

THE THREE-MOMENT EQUATION
for
CONTINUOUS BEAM ANALYSIS
CE 130L. Uncertainty, Design, and Optimization
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Consider a continuous beam over several supports carrying arbitrary loads, $w(x)$.

Using the Moment-Area Theorem, we will analyze two adjoining spans of this beam to find the relationship between the internal moments at each of the supports and the loads applied to the beam. We will label the left, center, and right supports of this two-span segment L , C , and R . The left span has length L_L and flexural rigidity EI_L ; the right span has length L_R and flexural rigidity EI_R (see figure (a)).

Applying the principle of superposition to this two-span segment, we can separate the moments caused by the applied loads from the internal moments at the supports. The two-span segment can be represented by two simply-supported spans (with zero moment at L , C , and R) carrying the external loads plus two simply-supported spans carrying the internal moments M_L , M_C , and M_R (figures (b), (c), and (d)). The applied loads are illustrated below the beam, so as not to confuse the loads with the moment diagram (shown above the beams). Note that we are being consistent with our sign convention: positive moments create positive curvature in the beam. The internal moments M_L , M_C , and M_R are drawn in the positive directions. The areas under the moment diagrams due to the applied loads on the simply-supported spans (figure (b)) are A_L and A_R ; \bar{x}_L represents

the distance from the left support to the centroid of A_L , and \bar{x}_R represents the distance from the right support to the centroid of A_R , as shown. The moment diagrams due to the unknown moments, M_L , M_C , and M_R are triangular, as shown in figures (c) and (d).

Examining the elastic curve of the continuous beam (figure (e)), we recognize that the rotation of the beam at the center support, θ_C , is continuous across support C . In other words, θ_C just to the left of point C is the same as θ_C just to the right of point C . This *continuity condition* may be expressed

$$\frac{\Delta_{L \tan C}}{L_L} = -\frac{\Delta_{R \tan C}}{L_R}, \quad (1)$$

where $\Delta_{L \tan C}$ is the distance from the tangent at C to point L , and $\Delta_{R \tan C}$ is the distance from the tangent at C to point R .

Using the second Moment-Area Theorem, and assuming that the flexural rigidity (EI) is constant within each span, we can find the terms $\Delta_{L \tan C}$, and $\Delta_{R \tan C}$ in terms of the unknown moments, M_L , M_C , and M_R and the known applied loads.

$$\Delta_{L \tan C} = \frac{1}{EI_L} \left[\bar{x}_L A_L + \frac{2}{3} L_L \frac{1}{2} M_C L_L + \frac{1}{3} L_L \frac{1}{2} M_L L_L \right], \quad (2)$$

