

# Introductory Example

## Optimal design of a two-bar truss

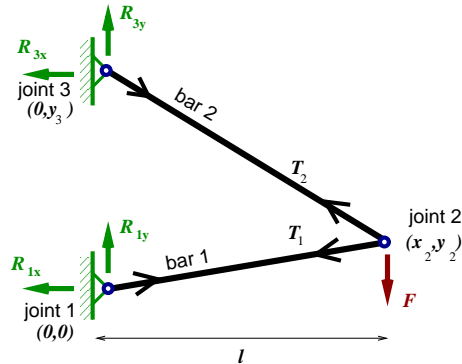
### CE 130L. Uncertainty, Design, and Optimization Spring, 2009

Consider a statically-determinate two-bar truss loaded by a vertical load  $F$ . The lengths of bars 1 and 2 depend on the locations of joints 2 and 3.

$$L_1 = \sqrt{X_1^2 + Y_1^2} = \sqrt{(x_2 - 0)^2 + (y_2 - 0)^2}$$

$$L_2 = \sqrt{X_2^2 + Y_2^2} = \sqrt{(x_2 - x_3)^2 + (y_3 - y_2)^2}$$

where  $X_i$  and  $Y_i$  are the positive-valued  $x$ -axis and  $y$ -axis projections of the bars.



The reactions are:  $R_{1x} = -F \frac{x_2}{y_3}$ ,  $R_{1y} = F \frac{y_2}{y_3}$ ,  $R_{3x} = F \frac{x_2}{y_3}$ ,  $R_{3y} = F(1 - \frac{y_2}{y_3})$ .  
The equilibrium equations at joint 1 are

$$\begin{aligned} \vec{\sum} F_x = 0 : c_1 T_1 &= R_{1x} , \\ \uparrow \sum F_y = 0 : s_1 T_1 &= -R_{1y} , \end{aligned}$$

The equilibrium equations at the loaded joint (joint 2) are

$$\begin{aligned} \vec{\sum} F_x = 0 : -c_1 T_1 - c_2 T_2 &= 0 , \\ \uparrow \sum F_y = 0 : -s_1 T_1 + s_2 T_2 &= F , \end{aligned}$$

The equilibrium equations at joint 3 are

$$\begin{aligned} \vec{\sum} F_x = 0 : c_2 T_2 &= R_{3x} , \\ \uparrow \sum F_y = 0 : -s_2 T_2 &= -R_{3y} , \end{aligned}$$

where  $c_i = X_i/L_i$  and  $s_i = Y_i/L_i$ . The bar tensions may be found by solving the matrix equation

$$\begin{bmatrix} c_1 & 0 \\ s_1 & 0 \\ -c_1 & -c_2 \\ -s_1 & s_2 \\ 0 & c_2 \\ 0 & -s_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -Fx_2/y_3 \\ -Fy_2/y_3 \\ 0 \\ F \\ Fx_2/y_3 \\ -F(1 - y_2/y_3) \end{bmatrix}$$

### Design Goal

Given fixed values for the material properties  $(E, S_y)$ , the load,  $F$ , the span,  $x_2 = l$ , and the bar cross section  $(A_1, I_1, A_2, I_2)$ , determine the values of parameters  $y_2$  and  $y_3$  that minimize the total weight of the truss, such that yielding and buckling are prevented, with some degree of safety. In other words,

$$\min J = f(y_2, y_3) = A_1 L_1 + A_2 L_2 = A_1 \sqrt{l^2 + y_2^2} + A_2 \sqrt{l^2 + (y_3 - y_2)^2}$$

