BEAM ELEMENT STIFFNESS MATRICES
CE 131 — Matrix Structural Analysis
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Local Element Flexibility Matrix: F:

The element flexibility matrix, **F** relates $\{\delta, \theta_1, \theta_2\}$ to $\{N, M_1, M_2\}$:

$$\left\{ \begin{array}{c} \delta \\ \theta_1 \\ \theta_2 \end{array} \right\} = \mathbf{F} \left\{ \begin{array}{c} N \\ M_1 \\ M_2 \end{array} \right\}$$

and can be obtained from Castigliano's Theorem.

First Column Second Column Third Column

$$N(x) = N \qquad V(x) = \frac{M_1 + M_2}{L} \qquad M(x) = M_1\left(\frac{x}{L} - 1\right) + M_2\frac{x}{L}$$

The total potential energy of a beam with these forces is:

$$U = \frac{1}{2}\int_0^L \frac{N^2}{EA}\,dx + \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx + \frac{1}{2}\int_0^L \frac{V^2}{G(A/\alpha)}\,dx$$

By Castigliano's Theorem,

$$\begin{aligned}
\delta &= \frac{\partial U}{\partial N} = \int_0^L \frac{N(x)\frac{\partial N(x)}{\partial N}}{EA}\,dx = \int_0^L \frac{N(x)\cdot 1}{EA}\,dx = N\int_0^L \frac{dx}{EA}. \\
\theta_1 &= \frac{\partial U}{\partial M_1} \\
&= \int_0^L \frac{M(x)\frac{\partial M(x)}{\partial M_1}}{EI}\,dx + \int_0^L \frac{V(x)\frac{\partial V(x)}{\partial M_1}}{G(A/\alpha)}\,dx \\
&= \left(\int_0^L \frac{\left(\frac{x}{L}-1\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_1 + \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right) dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_2
\end{aligned}$$

and

$$\begin{aligned}
\theta_2 &= \frac{\partial U}{\partial M_2} \\
&= \int_0^L \frac{M(x)\frac{\partial M(x)}{\partial M_2}}{EI}\,dx + \int_0^L \frac{V(x)\frac{\partial V(x)}{\partial M_2}}{G(A/\alpha)}\,dx \\
&= \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right) dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_1 + \left(\int_0^L \frac{\left(\frac{x}{L}\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) M_2
\end{aligned}$$

or, in matrix form,

$$\left\{\begin{array}{c} \delta \\ \\ \theta_1 \\ \\ \theta_2 \end{array}\right\} = \left[\begin{array}{ccc}
\int_0^L \frac{dx}{EA} & 0 & 0 \\
0 & \left(\int_0^L \frac{\left(\frac{x}{L}-1\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) & \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right)dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) \\
0 & \left(\int_0^L \frac{\frac{x}{L}\left(\frac{x}{L}-1\right)dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right) & \left(\int_0^L \frac{\left(\frac{x}{L}\right)^2 dx}{EI} + \int_0^L \frac{\alpha dx}{GAL^2}\right)
\end{array}\right] \left\{\begin{array}{c} N \\ \\ M_1 \\ \\ M_2 \end{array}\right\}$$

For prismatic members, E, A, and I are constant along the length, and the element flexibility matrix is

$$\mathbf{F} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} + \frac{1}{G(A/\alpha)L} & -\frac{L}{6EI} + \frac{1}{G(A/\alpha)L} \\ 0 & -\frac{L}{6EI} + \frac{1}{G(A/\alpha)L} & \frac{L}{3EI} + \frac{1}{G(A/\alpha)L} \end{bmatrix}$$

To neglect shear deformation, set $\alpha = 0$.