and

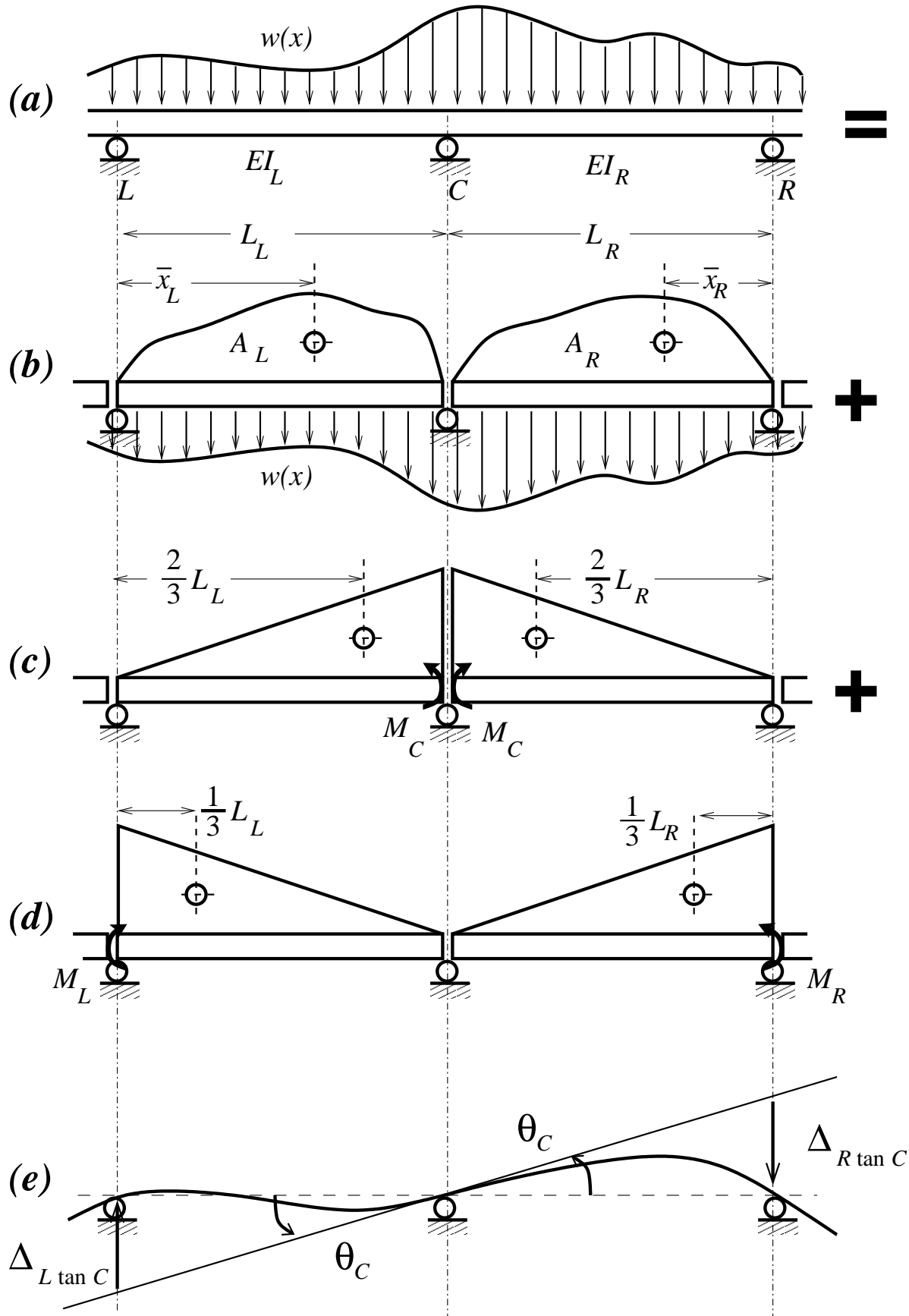
$$\Delta_{R \tan C} = \frac{1}{EI_R} \left[\bar{x}_R A_R + \frac{2}{3} L_R \frac{1}{2} M_C L_R + \frac{1}{3} L_R \frac{1}{2} M_R L_R \right]. \quad (3)$$

Substituting these expressions into equation (1) and re-arranging terms, leads to the three-moment equation.¹

$$\frac{L_L}{EI_L} M_L + 2 \left(\frac{L_L}{EI_L} + \frac{L_R}{EI_R} \right) M_C + \frac{L_R}{EI_R} M_R = -\frac{6\bar{x}_L A_L}{L_L EI_L} - \frac{6\bar{x}_R A_R}{L_R EI_R}. \quad (4)$$

Note that if $EI_L = EI_R = EI$, the three-moment equation is independent of EI .

¹The three-moment equation was derived by Émile Clapeyron in 1857 using the differential equations of beam bending.



To apply the three-moment equation numerically, the lengths, moments of inertia, and applied loads must be specified for each span. Two commonly applied loads are point loads and uniformly distributed loads. For point loads P_L and P_R acting a distance x_L and x_R from the left and right supports, respectively, the right hand side of the three-moment equation becomes

$$-\frac{6\bar{x}_L A_L}{L_L E I_L} - \frac{6\bar{x}_R A_R}{L_R E I_R} = -P_L \frac{x_L}{L_L E I_L} (L_L^2 - x_L^2) - P_R \frac{x_R}{L_R E I_R} (L_R^2 - x_R^2). \quad (5)$$

