONAL MOMENTS**

A three dimensional linear elastic solid with loads supplied by external forces,  $F_1, \dots, F_n$ , and through support reactions, can be considered to be made up of small cubic elements as shown below:

The incremental strain energy,  $dU$ , for this elemental cube of volume  $dV$  can be written:

$$dU = \frac{1}{2} \{ \sigma_{xx} \epsilon_{xx} + \sigma_{yy} \epsilon_{yy} + \sigma_{zz} \epsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz} \} dV.$$

Integrating the incremental strain energy,  $dU$ , over an entire volume,  $V$ , the total strain energy,  $U$ , is

$$U = \frac{1}{2} \int_V \{ \sigma_{xx} \epsilon_{xx} + \sigma_{yy} \epsilon_{yy} + \sigma_{zz} \epsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz} \} dV.$$

If the stresses and strains are re-written as vectors,

$$\begin{aligned} \{\sigma\}^T &= \{ \sigma_{xx} \ \sigma_{yy} \ \sigma_{zz} \ \tau_{xy} \ \tau_{xz} \ \tau_{yz} \} \\ \{\epsilon\}^T &= \{ \epsilon_{xx} \ \epsilon_{yy} \ \epsilon_{zz} \ \gamma_{xy} \ \gamma_{xz} \ \gamma_{yz} \}, \end{aligned}$$

then the total strain energy can be written compactly as

$$U = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} dV.$$

This equation is a general expression for the strain energy of a linear elastic structure of any type. It can be simplified significantly for structures made of prismatic members, such as trusses and frames.

**Strain Energy due to Axial Loads**  $\sigma_{xx} = E\epsilon_{xx}$ 

Consider a rod subjected to a normal force,  $N_x$ :

$$\sigma_{xx} = \frac{N_x}{A}$$
$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{N_x}{EA}$$

$$U = \frac{1}{2} \int_V \sigma_{xx} \epsilon_{xx} dV = \frac{1}{2} \int_l \iint_A \frac{N_x^2}{EA^2} dA dl = \frac{1}{2} \int_l \frac{N_x^2}{EA^2} \iint_A dA dl = \frac{1}{2} \int_l \frac{N_x^2}{EA} dl.$$

For a prismatic member, this becomes a truss element and the strain energy is

$$U = \frac{1}{2} \frac{N_x^2 L}{EA}$$

### Stain Energy due to Bending Moments $\sigma_{xx} = -M_z y / I_z$

Consider a beam subjected to a pure bending moment about the  $z$ -axis,  $M_z$ :

The normal stress on an element  $dA$  at a distance  $y$  from the neutral axis is

$$\sigma_{xx} = -\frac{M_z y}{I_z}.$$

The corresponding strain is

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} = -\frac{M_z y}{EI_z}.$$

Then the incremental internal strain energy,  $dU$ , in an incremental volume element,  $dV$ , is

$$dU = \frac{1}{2} \sigma_{xx} \epsilon_{xx} dV = \frac{1}{2} \frac{M_z^2 y^2}{EI_z^2} dV,$$

and the total strain energy in a beam under pure bending moments is

$$U = \frac{1}{2} \int_l \frac{M_z^2}{EI_z^2} \iint_A y^2 dA dl.$$

Note that the definition of the moment of inertia,  $I$ , is the same as  $\iint_A y^2 dA$ .

Therefore, the total internal strain energy simplifies to

$$U = \frac{1}{2} \int_l \frac{M_z^2}{EI_z} dl.$$

Note that this derivation assumes that the origin of the coordinate system lies on the neutral axis of the beam:

$$\iint_A yz dy dz = 0.$$

**Strain Energy due to Shear Loads**  $\tau_{xy} = V_y Q(y) / I_z t(y)$ 

Consider a beam subjected to a shear force,  $V$ , (and bending moment):

$$\tau_{xy}(y) = \frac{V_y Q(y)}{I_z t(y)}$$

$$Q(y) = \text{Moment of Area of Cross Section} = \int_y^d t(y)y \, dy$$

$$\begin{aligned} U &= \frac{1}{2} \int_V \tau_{xy} \gamma_{xy} \, dV = \frac{1}{2} \int_l \iint_A \frac{\tau_{xy}^2}{G} \, dA \, dl \\ &= \frac{1}{2} \int_l \iint_A \frac{V_y^2 Q(y)^2}{I_z^2 G t(y)^2} \, dA \, dl = \frac{1}{2} \int_l \frac{V_y^2}{I_z^2 G} \iint_A \frac{Q(y)^2}{t(y)^2} \, dA \, dl \\ &= \frac{1}{2} \int_l \frac{V_y^2}{GA} \frac{A}{I_z^2} \iint_A \frac{Q(y)^2}{t(y)^2} \, dA \, dl \end{aligned}$$

This last integral is a reduces to a constant which depends only upon the shape of the cross-section. This constant is given the variable name  $\alpha$ .

$$\alpha = \frac{A}{I_z^2} \iint_A \frac{Q(y)^2}{t(y)^2} \, dA$$

The constant  $\alpha$  for some common cross-section shapes is given below ( $\alpha$  is always greater than 1).