subject to the safety conditions

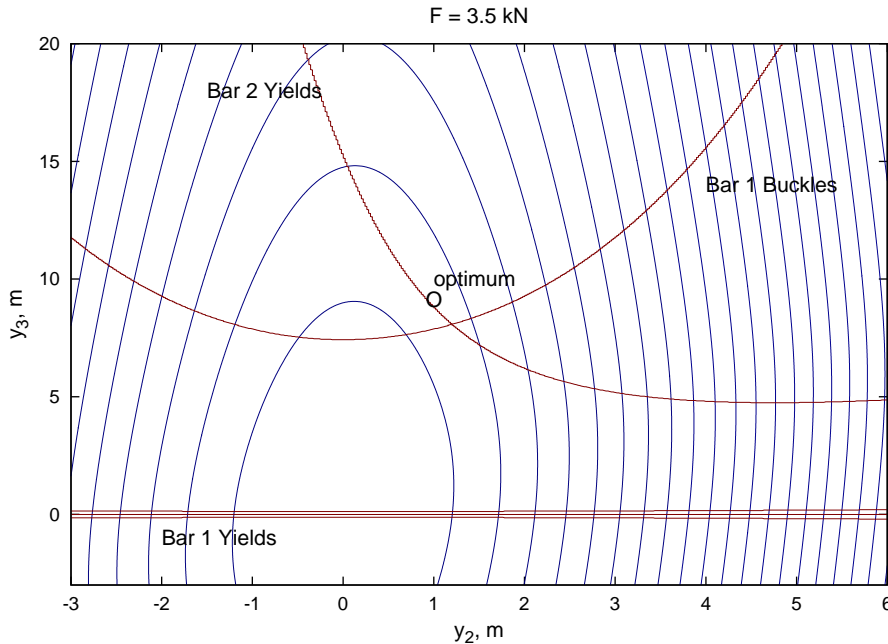
$$\begin{aligned} g_1(y_2, y_3) &= |T_1(y_2, y_3)| \phi_L - A_1 S_y \phi_R < 0 \\ g_2(y_2, y_3) &= |T_2(y_2, y_3)| \phi_L - A_2 S_y \phi_R < 0 \\ g_3(y_2, y_3) &= -T_1(y_2, y_3) \phi_L - \frac{\pi^2 E I_1}{L_1^2} \phi_R < 0 \\ g_4(y_2, y_3) &= -T_2(y_2, y_3) \phi_L - \frac{\pi^2 E I_2}{L_2^2} \phi_R < 0 \end{aligned} \tag{1}$$

where  $\phi_L$  is a safety factor on the load effects and  $\phi_R$  is a safety factor on the resistance to the loads ( $\phi_L > 0$  and  $\phi_R < 0$ ).

### Characteristics of the design problem

To illustrate the characteristics of this optimal design problem, consider a case in which both bars are made of mild steel. The elastic modulus,  $E$ , is  $200 \times 10^6$  kN/m<sup>2</sup>, and the yield stress,  $S_y$ , is  $250 \times 10^3$  kN/m<sup>2</sup>. Further, fix the length  $L$  to be 5 m, and the diameters of bars 1 and 2 to be 30 mm and 5 mm, respectively. The safety factors are  $\phi_R = 0.9$  and  $\phi_L = 1.2$ . These safety factors imply that there is a degree of uncertainty in the actual loads that may be applied to the system (the operating environment) and in the ability of the system to withstand a given load (the quality of the materials and workmanship).

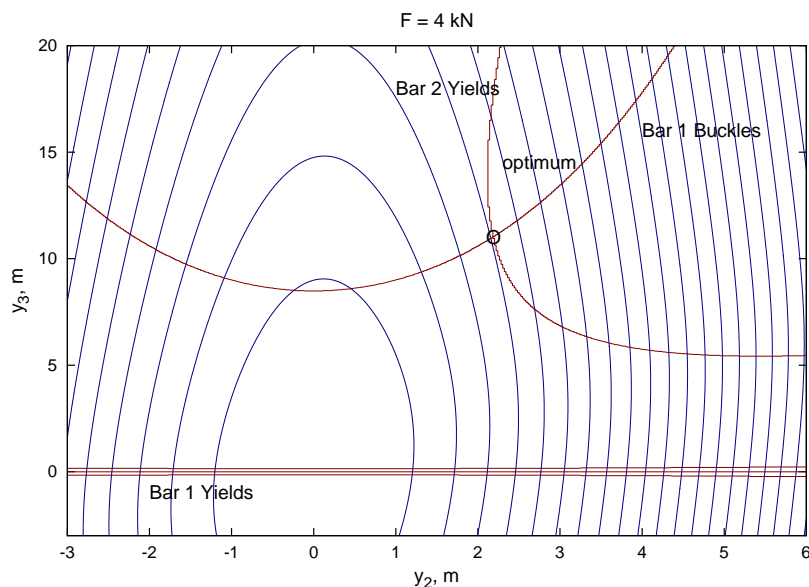
If the load,  $F$ , is 3.5 kN, then the contours of the cost function,  $f(y_2, y_3)$ , and the lines  $g_1 = 0$ ,  $g_2 = 0$ , and  $g_3 = 0$  are shown below. The ellipsoidal lines are the contours of the cost function  $f(y_2, y_3)$  and the parabolic lines show the constraint limits.