Local Element Stiffness Matrix: k

The element stiffness matrix is the inverse of the element flexibility matrix, and for prismatic members,

$$\mathbf{k} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{(4+\Phi)EI}{(1+\Phi)L} & \frac{(2-\Phi)EI}{(1+\Phi)L} \\ 0 & \frac{(2-\Phi)EI}{(1+\Phi)L} & \frac{(4+\Phi)EI}{(1+\Phi)L} \end{bmatrix}$$

where

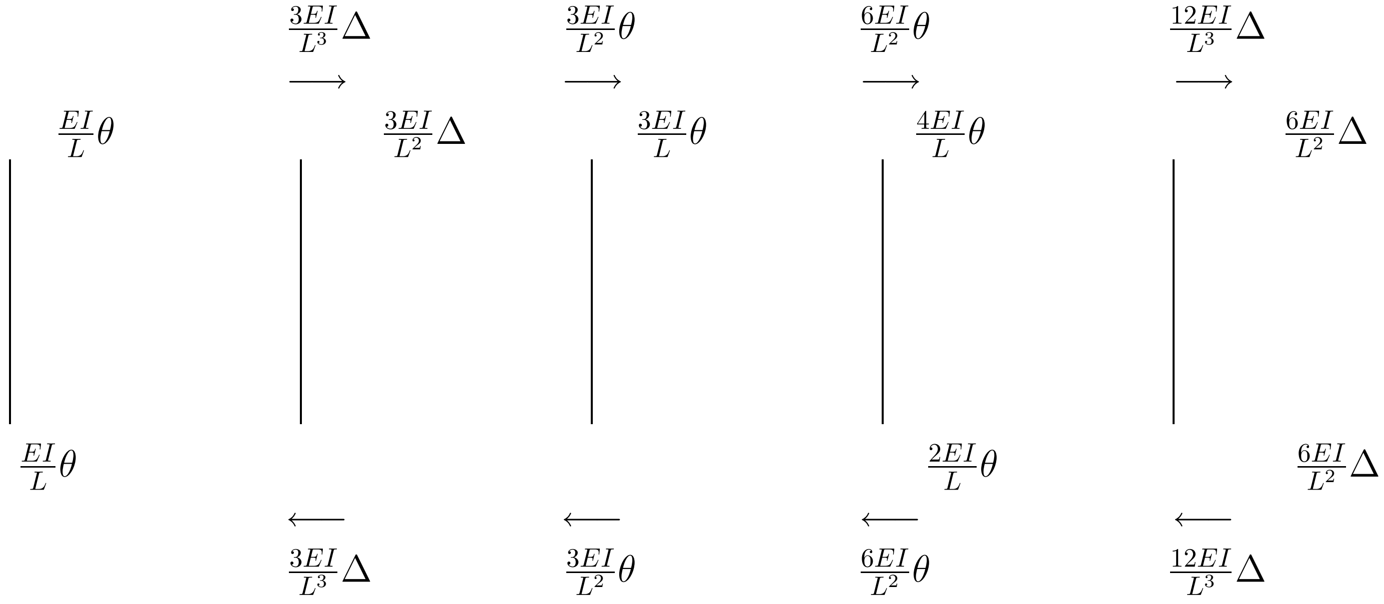
$$\Phi = \frac{12EI}{G(A/\alpha)L^2} = 24\alpha(1+\nu)\left(\frac{r}{L}\right)^2$$

and r is the "radius of gyration" of the cross section, $r = \sqrt{I/A}$.

To neglect shear deformation, set $\Phi = 0$:

$$\mathbf{k} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Beam Element Stiffness Matrix in Local Coordinates: k



$$
\left\{ \begin{array}{c} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{array} \right\}
=
\left[\begin{array}{cccccc}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
 & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
 & & & \frac{EA}{L} & 0 & 0 \\
 & \mathrm{SYM} & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
 & & & & & \frac{4EI}{L}
\end{array} \right]
\left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \right\}
$$

Relationships between Local Coordinates and Global Coordinates: T

The geometric relationship between local displacements, $\mathbf{u}$, and global displacements, $\mathbf{v}$, is

$$u_1 = v_1 \cos\theta + v_2 \sin\theta \qquad u_2 = -v_1 \sin\theta + v_2 \cos\theta \qquad u_3 = v_3$$

or, $\mathbf{u} = \mathbf{T}\ \mathbf{v}$.