For uniformly distributed loads w_L and w_R on the left and right spans,

$$-\frac{6\bar{x}_L A_L}{L_L E I_L} - \frac{6\bar{x}_R A_R}{L_R E I_R} = -\frac{w_L L_L^3}{4 E I_L} - \frac{w_R L_R^3}{4 E I_R}. \quad (6)$$

To find the internal moments at the $N + 1$ supports in a continuous beam with N spans, the three-moment equation is applied to $N - 1$ adjacent pairs of spans. For example, consider the application of the three-moment equation to a four-span beam. Spans a , b , c , and d carry uniformly distributed loads w_a , w_b , w_c , and w_d , and rest on supports 1, 2, 3, 4, and 5.

The three-moment equation is applied to spans $a - b$, spans $b - c$, and spans $c - d$.

$$\begin{aligned} \frac{L_a}{E I_a} M_1 + 2 \left(\frac{L_a}{E I_a} + \frac{L_b}{E I_b} \right) M_2 + \frac{L_b}{E I_b} M_3 &= -\frac{w_a L_a^3}{4 E I_a} - \frac{w_b L_b^3}{4 E I_b} \\ \frac{L_b}{E I_b} M_2 + 2 \left(\frac{L_b}{E I_b} + \frac{L_c}{E I_c} \right) M_3 + \frac{L_c}{E I_c} M_4 &= -\frac{w_b L_b^3}{4 E I_b} - \frac{w_c L_c^3}{4 E I_c} \\ \frac{L_c}{E I_c} M_3 + 2 \left(\frac{L_c}{E I_c} + \frac{L_d}{E I_d} \right) M_4 + \frac{L_d}{E I_d} M_5 &= -\frac{w_c L_c^3}{4 E I_c} - \frac{w_d L_d^3}{4 E I_d} \end{aligned}$$

We know that $M_1 = 0$ and $M_5 = 0$ because they are at the ends of the span. Applying these end-moment conditions to the three three-moment equations and casting the equations into matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{L_a}{EI_a} & 2\left(\frac{L_a}{EI_a} + \frac{L_b}{EI_b}\right) & \frac{L_b}{EI_b} & 0 & 0 \\ 0 & \frac{L_b}{EI_b} & 2\left(\frac{L_b}{EI_b} + \frac{L_c}{EI_c}\right) & \frac{L_c}{EI_c} & 0 \\ 0 & 0 & \frac{L_c}{EI_c} & 2\left(\frac{L_c}{EI_c} + \frac{L_d}{EI_d}\right) & \frac{L_d}{EI_d} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{w_a L_a^3}{4EI_a} - \frac{w_b L_b^3}{4EI_b} \\ -\frac{w_b L_b^3}{4EI_b} - \frac{w_c L_c^3}{4EI_c} \\ -\frac{w_c L_c^3}{4EI_c} - \frac{w_d L_d^3}{4EI_d} \\ 0 \end{bmatrix}. \quad (7)$$

The 5×5 matrix on the left hand side of equation (7) is called a *flexibility matrix* and is tri-diagonal and symmetric. This equation can be written symbolically as $\mathbf{F} \mathbf{m} = \mathbf{d}$. By examining the general form of this expression, we can write a matrix representation of the three-moment equation for arbitrarily many spans. If a numbering convention is adopted in which support j lies between span $j - 1$ and span j , the three non-zero elements in row j of matrix \mathbf{F} are given by

$$F_{j,j-1} = \frac{L_{j-1}}{EI_{j-1}}, \quad (8)$$

$$F_{j,j} = 2\left(\frac{L_{j-1}}{EI_{j-1}} + \frac{L_j}{EI_j}\right), \quad (9)$$

$$F_{j,j+1} = \frac{L_j}{EI_j}. \quad (10)$$

For the case of uniformly distributed loads, row j of vector \mathbf{d} is

$$d_j = -\frac{w_{j-1} L_{j-1}^3}{4EI_{j-1}} - \frac{w_j L_j^3}{4EI_j}. \quad (11)$$

The moments at the supports are computed by solving the system of equations $\mathbf{F} \mathbf{m} = \mathbf{d}$ for the vector \mathbf{m} .