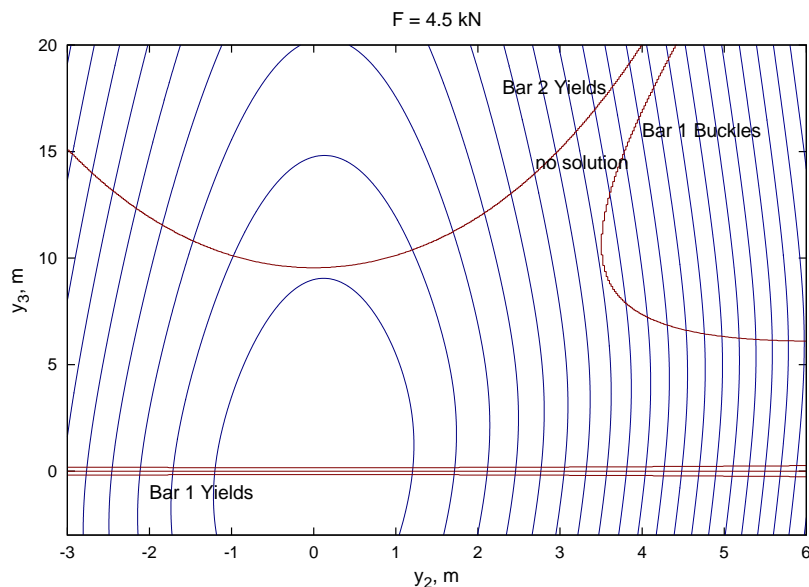
The constraint limits divide the design space  $(y_2, y_3)$  into *feasible* and *infeasible* regions. Values of  $(y_2, y_3)$  below the line labeled **Bar 1 Yields** will result in yielding of bar 1. Values of  $(y_2, y_3)$  below the line labeled **Bar 2 Yields** will result in yielding of bar 2. Values of  $(y_2, y_3)$  below the line labeled **Bar 1 Buckles** will result in buckling of bar 1. To prevent yielding and buckling in both bars, the design values  $(y_2, y_3)$  must lie above all three curves. The optimal design values  $(y_2^*, y_3^*)$  are the feasible values that result in the lowest value of the cost function (the lightest truss).

For a design load of 3.5 kN, the lightest truss has optimal dimensions  $(y_2^*, y_3^*) = (0.751, 9.970)$  m and uses  $3780 \text{ cm}^3$  of steel. This optimal design is constrained by the yielding of bar 2; a small increase in the load  $F$ , would cause failure through the yielding of bar 2.

Increasing the load from 3.5 kN to 4 kN decreases the feasible region, and requires a heavier truss. For  $F = 4$  kN, the design problem is characterized by the constraint equations shown below. For a design load of 4 kN, the lightest truss has optimal dimensions  $(y_2^*, y_3^*) = (2.193, 11.060)$  m, and uses  $4060 \text{ cm}^3$  of steel. This design is constrained by yielding of bar 2 *and* buckling of bar 1.



For loads greater than 4.45 kN, there is no feasible design. Given the limitations on the cross-section areas of the bars, the truss will fail regardless of the values of  $y_2$  and  $y_3$ .



## Safety Analysis

The safety criteria used in this design optimization applied a safety factor  $\phi_L$  to the truss bar loads and a safety factor  $\phi_R$  to the resistance of the truss bars to these loads. A load factor  $\phi_L > 1$  makes the design safer by designing for larger-than-expected loads. A resistance factor  $\phi_R < 1$  makes the design safer by designing for weaker-than-expected strength/performance.

In most design situations the designer is uncertain as to the operating environment of the designed system. The designer may also be uncertain as to the quality of the designed system, when it is eventually produced. If the designer can estimate:

- an expected average or nominal load level (mean load,  $\mu_L$ )
- the variability in the loads (standard deviation of the loads,  $\sigma_L$ )
- an expected average or nominal level of quality (mean of the strength or performance,  $\mu_R$ ), and
- the variability in the quality (standard deviation of the strength or performance,  $\sigma_R$ )

then the designer can carry out a risk analysis to determine the probability of a product failure. To do so, the safety of the optimized design is re-evaluated hundreds or thousands of times, each time using different randomly-generated values for the loading environment and the strength characteristics. These randomly-generated loads and strengths will conform to the designer's expectation regarding average load and strengths and the variability of the load and strengths.

For example, in this two-bar truss problem, a design was optimized using a load value of 3.5 kN and a strength value of  $250 \times 10^3$  kN/m<sup>2</sup>. These values would be the designer's expected (or mean or average) load,  $\mu_L$ , and strength,  $\mu_R$ , values. In evaluating the safety of the design, truss bar forces calculated on the basis of this expected load were multiplied by a factor  $\phi_L$  of 1.2, and truss bar strengths calculated on the basis of this expected yield stress were multiplied by a factor  $\phi_R$  of 0.9.

If the designer can assess the variability in the loading environment,  $\sigma_L$ , and the strength,  $\sigma_R$ , then the probability of failure of the optimized design ( $y_2^*$ ,  $y_3^*$ ) can be carried out as follows:

1. Generate a random value for the load,  $F$ , and strength,  $S_y$ , characteristics. These random values will have a mean  $\mu$  and standard deviation  $\sigma$  corresponding to the designers expectation of the loading environment and the strength properties of the manufactured product.
2. Analyze the safety of the optimized design with the random values of  $F$  and  $S_y$  using  $\phi_L = 1$  and  $\phi_R = 1$ .
3. Save the values of the safety constraints. ( $g < 0$  is o.k.,  $g > 0$  represents a failure).
4. After repeating steps 1, 2 and 3  $N$  times ( $10^2 < N < 10^6$ ), determine the number,  $n$ , of analyses resulting in one or more violated constraint.
5. The probability of failure is  $n/N$ .
6. The probability distribution of the safety constraints may be plotted, if desired.