The equilibrium relationship between local forces, $\mathbf{q}$, and global forces, $\mathbf{f}$, is

$$q_1 = f_1 \cos\theta + f_2 \sin\theta \qquad q_2 = -f_1 \sin\theta + f_2 \cos\theta \qquad q_3 = f_3$$

or, $\mathbf{q} = \mathbf{T}\ \mathbf{f}$, where, in both cases,

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} c = \cos\theta = \dfrac{x_2 - x_1}{L} \\[2em] s = \sin\theta = \dfrac{y_2 - y_1}{L} \end{array}$$

The coordinate transformation matrix, $\mathbf{T}$, is orthogonal, $\mathbf{T}^{-1} = \mathbf{T}^T$.

Beam Element Stiffness Matrix in Global Coordinates: K

Combining the coordinate transformation relationships,

$$\begin{aligned} \mathbf{q} &= \mathbf{k}\ \mathbf{u} \\ \mathbf{T}\ \mathbf{f} &= \mathbf{k}\ \mathbf{T}\ \mathbf{v} \\ \mathbf{f} &= \mathbf{T}^T\ \mathbf{k}\ \mathbf{T}\ \mathbf{v} \\ \mathbf{f} &= \mathbf{K}\ \mathbf{v} \end{aligned}$$

which provides the force-deflection relationships in global coordinates. The stiffness matrix in global coordinates is $\mathbf{K} = \mathbf{T}^T\ \mathbf{k}\ \mathbf{T}$

$$\mathbf{K} = \left[\begin{array}{cccccc}
\begin{array}{c} \frac{EA}{L}c^2 \\ +\frac{12EI}{L^3}s^2 \end{array} & \begin{array}{c} \frac{EA}{L}cs \\ -\frac{12EI}{L^3}cs \end{array} & -\frac{6EI}{L^2}s & \begin{array}{c} -\frac{EA}{L}c^2 \\ -\frac{12EI}{L^3}s^2 \end{array} & \begin{array}{c} -\frac{EA}{L}cs \\ +\frac{12EI}{L^3}cs \end{array} & -\frac{6EI}{L^2}s \\
 & \begin{array}{c} \frac{EA}{L}s^2 \\ +\frac{12EI}{L^3}c^2 \end{array} & \frac{6EI}{L^2}c & \begin{array}{c} -\frac{EA}{L}cs \\ +\frac{12EI}{L^3}cs \end{array} & \begin{array}{c} -\frac{EA}{L}s^2 \\ -\frac{12EI}{L^3}c^2 \end{array} & \frac{6EI}{L^2}c \\
 & & \frac{4EI}{L} & \frac{6EI}{L^2}s & -\frac{6EI}{L^2}c & \frac{2EI}{L} \\
 & & & \begin{array}{c} \frac{EA}{L}c^2 \\ +\frac{12EI}{L^3}s^2 \end{array} & \begin{array}{c} \frac{EA}{L}cs \\ -\frac{12EI}{L^3}cs \end{array} & \frac{6EI}{L^2}s \\
 & \text{SYM} & & & \begin{array}{c} \frac{EA}{L}s^2 \\ +\frac{12EI}{L^3}c^2 \end{array} & -\frac{6EI}{L^2}c \\
 & & & & & \frac{4EI}{L}
\end{array} \right]$$

Fixed-Pinned Beam, k

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & -\frac{3EI}{L^3} & 0 \\ & & \frac{3EI}{L} & 0 & -\frac{3EI}{L^2} & 0 \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{3EI}{L^3} & 0 \\ & & & & & 0 \end{bmatrix}$$

