# LARGE DEFORMATION ANALYSIS
# OF PLANAR FRAMES
## CE 131 — Matrix Structural Analysis
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## Fall, 2004

In situations in which deformations are not *infinitissimally*[1] small, linear elastic analyses may not capture the true structural response. In such cases, *finite*[2] deformation analysis is more accurate than linear, infinitessimal deformation analysis. Incorporating the details of finite deformation, the analysis may also be applied to a buckling analysis of the structural system.

In the derivation of the linear elastic stiffness matrix for frame elements, the potential energy function includes strain energy due to bending, axial and shear deformation effects. Axial effects are decoupled from shear and bending effects in the resulting linear elastic stiffness matrices.[3] In finite deformation analysis, on the other hand, the potential energy function includes additional terms, which accounts for the interaction between the axial load effects on the frame element and the lateral deformation of the frame element. These effects are often called "$P - \Delta$ effects".

We will separate the potential energy function $U$ into an elastic part $U_{\mathrm{E}}$ (which contains the infinitessimal strain energy) and a geometric part, $U_{\mathrm{G}}$ (which includes the interaction of lateral deformations and axial loads). The linear elastic strain energy results in the same frame element stiffness matrices $\mathbf{k}_{\mathrm{E}}$ that we have found previously. So, this document focuses only on the geometric component of the potential energy function. From this geometric part of the potential energy, we will derive the geometric stiffness matrix $\mathbf{k}_{\mathrm{G}}$.

As in the finite deformation analysis of trusses, we need to know the deformation of the structure in order to find the internal axial loads, but we need to know the internal axial loads to determine the geometric stiffness matrix and the deformations. This "chicken-and-egg" problem can be solved with the same type of Newton-Raphson iteration approach which we used previously for trusses.

To start with, we need to introduce the deformed shape of a beam. The deformed shape of a beam, $h(x)$, subjected to end-forces, $\mathbf{q}$, is a cubic polynomial. A cubic polynomial may be written in a power-polynomial form as follows:

$$h(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3. \tag{1}$$

Likewise, the slope of the beam, $h'(x)$ may be expressed

$$h'(x) = a_1 + 2a_2 \cdot x + 3a_3 \cdot x^2. \tag{2}$$

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[1] **infinitessimal** means arbitrarily close to zero, as in infinitessimal calculus.

[2] **finite** means neither infinite nor infinitessimal, as in a finite distance.

[3] The rows and columns corresponding to axial effects (1st and 4th) have non-zero elements only in the 1st and 4th columns and rows. Also, the rows and columns corresponding to bending and shear effects have non-zero elements only in the 2nd, 3rd, 5th, and 6th columns and rows.

Figure 1: The deformed shape of a beam, $h(x)$ is assumed to be a cubic function of $x$.

The polynomial coefficients, $a_0, \cdots, a_3$, satisfy the end displacements and rotations of the beam. Assuming small rotations, $\tan\theta \approx \theta$, and neglecting shear deformation effects,

$$
\begin{aligned}
h(0) &= u_2 & &\rightarrow & & a_0 = u_2 \\
h'(0) &= u_3 & &\rightarrow & & a_1 = u_3 \\
h(L) &= u_5 & &\rightarrow & & a_0 + a_1 L + a_2 L^2 + a_3 L^3 = u_5 \\
h'(L) &= u_6 & &\rightarrow & & a_1 + 2a_2 L + 3a_3 L^2 = u_6
\end{aligned}
$$

These four equations with four unknowns $(a_0, \cdots, a_3)$ have the solution

$$\begin{aligned} a_0 &= u_2 && (3) \\ a_1 &= u_3 && (4) \\ a_2 &= 3(u_5 - u_2)/L^2 - (2u_3 + u_6)/L && (5) \\ a_3 &= -2(u_5 - u_2)/L^3 + (u_3 + u_6)/L^2. && (6) \end{aligned}$$