With this simplification, the internal strain energy due to shear loads is

$$U = \frac{1}{2} \int_l \frac{\alpha V_y^2}{GA} dl = \frac{1}{2} \int_l \frac{V_y^2}{G(A/\alpha)} dl.$$

The term  $A/\alpha$  is called the *effective shear area*.

As a review of shear stresses in beams, consider the shear stress in a rectangular section.

$$\tau_{xy} = \frac{V_y Q(y)}{I_z t(y)}$$

$$Q(y) = \int_y^{d/2} t(y) y dy = b \int_y^{d/2} y dy = b \left[ \frac{y^2}{2} \right]_y^{d/2} = b \left[ \frac{d^2}{8} - \frac{y^2}{2} \right]$$

$$\tau_{xy} = \frac{V_y}{2I_z} \left( \frac{d^2}{4} - y^2 \right).$$

This stress varies parabolically along the direction of the applied shear. It is maximum at the centroid of the section and zero at the ends.

**Strain Energy due to Torsion**  $\tau = T_x r / J$ 

Consider a circular bar subjected to a constant torsional moment,  $T_x$ :

The circumferential shear stress  $\tau(r)$  is

$$\tau(r) = \frac{T_x r}{J}$$

and the corresponding shear strain is

$$\gamma(r) = \frac{\tau(r)}{G}.$$

The incremental internal strain energy,  $dU$ , is

$$dU = \frac{1}{2} \frac{\tau}{G} dV = \frac{1}{2} \frac{\tau^2}{G} dV,$$

and the total strain energy for the whole bar is

$$U = \frac{1}{2} \int_V \frac{\tau^2}{G} dV = \frac{1}{2} \int_V \frac{T_x^2 r^2}{J^2 G} dV = \frac{1}{2} \int_l \frac{T_x^2}{J^2 G} \iint_A r^2 dA dl,$$

in which the double integral term is the same as the polar moment of inertia

$J = \iint_A r^2 dA$ . So the total internal strain energy simplifies to:

$$U = \frac{1}{2} \int_l \frac{T_x^2}{GJ} dl.$$

## Total Strain Energy due to Combined Axial Stresses

As a review of the material above, consider a three-dimensional bending problem with a super-imposed normal force,  $N_x$ .

$$\sigma_{xx} = \frac{N_x}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}.$$

The total strain energy due to axial and pure bending effects is

$$U_n = \frac{1}{2} \int_V \sigma_{xx} \epsilon_{xx} dV = \frac{1}{2} \int_V \frac{\sigma_{xx}^2}{E} dV = \frac{1}{2} \int_l \frac{1}{E} \iint_A \sigma_{xx}^2 dA dl.$$

The term  $\sigma_{xx}^2$  in the integral above can be expanded as follows.

$$\iint_A \sigma_{xx}^2 dA = \iint_A \left\{ \frac{N_x^2}{A^2} + \frac{M_z^2 y^2}{I_z^2} + \frac{M_y^2 z^2}{I_y^2} - 2 \frac{N_x M_z y}{A I_z} + 2 \frac{N_x M_y z}{A I_y} - 2 \frac{M_z M_y z y}{I_z I_y} \right\} dA.$$

But, since the coordinate axes are assumed to pass through the centroid of the cross-sectional area,

$$\iint_A y dA = \iint_A z dA = \iint_A yz dA = 0$$

Therefore, the total potential energy is simply the sum of the potential energies due to axial and bending moments individually.

$$U_n = \frac{1}{2} \left\{ \int_l \frac{N_x^2}{EA} dl + \int_l \frac{M_z^2}{EI_z} dl + \int_l \frac{M_y^2}{EI_y} dl \right\}.$$

## Total Strain Energy due to Combined Shear Stresses

Just as a structural element can be subjected to combined normal and bending stresses, combined shear stresses can also act together.

$$\tau_{xy} = \frac{V_y Q_y(y)}{I_z t_z(y)}$$

$$\tau_{xz} = \frac{V_z Q_z(z)}{I_y t_y(z)}$$

$$\tau_r = \frac{T_x r}{J}$$

Through mathematical manipulations similar to those above, it can be shown that

$$U_v = \frac{1}{2} \left\{ \int_l \frac{V_y^2}{G(A/\alpha_y)} dl + \int_l \frac{V_z^2}{G(A/\alpha_z)} dl + \int_l \frac{T_x^2}{GJ} dl \right\},$$

where

$$\alpha_y = \frac{A}{I_z^2} \iint_A \left( \frac{Q_y(y)}{t_z(y)} \right)^2 dA$$

$$\alpha_z = \frac{A}{I_y^2} \iint_A \left( \frac{Q_z(z)}{t_y(z)} \right)^2 dA$$

## Total Strain Energy

The total strain energy for solids subjected to axial, bending, shear, and torsional forces is the sum of  $U_n$  and  $U_v$  above.