Pinned-Fixed Beam, k

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{3EI}{L^3} & -\frac{3EI}{L^2} \\ & & & & & \frac{3EI}{L} \end{bmatrix}$$

For beam element stiffness matrices including shear deformations, see:
Theory of Matrix Structural Analysis, by J.S. Przemieniecki (Dover Pub., 1985).
(... a steal at \$12.95)

Fixed-Pinned Beam in Global Coordinates, K

$$\mathbf{K} = \left[\begin{array}{cccccc}
\begin{array}{r} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{r} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{array} & -\frac{3EI}{L^2}s & \begin{array}{r} -\frac{EA}{L}c^2 \\ -\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{r} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{array} & 0 \\
 & \begin{array}{r} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{array} & \frac{3EI}{L^2}c & \begin{array}{r} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{array} & \begin{array}{r} -\frac{EA}{L}s^2 \\ -\frac{3EI}{L^3}c^2 \end{array} & 0 \\
 & & \frac{3EI}{L} & \frac{3EI}{L^2}s & -\frac{3EI}{L^2}c & 0 \\
 & & & \begin{array}{r} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{r} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{array} & 0 \\
 & \text{SYM} & & & \begin{array}{r} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{array} & 0 \\
 & & & & & 0
\end{array} \right]$$

Pinned-Fixed Beam in Global Coordinates, K

$$\mathbf{K} = \left[\begin{array}{cccccc}
\begin{array}{r} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{r} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{array} & 0 & \begin{array}{r} -\frac{EA}{L}c^2 \\ -\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{r} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{array} & -\frac{3EI}{L^2}s \\
 & \begin{array}{r} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{array} & 0 & \begin{array}{r} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{array} & \begin{array}{r} -\frac{EA}{L}s^2 \\ -\frac{3EI}{L^3}c^2 \end{array} & \frac{3EI}{L^2}c \\
 & & 0 & 0 & 0 & 0 \\
 & & & \begin{array}{r} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{array} & \begin{array}{r} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{array} & \frac{EI}{L^2}s \\
 & \text{SYM} & & & \begin{array}{r} \frac{EA}{L}s^2 \\ \frac{3EI}{L^3}c^2 \end{array} & -\frac{3EI}{L^2}c \\
 & & & & & \frac{3EI}{L}
\end{array} \right]$$

Notation

$\mathbf{u}$ = Element deflection vector in the Local coordinate system

$\mathbf{q}$ = Element force vector in the Local coordinate system

$\mathbf{k}$ = Element stiffness matrix in the Local coordinate system

... $\mathbf{q} = \mathbf{k}\ \mathbf{u}$

$\mathbf{T}$ = Coordinate Transformation Matrix

... $\mathbf{T}^{-1} = \mathbf{T}^T$

$\mathbf{v}$ = Element deflection vector in the Global coordinate system

... $\mathbf{u} = \mathbf{T}\ \mathbf{v}$

$\mathbf{f}$ = Element force vector in the Global coordinate system

... $\mathbf{q} = \mathbf{T}\ \mathbf{f}$

$\mathbf{K}$ = Element stiffness matrix in the Global coordinate system

... $\mathbf{K} = \mathbf{T}^T\ \mathbf{k}\ \mathbf{T}$

$\mathbf{d}$ = Structural deflection vector in the Global coordinate system

$\mathbf{p}$ = Structural load vector in the Global coordinate system

$\mathbf{K_s}$ = Structural stiffness matrix in the Global coordinate system

... $\mathbf{p} = \mathbf{K_s}\ \mathbf{d}$

	Local	**Global**
Element Deflection	$\mathbf{u}$	$\mathbf{v}$
Element Force	$\mathbf{q}$	$\mathbf{f}$
Element Stiffness	$\mathbf{k}$	$\mathbf{K}$
Structural Deflection	-	$\mathbf{d}$
Structural Loads	-	$\mathbf{p}$
Structural Stiffness	-	$\mathbf{K_s}$