You should be able to confirm this solution for the polynomial coefficients. Note that the cubic deformation function $h(x)$ may also be written in another way:

$$h(x) = u_2 \cdot b_2(x) + u_3 \cdot b_3(x) + u_5 \cdot b_5(x) + u_6 \cdot b_6(x), \tag{7}$$

where the functions $b_i(x)$ are all cubic functions in $x$. If shear deformation effects are neglected, these cubic shape functions are:

$$\begin{aligned} b_2(x) &= 1 - 3\,(x/L)^2 + 2\,(x/L)^3 \\ b_3(x) &= x\,(1 - x/L)^2 \\ b_5(x) &= 3\,(x/L)^2 - 2\,(x/L)^3 \\ b_3(x) &= (x/L)^2\,(x/L - 1) \end{aligned}$$

Equations (1) and (7) are two different ways of expressing *exactly* the same equation, $h(x)$. The finite element method makes use of the form of equation (7). To complete the picture, for axial deformations, (which contribute to transverse deformations only through the axial load in the geometric stiffness matrix),

$$\begin{aligned} b_1(x) &= (1 - x/L) \\ b_4(x) &= (x/L)\,. \end{aligned}$$

You should confirm that with the given definitions of $b_i(x)$, and the coefficients $a_i$, that equations (1) and (7) are equivalent.

Now, to introduce how this assumed deformation shape function can be used to find a potential energy function, let's recall the internal elastic strain energy of a beam due to bending effects,

$$U_{\mathrm{B}} = \frac{1}{2}\int_0^L \frac{M^2(x)}{EI}\,dx. \tag{8}$$

Now, since the curvature of the beam is $M(x)/(EI)$ and assuming infinitessimal deformation, $h''(x)$ is practically the same as the curvature, and

$$\begin{aligned} U_{\mathrm{B}} &= \frac{1}{2}\int_0^L M(x)h''(x)\,dx && (9) \\ &= \frac{1}{2}\,EI\int_0^L h''(x)\cdot h''(x)\,dx\,. && (10) \end{aligned}$$

Equation (8) is an expression of the strain energy in terms of the internal bending moment; equation (9) is an expression of the strain energy in terms of the internal bending moment and the assumed cubic deformation function; and equation (10) is an expression of the strain energy in terms of the assumed cubic deformation function only. In the finite element method it is common to express the potential energy using forms like equation (10). If the internal bending moment $M(x)$ does indeed generate the assumed deformation function $h(x)$ then all three forms of the elastic strain energy are exactly equivalent and completely interchangeable.

Turning now to our problem of determining the potential energy associated with axial loads and transverse displacements, recall the elastic strain energy due to axial loads:

$$U_{\mathrm{A}} = \frac{1}{2}\int_0^L \frac{N^2(x)}{EA}\, dx. \tag{11}$$

Since the incremental displacement $du$ within a segment of length $dx$ is

$$du = \frac{N(x)}{EA} dx,$$

the internal strain energy due to axial effects may be written

$$U_{\mathrm{A}} = \frac{1}{2}\int_0^L N(x)\, du. \tag{12}$$

In this example, $du$ is the elastic extension of the beam due to axial loads. If the beam element also has transverse displacements, $h(x)$, the beam will also shorten in the direction of the axial load. With the usual small-angle approximation, $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$, Figure 2 shows that

Figure 2: Shortening effects due to transverse deflections, and ignoring axial deformation.

$$du = \frac{dh}{dx} \cdot dh = \frac{dh}{dx} \cdot \frac{dh}{dx} \cdot dx \tag{13}$$

So, the potential energy due to axial loads, $N(x)$ and transverse displacements, $h(x)$, is

$$U_{\mathrm{G}} = \frac{1}{2}\int_0^L N(x)\ \frac{dh}{dx}\ \frac{dh}{dx}\ dx\ . \tag{14}$$

If the axial load $N(x)$ is constant over the length of the beam then the tensile force is $T = N(x) = \text{const.}$, and

$$U_{\mathrm{G}} = \frac{1}{2}\ T\int_0^L h'(x)\ h'(x)\ dx\ . \tag{15}$$

Substituting, and carrying out the integral leads to the potential energy function in terms of transverse end displacements, $u_2$ and $u_5$, and end rotations, $u_3$ and $u_6$,

$$\begin{aligned} U_{\mathrm{G}} = \tfrac{T}{30L}\ & (-Lu_3u_6 - 3u_5Lu_3 - 3u_5Lu_6 + 3u_2Lu_3 + 3u_2Lu_6 \\ & +18u_5^2 - 36u_5u_2 + 18u_2^2 + 2L^2u_3^2 + 2L^2u_6^2)\ . \end{aligned} \tag{16}$$

