THE PRINCIPLE OF VIRTUAL WORK
CE 130 — Structural Design and Optimization
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Definitions:

Virtual work is the work done by a *real force* acting through a *virtual displacement* or a *virtual force* acting through a *real displacement.*

A *virtual displacement* is any displacement consistent with the constraints of the structure, i.e., that satisfy the boundary conditions at the supports.

A *virtual force* is any system of forces in equilibrium.

Example:

$f(x)$ and $y(x)$ are real forces and associated displacements.
$\delta y(x)$ is a virtual displacement consistent with the boundary conditions.

Consider a structure deformed by the effect of n external forces, denoted by the vector $\{F\}$. The actual (real) displacements at the same n coordinates are contained in the vector $\{D\}$.

The stresses and strains at any point in the structure are elements of the vectors $\{\sigma\}$ and $\{\epsilon\}$:

$$\begin{aligned}\{\sigma\}^T &= \{\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \tau_{xy}\ \tau_{xz}\ \tau_{yz}\} \\ \{\epsilon\}^T &= \{\epsilon_{xx}\ \epsilon_{yy}\ \epsilon_{zz}\ \gamma_{xy}\ \gamma_{xz}\ \gamma_{yz}\}.\end{aligned}$$

The total external work done by $\{F\}$ is

$$W_{\mathrm{E}} = \frac{1}{2}\sum_{i=1}^{n} F_i\ D_i = \frac{1}{2}\{F\}^T\{D\},$$

and the total internal work done by $\{F\}$ is the total strain energy, which can be written compactly as

$$W_{\mathrm{I}} = U = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}dV.$$

Setting W_{E} equal to W_{I} gives

$$\frac{1}{2}\{F\}^T\{D\} = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}dV. \tag{1}$$

Suppose, that *after* the structure is subjected to the system of n external forces, $\{F\}$, producing internal stresses $\{\sigma\}$, a system of m *virtual forces* $\{\delta F\}$ are applied, producing additional *virtual deformations* $\{\delta D\}$, *virtual stresses* $\{\delta\sigma\}$, and virtual strains $\{\delta\epsilon\}$.

The external work done by the application of $\{\delta F\}$ is

$$W_{\mathrm{E}} = \frac{1}{2}\{\delta F\}^T\{\delta D\} + \sum_{j=1}^{m} \delta F_j \delta D_j = \frac{1}{2}\{\delta F\}^T\{\delta D\} + \{F\}^T\{\delta D\},$$

and the internal work done by the application of $\{\delta F\}$ is

$$W_{\mathrm{I}} = \frac{1}{2}\int_V \{\delta\sigma\}^T\{\delta\epsilon\}dV + \int_V \{\sigma\}^T\{\delta\epsilon\}dV.$$

Setting the external work equal to the internal work,

$$\frac{1}{2}\{\delta F\}^T\{\delta D\} + \{F\}^T\{\delta D\} = \frac{1}{2}\int_V \{\delta\sigma\}^T\{\delta\epsilon\}dV + \int_V \{\sigma\}^T\{\delta\epsilon\}dV. \qquad (2)$$

If we consider the virtual system alone,

The external work is $\frac{1}{2}\{\delta F\}^T\{\delta D\}$, and the internal work is $\frac{1}{2}\int_V \{\delta\sigma\}^T\{\delta\epsilon\}dV$, or

$$\frac{1}{2}\{\delta F\}^T\{\delta D\} = \frac{1}{2}\int_V \{\delta\sigma\}^T\{\delta\epsilon\}dV. \qquad (3)$$

Substituting equation (3) into equation (2) gives

$$\{F\}^T\{\delta D\} = \int_V \{\sigma\}^T\{\delta\epsilon\}dV. \qquad (4)$$

Suppose, instead, that *before* the actual loads $\{F\}$ and deformations $\{D\}$ are introduced, the structure was subjected to a system of m virtual forces, $\{\delta F\}$, producing internal stresses $\{\delta\sigma\}$.

The external work done by the application of $\{F\}$ is now

$$W_{\mathrm{E}} = \frac{1}{2}\{F\}^T\{D\} + \sum_{j=1}^{m} \delta F_j D_j = \frac{1}{2}\{F\}^T\{D\} + \{\delta F\}^T\{D\}.$$

Note here that the actual deflections $\{D\}$ are unrelated to the virtual forces $\{\delta F\}$. The internal work done by the application of $\{F\}$ is now

$$W_{\mathrm{I}} = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}dV + \int_V \{\delta\sigma\}^T\{\epsilon\}dV.$$

Setting the external work equal to the internal work,

$$\frac{1}{2}\{F\}^T\{D\} + \{\delta F\}^T\{D\} = \frac{1}{2}\int_V \{\sigma\}^T\{\epsilon\}dV + \int_V \{\delta\sigma\}^T\{\epsilon\}dV, \qquad (5)$$

and substituting equation (1) into equation (5) gives

$$\{\delta F\}^T\{D\} = \int_V \{\delta\sigma\}^T\{\epsilon\}dV. \qquad (6)$$

Equation (6) is used in the unit load method to find redundant forces or reactions, and to find real structural displacements, as will be shown shortly. The left hand side of this equation, $\{\delta F\}^T\{D\}$, is called the *external virtual work*, δW_{E}. The right hand side of this equation, $\int_V \{\delta\sigma\}^T\{\epsilon\}dV$, is called the *internal virtual work*, δW_{I}. Note that equation (6) is valid for both linear and nonlinear elastic structures (why?).