Once the internal moments are found, the reactions at the supports can be computed from static equilibrium

$$R_j = \frac{1}{2}w_{j-1}L_{j-1} + \frac{1}{2}w_jL_j - \frac{M_j}{L_{j-1}} - \frac{M_j}{L_j} + \frac{M_{j-1}}{L_{j-1}} + \frac{M_{j+1}}{L_j} + P_L \frac{x_L}{L_L} + P_R \frac{x_R}{L_R} \quad (12)$$

where the first two terms on the right hand side correspond to a uniformly distributed load and the last two terms correspond to interior point loads.

Having computed the reactions and internal moments, we can find the shear and moment diagrams from equilibrium equations. For example, consider span j between support j and support $j + 1$. The internal shear force at support j in span j is

$$V_{j,j} = \frac{M_j - M_{j+1}}{L_j} - \frac{1}{2}w_jL_j - P_k \frac{x_k}{L_j}, \quad (13)$$

and the internal shear force at support $j + 1$ in span j is

$$V_{j+1,j} = \frac{M_j - M_{j+1}}{L_j} + \frac{1}{2}w_jL_j + P_k \left(1 - \frac{x_k}{L_j}\right). \quad (14)$$

The term $w_jL_j/2$ in these two equations corresponds to a uniformly distributed load, P_k is a point load within span j , and x_k is the distance from the right end of span j to the point load P_k .

Beam rotations at the supports may be computed from equations (1), (2), and (3). The slope of the beam at support j is $\tan \theta_j$. From the second Moment-Area Theorem,

$$\tan \theta_j = -\frac{1}{EI_j} \left[\frac{1}{24}w_jL_j^3 + \frac{1}{6}P_k \frac{x_k}{L_j}(L_j^2 - x_k^2) + \frac{1}{3}M_jL_j + \frac{1}{6}M_{j+1}L_j \right], \quad (15)$$

where span j lies between support j and support $j+1$. The first term inside the brackets corresponds to a uniformly distributed load. The second term inside the brackets corresponds to a point load P_k within span j , located a distance x_k from the right end of span j , (support $j + 1$).

These ideas are implemented in the MATLAB function `three_moment.m`.

```

1 function [M,R,V, xs, M_Diag, V_Diag, d_Diag, D_max] = three_moment(L,I,E,w,P,x)
2 % [M,R,V, xs, M_Diag, V_Diag, d_Diag, D_max] = three_moment(L,I,E,w,P,x)
3 % solve the three-moment equations for a continuous beam of N spans.
4 %
5 % INPUT:
6 % L is a vector of length N containing the lengths of each span.
7 % I is a vector of length N containing the moments of inertia of each span.
8 % E is a scalar constant for the modulus of elasticity.
9 % w is a vector of length N containing the uniform loads on each span.
10 % P is a vector of point loads magnitudes
11 % x is a vector of point load locations, from the start of the first span
12 %
13 % OUTPUT:
14 % M is a vector of length N+1 containing the moments at each support
15 % R is a vector of length N+1 containing the reactions at each support
16 % V is a matrix of size 2xN+1 containing end-shears of each span
17 % xs is a vector of the x-axis for the shear, moment, and displ plots
18 % M_Diag is a vector of the moment diagram
19 % V_Diag is a vector of the shear diagram
20 % d_Diag is a vector of the displacement diagram
21 % D_max is a vector of the max absolute displacement of each span
22 %
23 % This program assumes that none of the supports are moment resisting,
24 % that there are no hinges in the beam, that all the spans are made of
25 % the same material, and that each spans is prismatic.
26
27 % H.P. Gavin, Civil and Environmental Engineering, Duke University, 3/24/09
28
29 N = length(L); % number of spans
30
31 nP = length(x); % number of concentrated point loads
32 span = zeros(1,nP); % spans containing the point loads
33 sumL = cumsum(L);
34 xL = zeros(1,nP); % distance from point load to left rctn
35 xR = zeros(1,nP); % distance from point load to right rctn
36 for i=1:length(x)
37     span(i) = min( find( x(i) < sumL ) );
38 end
39
40 for k=1:nP % loop over all concentrated point loads
41     if span(k) == 1 % the point load is in the first span
42         xL(k) = x(k);
43     else
44         xL(k) = x(k) - sumL(span(k)-1);
45     end
46     xR(k) = sumL(span(k)) - x(k);
47 end
48
49 F = zeros(N+1,N+1); % initialize the flexibility matrix
50
51 for j=2:N % create the flexibility matrix (8)-(10)
52
53     F(j,j-1) = L(j-1) / I(j-1);
54
55     F(j,j) = 2 * ( L(j-1) / I(j-1) + L(j) / I(j) );
56
57     F(j,j+1) = L(j) / I(j);
58 end
59
60 F(1,1) = 1.0;
61 F(N+1,N+1) = 1.0;
62
63 d = zeros(N+1,1);
64 for j=2:N % create the right-hand-side vector
65
66     l = j-1; % j-1 is the number of the left span
67     r = j; % j is the number of the right span