Invoking Castigliano's theorem, the partial derivative of the potential energy function with respect to a displacement coordinate is the force in the direction of that displacement coordinate. The end forces due to geometric stiffness effects can then be found as follows:

$$\begin{aligned} q_2 &= \frac{\partial U_{\mathrm{G}}}{\partial u_2} = \frac{T}{30L}(36u_2 + 3Lu_3 - 36u_5 + 3Lu_6) \\ q_3 &= \frac{\partial U_{\mathrm{G}}}{\partial u_3} = \frac{T}{30L}(3Lu_2 + 4L^2u_3 - 3Lu_5 - L^2u_6) \\ q_5 &= \frac{\partial U_{\mathrm{G}}}{\partial u_5} = \frac{T}{30L}(-36u_2 - 3Lu_3 + 36u_5 - 3Lu_6) \\ q_6 &= \frac{\partial U_{\mathrm{G}}}{\partial u_6} = \frac{T}{30L}(3Lu_2 - L^2u_3 - 3Lu_5 + 4L^2u_6). \end{aligned}$$

Writing these expressions in matrix form, we arrive at the geometric stiffness matrix for a frame element:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \frac{T}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & -\frac{2L^2}{15} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \tag{17}$$

where the tension in the beam is given by $T = EA(u_4 - u_1)/L$. The geometric stiffness matrix for a planar frame element in local coordinates is:

$$\mathbf{k}_{\mathrm{G}} = \frac{T}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & \frac{-6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2L^2}{15} & 0 & \frac{-L}{10} & \frac{-L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6}{5} & \frac{-L}{10} & 0 & \frac{6}{5} & \frac{-L}{10} \\ 0 & \frac{L}{10} & \frac{-L^2}{30} & 0 & \frac{-L}{10} & \frac{2L^2}{15} \end{bmatrix}\ . \tag{18}$$

The coordinate transformation process is identical to the process carried out before for the elastic element stiffness matrix. The coordinate transformation matrix, $\mathbf{T}$, is

$$\mathbf{T} = \left[ \begin{array}{cccccc} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] , \tag{19}$$

where $s$ and $c$ are the sine and cosine of the counter-clockwise angle from global element coordinate number 1 to the frame element. Here we are making the approximation that the deformed inclination of the frame element is approximately the same as the original inclination of the frame element. The element stiffness matrix in global coordinates is found by applying the coordinate transformation matrix.

$$\mathbf{K}_\mathrm{G} = \mathbf{T}^T \; \mathbf{k}_\mathrm{G} \; \mathbf{T} = \frac{T}{L} \left[ \begin{array}{cccccc} \frac{6}{5}s^2 & \frac{-6}{5}sc & \frac{-L}{10}s & \frac{-6}{5}s^2 & \frac{6}{5}sc & \frac{-L}{10}s \\ \frac{-6}{5}sc & \frac{6}{5}c^2 & \frac{L}{10}c & \frac{6}{5}sc & \frac{-6}{5}c^2 & \frac{L}{10}c \\ \frac{-L}{10}s & \frac{L}{10}c & \frac{2L^2}{15} & \frac{L}{10}s & \frac{-L}{10}c & \frac{-L^2}{30} \\ \frac{-6}{5}s^2 & \frac{6}{5}sc & \frac{L}{10}s & \frac{6}{5}s^2 & \frac{-6}{5}sc & \frac{L}{10}s \\ \frac{6}{5}sc & \frac{-6}{5}c^2 & \frac{-L}{10}c & \frac{-6}{5}sc & \frac{6}{5}c^2 & \frac{-L}{10}c \\ \frac{-L}{10}s & \frac{L}{10}c & \frac{-L^2}{30} & \frac{L}{10}s & \frac{-L}{10}c & \frac{2L^2}{15} \end{array} \right] \tag{20}$$

It is not hard to confirm this expression for $\mathbf{K}_\mathrm{G}$, and you should feel encouraged to do so. The assembly of the structural stiffness matrix $\mathbf{K_s}$ with elastic and geometric effects proceeds exactly as with the elastic stiffness matrix.