Example: Unit Load Method

Find the deflection of a bar under axial tension.

The Unit Load Method

When the principle of virtual work is used to calculate the displacement D_i, at coordinate i, the system of external forces, $\{\delta F\}$ is chosen so as to consist only of a single **unit force** at coordinate i:

equation (6) becomes:

$$1 \cdot D_i = \int_V \{\delta\sigma\}^T \{\epsilon\} dV,$$

in which $\{\delta\sigma\}$ are the virtual stresses arising from the single unit force at i, and $\{\epsilon\}$ are the real strains due to the actual loading.

FORMS OF INTERNAL VIRTUAL WORK FOR FRAMED STRUCTURES

Virtual Axial Force

Consider a rod subjected to a virtual normal force n, and a real normal force, N:

$$\begin{aligned} \text{Virtual Stress} = \{\delta\sigma\}^T &= \{\delta\sigma_{xx}\ 0\ 0\ 0\ 0\ 0\} \\ \text{Real Strain} = \{\epsilon\}^T &= \{\epsilon_{xx}\ \epsilon_{yy}\ \epsilon_{zz}\ 0\ 0\ 0\} \end{aligned}$$

But we only need ϵ_{xx} because we are interested in $\{\delta\sigma\}^T\{\epsilon\}$.

$$\text{Virtual Stress } = \delta\sigma_{xx} = \frac{n}{A} \qquad \text{Real Strain } = \epsilon_{xx} = \frac{N}{EA}$$

The internal virtual work due to an axial force is

$$\delta W_{\text{I}} = \int_V \{\delta\sigma\}^T\{\epsilon\} dV = \int_l \iint_A \frac{nN}{EA^2}\, dA\, dl = \int_l \frac{nN}{EA}\, dl.$$

For a structure made up entirely of prismatic truss members,

$$\delta W_{\text{I}} = \sum_{m=1}^{M} \frac{n_m N_m L_m}{E_m A_m}.$$

Virtual Bending Moment

Consider a beam subjected to pure virtual and real bending moments about the z-axis, m_z and M_z:

$$\text{Virtual Stress} = \delta\sigma_{xx} = -\frac{m_z y}{I_z} \qquad \text{Real Strain} = \epsilon_{xx} = -\frac{M_z y}{EI_z}$$

The internal virtual work due to a bending moments is

$$\delta W_{\text{I}} = \int_V \{\delta\sigma\}^T\{\epsilon\} dV = \int_l \iint_A \frac{m_z M_z y^2}{EI_z^2}\, dA\, dl = \int_l \frac{m_z M_z}{EI_z}\, dl.$$

Recall that $I_z = \iint_A y^2\, dA$ when the origin of the coordinate system lies on the neutral axis of the beam.

Virtual Shear Force

Consider a beam subjected to a pure virtual and real shear forces in the y-direction, v_y and V_y:

$$\text{Virtual Stress} = \delta\tau_{xy} = \frac{v_y Q(y)}{I_z t(y)} \qquad \text{Real Strain} = \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{V_y Q(y)}{G I_z t(y)},$$

where $Q(y)$ is called the *moment of area*. The internal virtual work due to shear forces is

$$\delta W_{\mathrm{I}} = \int_V \delta\tau_{xy}\gamma_{xy} dV = \int_l \iint_A \frac{v_y V_y Q(y)^2}{G I_z^2 t(y)^2}\, dA\, dl = \int_l \frac{v_y V_y}{G(A/\alpha_y)}\, dl,$$

where

$$\alpha_y = \frac{A}{I_z^2} \iint_A \frac{Q(y)^2}{t(y)^2}\, dA.$$

Virtual Torsion

Consider a circular bar subjected to a virtual and real torsional moments, t and T:

$$\text{Virtual Stress} = \delta\tau = \frac{tr}{J} \qquad \text{Real Strain} = \gamma = \frac{\tau}{G} = \frac{Tr}{GJ},$$

The internal virtual work due to torsional moments is

$$\delta W_{\mathrm{I}} = \int_V \delta\tau\, \gamma\, dV = \int_l \iint_A \frac{tTr^2}{GJ^2}\, dA\, dl = \int_l \frac{tT}{GJ}\, dl,$$

$(J = \iint_A r^2\, dA)$

Total Internal Virtual Work

As a review of the material above, consider general three-dimensional superimposed real and virtual forces

The total virtual strain energy due to these combined effects is

$$\begin{aligned}\delta W_{\mathrm{I}} \ &= \int_l \frac{nN}{EA}\, dl + \int_l \frac{m_z M_z}{EI_z}\, dl + \int_l \frac{m_y M_y}{EI_y}\, dl + \\ &\quad \int_l \frac{v_y V_y}{G(A/\alpha_y)}\, dl + \int_l \frac{v_z V_z}{G(A/\alpha_z)}\, dl + \int_l \frac{tT}{GJ}\, dl\end{aligned}$$

where

$$\alpha_y = \frac{A}{I_z^2} \iint_A \left(\frac{Q_y(y)}{t_z(y)}\right)^2 dA$$

$$\alpha_z = \frac{A}{I_y^2} \iint_A \left(\frac{Q_z(z)}{t_y(z)}\right)^2 dA$$