Duke University
Department of Civil and Environmental Engineering
CE 131L. Matrix Structural Analysis
Henri P. Gavin
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Frame Element Stiffness Matrices

Beam elements carry shear forces and bending moments. Frame elements carry shear forces, bending moments, and axial forces. This document picks up with the previously-derived beam element stiffness matrices in local coordinates and proceeds through frame element stiffness matrices in global coordinates.

1 Frame Element Stiffness Matrix in Local Coordinates, $\mathbf{k}$

A frame element is the same as a beam element with the addition of a coordinate in the axial direction. The forces and displacements in the local axial direction are independent of the shear forces and bending moments.

$$\left\{\begin{array}{c} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{array}\right\} = \left\{\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array}\right\} = \left[\begin{array}{cccccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{array}\right] \left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}\right\}$$

2 Relationships between Local Coordinates and Global Coordinates: $\mathbf{T}$

The geometric relationship between local displacements, $\mathbf{u}$, and global displacements, $\mathbf{v}$, is

$$u_1 = v_1 \cos\theta + v_2 \sin\theta \qquad u_2 = -v_1 \sin\theta + v_2 \cos\theta \qquad u_3 = v_3$$

or, $\mathbf{u} = \mathbf{T}\ \mathbf{v}$.

The equilibrium relationship between local forces, $\mathbf{q}$, and global forces, $\mathbf{f}$, is

$$q_1 = f_1 \cos\theta + f_2 \sin\theta \qquad q_2 = -f_1 \sin\theta + f_2 \cos\theta \qquad q_3 = f_3$$

or, $\mathbf{q} = \mathbf{T}\ \mathbf{f}$, where, in both cases,

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{array} \right] \qquad c = \cos\theta = \frac{x_2 - x_1}{L} \qquad s = \sin\theta = \frac{y_2 - y_1}{L}$$

The coordinate transformation matrix, $\mathbf{T}$, is orthogonal, $\mathbf{T}^{-1} = \mathbf{T}^T$.

3 Frame Element Stiffness Matrix in Global Coordinates: $\mathbf{K}$

Combining the coordinate transformation relationships,

$$
\begin{aligned}
\mathbf{q} &= \mathbf{k}\ \mathbf{u} \\
\mathbf{T}\ \mathbf{f} &= \mathbf{k}\ \mathbf{T}\ \mathbf{v} \\
\mathbf{f} &= \mathbf{T}^T\ \mathbf{k}\ \mathbf{T}\ \mathbf{v} \\
\mathbf{f} &= \mathbf{K}\ \mathbf{v}
\end{aligned}
$$

which provides the force-deflection relationships in global coordinates. The stiffness matrix in global coordinates is $\mathbf{K} = \mathbf{T}^T\ \mathbf{k}\ \mathbf{T}$

$$
\mathbf{K} = \begin{bmatrix}
\frac{EA}{L}c^2 + \frac{12EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{12EI}{L^3}cs & -\frac{6EI}{L^2}s & -\frac{EA}{L}c^2 - \frac{12EI}{L^3}s^2 & -\frac{EA}{L}cs + \frac{12EI}{L^3}cs & -\frac{6EI}{L^2}s \\
 & \frac{EA}{L}s^2 + \frac{12EI}{L^3}c^2 & \frac{6EI}{L^2}c & -\frac{EA}{L}cs + \frac{12EI}{L^3}cs & -\frac{EA}{L}s^2 - \frac{12EI}{L^3}c^2 & \frac{6EI}{L^2}c \\
 & & \frac{4EI}{L} & \frac{6EI}{L^2}s & -\frac{6EI}{L^2}c & \frac{2EI}{L} \\
 & & & \frac{EA}{L}c^2 + \frac{12EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{12EI}{L^3}cs & \frac{6EI}{L^2}s \\
 & \text{SYM} & & & \frac{EA}{L}s^2 + \frac{12EI}{L^3}c^2 & -\frac{6EI}{L^2}c \\
 & & & & & \frac{4EI}{L}
\end{bmatrix}
$$

4 Fixed-Pinned Frame Element, $\mathbf{k}$

$$
\mathbf{k} = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 & \frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & -\frac{3EI}{L^3} & 0 \\
 & & \frac{3EI}{L} & 0 & -\frac{3EI}{L^2} & 0 \\
 & & & \frac{EA}{L} & 0 & 0 \\
 & \text{SYM} & & & \frac{3EI}{L^3} & 0 \\
 & & & & & 0
\end{bmatrix}
$$