Note that the $i, j$ component of the stiffness matrix is

$$k_{ij} = \frac{\partial}{\partial u_j} q_i = k_{ji} = \frac{\partial}{\partial u_i} q_j \; ,$$

and that the $i^{\text{th}}$ component of the end force, $q_i$, is

$$q_i = \frac{\partial}{\partial u_i} U \; .$$

Therefore, the stiffness coefficients may be written

$$k_{ij} = \frac{\partial^2 \, U}{\partial u_i \, \partial u_j} \; .$$

If the stiffness matrix to be determined is for bending effects only, then, as seen before,

$$U = U_\mathrm{B} = \frac{1}{2} \, EI \int_0^L h''(x) \cdot h''(x) \; dx \; .$$

Now, since integration and differentiation are both linear operations, it does not matter which is done first, integration or differentiation. Therefore, the stiffness coefficient may be written,

$$k_{ij} = \frac{1}{2} \, EI \int_0^L \frac{\partial h''(x)}{\partial u_i} \cdot \frac{\partial h''(x)}{\partial u_j} \; dx \; .$$

The elastic stiffness matrix incorporating bending effects only may be determined directly from this expression. Likewise, the geometric stiffness matrix may be determined directly from

$$k_{\mathrm{G}_{ij}} = \frac{1}{2}\, T \int_0^L \frac{\partial h'(x)}{\partial u_i} \cdot \frac{\partial h'(x)}{\partial u_j} \; dx \; .$$

The three-dimensional elastic stiffness matrix for frame elements in local coordinates including bending and shear deformation effects:

$$
\mathbf{k}_{\mathrm{E}} = \left[\begin{array}{cccccccccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} \\
\frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & \frac{12EI_z}{L^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} \\
0 & 0 & \frac{-12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_y}{L^3(1+\Phi_z)} & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 \\
0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{L(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{L^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{L(1+\Phi_z)} & 0 \\
0 & \frac{6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} & 0 & \frac{-6EI_z}{L^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}
\end{array}\right],
$$

where

$$
\Phi_y = \frac{12EI_z}{GA_{sy}L^2}\ , \qquad \text{and} \qquad \Phi_z = \frac{12EI_y}{GA_{sz}L^2}\ .
$$

## Cubic shape functions for beams including shear deformations

The transverse displacements in the local $x-y$ plane, $h_y(x)$, of an elastic beam may be separated into a shear-related component, $h_s(x)$ and a bending-related component $h_b(x)$,

$$h_y(x) = h_s(x) + h_b(x). \tag{21}$$

Considering the elastic stiffness matrix for a beam incluing shear deformations, given on page 8, the shear force at the end of the beam in the local $y$ direction, $q_2$ may be found in terms of the beam end-displacements,

$$q_2 = k_{22}\ u_2 + k_{26}\ u_6 + k_{28}\ u_8 + k_{2\ 12}\ u_{12}\ .$$

The shear strain is simply

$$h'_s(x) = -\frac{q_2}{GA_{sy}} = -\frac{1}{GA_{sy}}(k_{22}\ u_2 + k_{26}\ u_6 + k_{28}\ u_8 + k_{2\ 12}\ u_{12})\ . \tag{22}$$

The internal bending moment, $M_z(x)$, due to the effects of the end displacements and end rotations is simply

$$M_z(x) = q_2\ x - q_6,$$

and the curvature is approximately

$$\begin{aligned} h''_b(x) \ &= \tfrac{1}{EI_z} \ [\ (k_{22}\ u_2 + k_{26}\ u_6 + k_{28}\ u_8 + k_{2\ 12}\ u_{12})\ x \\ &\qquad -(k_{62}\ u_2 + k_{66}\ u_6 + k_{68}\ u_8 + k_{6\ 12}\ u_{12})\ ]. \end{aligned} \tag{23}$$

in which small angles are assumed. By computing the potential energy function for shear and bending deformations,

$$U = \frac{1}{2}EI_z\int_0^L (h''_b(x))^2\ dx + \frac{1}{2}GA_{sy}\int_0^L (h'_s(x))^2\ dx,$$

and taking the partial derivatives of the potential energy function with respect to the displacements coordinates $u_2, u_6, u_8$ and $u_{12}$, rows 2, 6, 8, and 12 of the elastic stiffness matrix may be recovered.

