TEMPERATURE EFFECTS IN STATICALLY DETERMINATE STRUCTURES
CE 130 — Structural Design and Optimization
Spring, 2003

Recall that the axial strain, ϵ, due to a change in temperature, ΔT, is

$$\epsilon = \alpha \, \Delta T,$$

where α is a material property called the *coefficient of linear expansion.*

In a statically determinate structure, when the change in temperature, ΔT, is constant across the cross-section, the temperature change produces axial strains similar to a normal force or a bending moment. In general, the change in temperature can vary over the height of the cross section. For example, sunlight shining on a bridge will cause the top surface of the bridge to be hotter than the bottom (shady) surface. So we can write that ΔT is a function of y: $\Delta T = \Delta T(y)$.

To solve temperature change problems with the principle of virtual work, think of the temperature change as a "real force" acting on the structure. We can then obtain the internal virtual work for virtual axial forces and virtual bending moments.

Axial Effects

System of Virtual Stresses (n) System of Real Strains ($\alpha\Delta T$)

In general, the internal virtual work due to virtual axial stresses and a real temperature change is

$$\bar{W}_{\mathrm{I}} = \int_V \bar{\sigma}_{xx}\epsilon_{xx}\, dV = \int_l \iint_A \frac{n}{A}\alpha\, \Delta T\, dA\, dl = \int_l \frac{n}{A}\alpha \left(\iint_A \Delta T\, dA \right)\, dl.$$

If we assume that the temperature between the top and the bottom of the beam varies linearly,

then

$$\Delta T(y) = \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right) y + \left[\Delta T_{\mathrm{t}} - \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right) h_2\right],$$

and

$$\iint_A \Delta T\, dA = \iint_A \left\{ \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right) y + \left[\Delta T_{\mathrm{t}} - \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right) h_2\right] \right\}\, dA.$$

Note that the first term of the integral is zero since $\iint_A y\, dA = 0$. Therefore

$$\iint_A \Delta T\, dA = \left[\Delta T_{\mathrm{t}} - \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right) h_2\right] A$$

and the internal virtual work due to axial forces and temperature change is

$$\bar{W}_{\mathrm{I}} = \int_l n\alpha \left[\Delta T_{\mathrm{t}} - \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right) h_2\right]\, dl.$$

Note that if the cross section is symmetric about the z-axis, then $h_2 = -h_1 = -h/2$, and

$$\bar{W}_{\mathrm{I}} = \int_l n\alpha \left(\frac{\Delta T_{\mathrm{t}} + \Delta T_{\mathrm{b}}}{2}\right)\, dl,$$

and if the temperature change is constant over the cross section, $\Delta T_{\mathrm{t}} = \Delta T_{\mathrm{b}}$, and

$$\bar{W}_{\mathrm{I}} = \int_l n\ \alpha\ \Delta T\ dl.$$

Bending Effects

Again, think of the temperature change a "real force" acting on the structure. We can then obtain the internal virtual work for bending moments.

System of Virtual Stresses (m) System of Real Strains ($\alpha \Delta T$)

In general, the internal virtual work due to virtual bending stresses and a real temperature change is

$$\bar{W}_{\mathrm{I}} = \int_V \bar{\sigma}_{xx}\epsilon_{xx}\, dV = \int_l \iint_A \frac{-m_z y}{I_z}\alpha\ \Delta T\ dA\ dl = \int_l \frac{-m_z}{I_z}\alpha \iint_A y\ \Delta T\ dA\ dl.$$

If we assume that the temperature varies linearly between the top and the bottom of the beam,

$$\iint_A y\ \Delta T(y)\ dA = \frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h} \iint_A y^2 dA.$$

And since $\iint_A y^2 dA = I_z$,

$$\bar{W}_{\mathrm{I}} = -\int_l m_z \alpha \left(\frac{\Delta T_{\mathrm{t}} - \Delta T_{\mathrm{b}}}{h}\right)\, dl = \int_l m_z \alpha \left(\frac{\Delta T_{\mathrm{b}} - \Delta T_{\mathrm{t}}}{h}\right)\, dl.$$

To solve temperature problems using the principle of virtual work, set the total external virtual work, $\bar{W}_{\mathrm{E}}$, equal to the internal virtual work due to external loads plus the internal virtual work due to temperature changes.