```

```

68
69     d(j) = -w(1)*L(1)^3 / (4*I(1)) - w(r)*L(r)^3 / (4*I(r));
70
71     for k=1:nP                                     % loop over all concentrated point loads
72         if span(k) == l                             % the point load is in the left span
73             d(j) = d(j) - P(k)*xL(k)/(L(1)*I(1))*(L(1)^2-xL(k)^2);
74         end
75         if span(k) == r                             % the point load is in the right span
76             d(j) = d(j) - P(k)*xR(k)/(L(r)*I(r))*(L(r)^2-xR(k)^2);
77         end
78     end
79 end
80
81 M = ( inv(F) * d )';                               % compute the internal moments (7)
82
83 R = zeros(1,N+1);                                  % build the vector of reaction forces
84 for j=1:N+1                                         % j is the reaction number
85
86     l = j-1;                                       % j-1 is the number of the left span
87     r = j;                                         % j is the number of the right span
88
89     if j == 1
90         R(j) = w(r)*L(r)/2 - M(j)/L(r) + M(j+1)/L(r);
91     end
92     if j == N+1
93         R(j) = w(1)*L(1)/2 - M(j)/L(1) + M(j-1)/L(1);
94     end
95     if j > 1 && j < N+1
96         R(j) = w(1)*L(1)/2 + w(r)*L(r)/2 ...
97             - M(j)/L(1) - M(j)/L(r) + M(j-1)/L(1) + M(j+1)/L(r);
98     end
99
100    for k=1:nP                                       % loop over all concentrated point loads
101        if span(k) == l                             % the point load is in the left span
102            R(j) = R(j) + P(k)*xL(k)/L(1);
103        end
104        if span(k) == r                             % the point load is in the right span
105            R(j) = R(j) + P(k)*xR(k)/L(r);
106        end
107    end
108
109 end
110
111 slope = zeros(1,N);
112 for j=1:N                                           % compute the slopes (15)
113
114     r = j;                                         % j is the span to the right of reaction j
115
116     slope(j) = w(r)*L(r)^3 / 24 + M(j+1)*L(r) / 6 + M(j)*L(r) / 3;
117
118     for k=1:nP                                       % loop over all concentrated point loads
119         if span(k) == r                             % the point load is in the right span
120             slope(j) = slope(j)+P(k)*xR(k)/L(r)*(L(r)^2-xR(k)^2)/6;
121         end
122     end
123
124     slope(j) = -slope(j) / ( E*I(r) );
125
126 end
127
128 if ( abs ( sum(R) - sum ( w .* L ) - sum(P) ) < 1e-9 )
129     disp ( ' yes! ' )                               % equilibrium check ... should be close to zero
130 end
131
132 %----- shear, moment, slope, and deflection data and plots -----
133
134 for j=1:N                                           % x-axis data for shear, moment, slope, and deflection diagrams
135     xs(:,j) = [ 0:L(j)/157:L(j) ]';

