ENERGY METHODS AND CASTIGLIANO'S THEOREMS
CE 131L. Matrix Structural Analysis
Fall, 2011

- Strain Energy: $U = \frac{1}{2}\int_V \{\sigma\}^T \ \{\epsilon\} \ dV$
- External Work: $W_{\mathrm{E}} = U = \int F \ dD$; Complementary Work: $W_{\mathrm{E}}^* = U^* = \int D \ dF$
- Superposition: $N = N_o + \sum n_i F_i$; $M = M_o + \sum m_i F_i$; $V = V_o + \sum v_i F_i$; etc.
- Castigliano's First Theorem: $F_i = \partial U \ / \ \partial D_i$
- Castigliano's Second Theorem: $D_i = \partial U^* \ / \ \partial F_i$
- Linear Elastic Systems: $U = U^*$

Mechanical Loads

Axial	$U = \frac{1}{2}\int_l \frac{N^2}{EA} \ dl = \frac{1}{2}\sum \frac{N^2 L}{EA}$	$n_i = \partial N / \partial F_i$	$D_i = \int_l \frac{N n_i}{EA} \ dl = \sum \frac{N n_i L}{EA}$
Bending	$U = \frac{1}{2}\int_l \frac{M^2}{EI} \ dl$	$m_i = \partial M / \partial F_i$	$D_i = \int_l \frac{M m_i}{EI} \ dl$
Shear	$U = \frac{1}{2}\int_l \frac{V^2}{G(A/\alpha)} \ dl$	$v_i = \partial V / \partial F_i$	$D_i = \int_l \frac{V v_i}{G(A/\alpha)} \ dl$
Torsion	$U = \frac{1}{2}\int_l \frac{T^2}{GJ} \ dl$	$t_i = \partial T / \partial F_i$	$D_i = \int_l \frac{T t_i}{GJ} \ dl$

Temperature Loads

Axial	$U = \sum N \ \alpha \Delta T L$	$\frac{\partial U}{\partial F_i} = \sum \frac{\partial N}{\partial F_i} \ \alpha \Delta T L$
Bending	$U = \int_l M \alpha \left[\frac{\Delta T_{\mathrm{b}} - \Delta T_{\mathrm{t}}}{h}\right] \ dl$	$\frac{\partial U}{\partial F_i} = \int_l \frac{\partial M}{\partial F_i} \ \alpha \left[\frac{\Delta T_{\mathrm{b}} - \Delta T_{\mathrm{t}}}{h}\right] \ dl$

Statically Indeterminate Structures and Superposition

1. Remove I redundant forces, R_i, $i = 1, \ldots, I$, where I is the degree of indeterminacy.
2. Solve for the internal forces, M_o, N_o, V_o, in the resulting statically determinate structure (without the redundant forces), due to the real applied loads.
3. Now, remove all of the real applied loads, and apply I unit loads to the structure, collocated with the redundant forces, one at a time.
4. Solve for I sets of internal forces, m_i, n_i, v_i, in each of the I different statically determinate systems.
5. Apply superposition for bending moments, axial forces, and shear forces.

$$M = M_o + \sum_{i=1}^{I} R_i m_i \qquad N = N_o + \sum_{i=1}^{I} R_i n_i \qquad V = V_o + \sum_{i=1}^{I} R_i v_i$$

6. Write I statements of Castigliano's Second Theorem, one for each virtual system, enforce compatibility with respect to support settlement and relative positions, and solve for the redundant forces, R_i.