In this two-bar truss design problem, a risk analysis may be carried out by assuming that the variability in the loads is twenty percent of the mean load and that the variability in the strength is ten percent of the mean strength,

$$F = \mu_F(1 + 0.2z) ,$$

and

$$S_y = \mu_{S_y}(1 + 0.1z) ,$$

where  $z$  is a normally-distributed random number with a mean of zero and a standard deviation of unity. A different random number  $z$  is used each time a random value is generated. The safety criteria ( $g_1, g_2, g_3$ ) are evaluated using  $\phi_L = 1$ ,  $\phi_R = 1$ ,  $y_2 = y_2^*$ , and  $y_3 = y_3^*$  for each pair of random load,  $F$ , and strength,  $S_y$ , values. The probability of failure is the number of evaluations resulting in a failure,  $n$ , (buckling of bar 1, yielding of bar 1, or yielding of bar 2) divided by the total number of evaluations,  $N$

$$P_f = n/N.$$

Results for the two feasible designs are tabulated below.

### Design

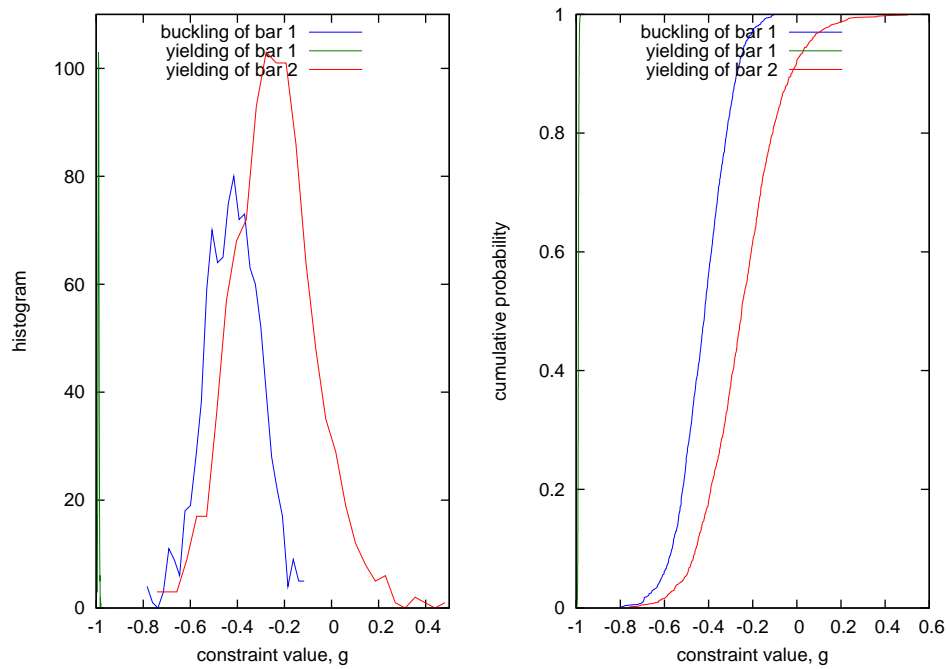
load	$F$	3.5	4.0	kN
yield stress	$S_y$	$250 \times 10^3$	$250 \times 10^3$	kN/m <sup>2</sup>
load factor	$\phi_L$	1.2	1.2	
resistance factor	$\phi_R$	0.9	0.9	
optimum $y_2$ value	$y_2^*$	0.751	2.193	m
optimum $y_3$ value	$y_3^*$	9.970	11.060	m
amount of material	$J$	3780	4060	cm <sup>3</sup>

### Safety Analysis

mean load	$\mu_F$	3.5	4.0	kN
std.dev. load	$\sigma_F$	0.7	0.8	kN
mean strength	$\mu_{S_y}$	$250 \times 10^3$	$250 \times 10^3$	kN/m <sup>2</sup>
std.dev. strength	$\sigma_{S_y}$	$250 \times 10^2$	$250 \times 10^2$	kN/m <sup>2</sup>
prob. failure	$P_f$	8.3%	9.3%	

If an eight or nine percent probability of failure is not acceptable, then a re-design would be carried out using a larger  $\phi_L$  and/or a smaller  $\phi_R$ .

Probability distribution plots of the three safety criteria for the 3.5 kN load.



Probability distribution plots of the three safety criteria for the 4.0 kN load.