5 Pinned-Fixed Frame Element, $\mathbf{k}$

$$
\mathbf{k} = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} \\
 & & 0 & 0 & 0 & 0 \\
 & & & \frac{EA}{L} & 0 & 0 \\
 & \text{SYM} & & & \frac{3EI}{L^3} & -\frac{3EI}{L^2} \\
 & & & & & \frac{3EI}{L}
\end{bmatrix}
$$

For frame element stiffness matrices including shear deformations, see: *Theory of Matrix Structural Analysis,* by J.S. Przemieniecki (Dover Pub., 1985). (... a steal at $12.95)

6 Fixed-Pinned Frame Element in Global Coordinates, $\mathbf{K}$

$$
\mathbf{K} = \begin{bmatrix}
\begin{matrix} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{matrix} & \begin{matrix} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{matrix} & -\frac{3EI}{L^2}s & \begin{matrix} -\frac{EA}{L}c^2 \\ -\frac{3EI}{L^3}s^2 \end{matrix} & \begin{matrix} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{matrix} & 0 \\
 & \begin{matrix} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{matrix} & \frac{3EI}{L^2}c & \begin{matrix} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{matrix} & \begin{matrix} -\frac{EA}{L}s^2 \\ -\frac{3EI}{L^3}c^2 \end{matrix} & 0 \\
 & & \frac{3EI}{L} & \frac{3EI}{L^2}s & -\frac{3EI}{L^2}c & 0 \\
 & & & \begin{matrix} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{matrix} & \begin{matrix} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{matrix} & 0 \\
 & \text{SYM} & & & \begin{matrix} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{matrix} & 0 \\
 & & & & & 0
\end{bmatrix}
$$

7 Pinned-Fixed Frame Element in Global Coordinates, $\mathbf{K}$

$$
\mathbf{K} = \begin{bmatrix}
\begin{matrix} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{matrix} & \begin{matrix} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{matrix} & 0 & \begin{matrix} -\frac{EA}{L}c^2 \\ -\frac{3EI}{L^3}s^2 \end{matrix} & \begin{matrix} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{matrix} & -\frac{3EI}{L^2}s \\
 & \begin{matrix} \frac{EA}{L}s^2 \\ +\frac{3EI}{L^3}c^2 \end{matrix} & 0 & \begin{matrix} -\frac{EA}{L}cs \\ +\frac{3EI}{L^3}cs \end{matrix} & \begin{matrix} -\frac{EA}{L}s^2 \\ -\frac{3EI}{L^3}c^2 \end{matrix} & \frac{3EI}{L^2}c \\
 & & 0 & 0 & 0 & 0 \\
 & & & \begin{matrix} \frac{EA}{L}c^2 \\ +\frac{3EI}{L^3}s^2 \end{matrix} & \begin{matrix} \frac{EA}{L}cs \\ -\frac{3EI}{L^3}cs \end{matrix} & \frac{EI}{L^2}s \\
 & \text{SYM} & & & \begin{matrix} \frac{EA}{L}s^2 \\ \frac{3EI}{L^3}c^2 \end{matrix} & -\frac{3EI}{L^2}c \\
 & & & & & \frac{3EI}{L}
\end{bmatrix}
$$

8 Notation

$\mathbf{u}$ = Element deflection vector in the Local coordinate system

$\mathbf{q}$ = Element force vector in the Local coordinate system

$\mathbf{k}$ = Element stiffness matrix in the Local coordinate system

... $\mathbf{q} = \mathbf{k}\ \mathbf{u}$

$\mathbf{T}$ = Coordinate Transformation Matrix

... $\mathbf{T}^{-1} = \mathbf{T}^T$

$\mathbf{v}$ = Element deflection vector in the Global coordinate system

... $\mathbf{u} = \mathbf{T}\ \mathbf{v}$

$\mathbf{f}$ = Element force vector in the Global coordinate system

... $\mathbf{q} = \mathbf{T}\ \mathbf{f}$

$\mathbf{K}$ = Element stiffness matrix in the Global coordinate system

... $\mathbf{K} = \mathbf{T}^T\ \mathbf{k}\ \mathbf{T}$

$\mathbf{d}$ = Structural deflection vector in the Global coordinate system

$\mathbf{p}$ = Structural load vector in the Global coordinate system

$\mathbf{K_s}$ = Structural stiffness matrix in the Global coordinate system

... $\mathbf{p} = \mathbf{K_s}\ \mathbf{d}$

	Local	**Global**
Element Deflection	$\mathbf{u}$	$\mathbf{v}$
Element Force	$\mathbf{q}$	$\mathbf{f}$
Element Stiffness	$\mathbf{k}$	$\mathbf{K}$
Structural Deflection	-	$\mathbf{d}$
Structural Loads	-	$\mathbf{p}$
Structural Stiffness	-	$\mathbf{K_s}$