```

```

136 end
137
138 for j=1:N % j is the span number
139     Vo = ( M(j) - M(j+1) ) / L(j) - w(j)*L(j)/2; % shear at left
140     V_Diag(:,j) = Vo + w(j)*xs(:,j);
141     for k=1:nP % loop over all concentrated point loads
142         if span(k) == j % the point load is in span to the right
143             i1 = find(xs(:,j)<xL(k));
144             i2 = find(xs(:,j)>xL(k));
145             V_Diag(i1,j) = V_Diag(i1,j) - P(k)*xR(k)/L(j);
146             V_Diag(i2,j) = V_Diag(i2,j) + P(k)*(1-xR(k)/L(j));
147         end
148     end
149     M_Diag(:,j) = M(j) + cumtrapz( -V_Diag(:,j) ) * xs(2,j);
150     s_Diag(:,j) = cumtrapz( M_Diag(:,j) ) * xs(2,j) / (E*I(j)) + slope(j) ;
151     d_Diag(:,j) = cumtrapz( s_Diag(:,j) ) * xs(2,j) ;
152 end
153
154 % ----- display key results to the screen -----
155 fprintf('-----\n');
156 fprintf('          Moment          Shear          Deflection\n');
157 fprintf(' Maximum    %12.5e    %12.5e    %12.5e\n', ...
158     max(max(M_Diag)), max(max(-V_Diag)), max(max(d_Diag)) );
159 fprintf(' Minimum    %12.5e    %12.5e    %12.5e\n', ...
160     min(min(M_Diag)), min(min(-V_Diag)), min(min(d_Diag)) );
161 fprintf('-----\n');
162
163 for j=1:N % j is the span number
164     V(1,j) = -V_Diag(1,j); % shear force at left end of span
165     V(2,j) = -V_Diag(158,j); % shear force at right end of span
166 end
167
168 for j=2:N % x-axis data for shear and moment diagram plots
169     xs(:,j) = xs(:,j) + sumL(j-1);
170 end
171
172 % Plotting
173
174 xs = xs(:);
175 M_Diag = M_Diag(:);
176 V_Diag = -V_Diag(:);
177 s_Diag = s_Diag(:);
178 d_Diag = d_Diag(:);
179 z = zeros(1,length(xs));
180
181 D_max = max(abs(d_Diag));
182
183 figure(1)
184 clf
185 subplot(411)
186 plot ( xs, z, '-k', xs, V_Diag, '-b', 'LineWidth', 2 )
187 ylabel('Internal Shear')
188
189 subplot(412)
190 plot ( xs, z, '-k', xs, M_Diag, '-b', 'LineWidth', 2 )
191 ylabel('Internal Moment')
192
193 subplot(413)
194 plot ( xs, z, '-k', xs, s_Diag, '-b', 'LineWidth', 2 )
195 ylabel('Slope')
196
197 subplot(414)
198 plot ( xs, z, '-k', xs, d_Diag, '-b', 'LineWidth', 2 )
199 ylabel('Deflection')
200
201 % ----- three_moment.m -----

```

Example

```
L = [100 150 150 50];           % lengths of each span
I = [ 500 1000 2000 100 ];      % bending moment of inertia of each span
E = 1000;                       % elastic modulus
w = [ 0.10 0.20 0.10 0.30 ];   % uniformly distributed load, each span
P = [ 10 20 40 20 5 ];         % concentrated interior point loads
x = [ 110 130 300 330 420 ];   % location of point loads from x=0
```

```
[M,R,V] = three_moment(L,I,E,w,P,x)
yes!
```

```
-----
                Moment                Shear                Deflection
Maximum        1.30355e+03            4.89758e+01            1.73453e-01
Minimum        -1.10017e+03           -2.60242e+01           -1.25557e+00
-----
```

```
M = 0.0000e+00 -3.0056e+02 -1.1002e+03 -2.7880e+02 0.0000e+00
```

```
R = 1.9944 43.0082 73.9732 42.1003 3.9239
```

```
V = 1.9944 35.0026 48.9758 16.0761
     -8.0056 -24.9974 -26.0242 -3.9239
```