The shape of the deformed beam may therefore by found by integrating equations (22) and (23) to obtain $h_s(x)$ and $h_b(x)$ and by solving for the constants of integration using the end conditions. So doing,

$$h'_y(x) = h'_s(x) + h'_b(x) = -\frac{1}{GA_{sy}}q_2 + \frac{1}{EI_z}\left(\frac{1}{2}q_2x^2 - q_6x + C_1\right).$$

Inserting the end condition $h'_y(0) = u_6$ the constant of integration, $C_1$, is $EI_zu_6$. Integrating again,

$$h_y(x) = h_s(x) + h_b(x) = -\frac{1}{GA_{sy}}q_2x + \frac{1}{EI_z}\left(\frac{1}{6}q_2x^3 - \frac{1}{2}q_6x^2\right) + u_6x + C_2.$$

Inserting the boundary condition $h(0) = u_2$, and noting that $-q_2x/(GA_{sy}) = -\Phi_yL^2q_2x/(12EI_z)$, the deformed shape of the beam becomes

$$h_y(x) = \frac{1}{EI_z}\left(\frac{1}{6}q_2\ x^3 - \frac{1}{2}q_6\ x^2 - \frac{1}{12}\Phi_yL^2q_2\ x\right) + u_6\ x + u_2\ .$$

Substituting expressions for $q_2$ and $q_6$ from the elastic stiffness matrix on page 8, one obtains the transverse deformation shape function for a beam with bending and shear deformation in the $x-y$ plane.

$$\begin{aligned} h_y(x) \;&= \tfrac{1}{L^3}\tfrac{1}{1+\Phi_y}\; \{ \;\; [2x^3 - 3Lx^2 - \Phi_y L^2 x + L^3(1+\Phi_y)] \; u_2 + \\ &\qquad [Lx^3 - L^2(2+\Phi_y/2)x^2 + L^3(1+\Phi_y/2)x] \; u_6 + \\ &\qquad [-2x^3 + 3Lx^2 + \Phi_y L^2 x] \; u_8 + \\ &\qquad [Lx^3 - L^2(1-\Phi_y/2)x^2 - \Phi_y L^3 x/2] \; u_{12} \;\; \} \; . \end{aligned} \tag{24}$$

For bending and shear deformations in the $x-z$ plane, the shape function may be found using an analous method, while respecting the right-hand coordinate system. The transverse deflection, $h_z(x)$ will consist of shear and bending components,

$$h_z(x) = h_s(x) + h_b(x) \; .$$

The end-shear force and the end-bending moment are provided by the elastic stiffness matrix,

$$q_3 = k_{33} \; u_3 + k_{35} \; u_5 + k_{39} \; u_9 + k_{3\;11} \; u_{11} \; ,$$

and

$$q_5 = k_{53} \; u_3 + k_{55} \; u_5 + k_{59} \; u_9 + k_{5\;11} \; u_{11} \; .$$

The shear strain is again

$$h_s'(x) = -\frac{q_3}{GA_{sz}} = -\frac{1}{GA_{sz}}( \; k_{33} \; u_3 + k_{35} \; u_5 + k_{39} \; u_9 + k_{3\;11} \; u_{11} \; ),$$

and the bending curvature is approximately

$$\begin{aligned} h_b''(x) \;&= \tfrac{1}{EI_y} \;\; (q_3 \; x + q_5) \\ &= \tfrac{1}{EI_y} \;\; [ \; (k_{33} \; u_3 + k_{35} \; u_5 + k_{39} \; u_9 + k_{3\;11} \; u_{11}) \; x \\ &\qquad +(k_{53} \; u_3 + k_{55} \; u_5 + k_{59} \; u_9 + k_{5\;11} \; u_{11}) \; ], \end{aligned}$$

where small angles are again assumed. Integrating $h_b''(x)$ and combining with $h_s'(x)$,

$$h_z'(x) = -\frac{1}{GA_{sz}} q_3 + \frac{1}{EI_y}\left(\frac{1}{2} q_3 \; x^2 + q_5 \; x + C_1\right) \; .$$

Inserting the end condition, $h_z'(0) = -u_5$ gives $C_1 = -EI_y u_5$. Integrating again,

$$h_z(x) = -\frac{1}{GA_{sz}} q_3 \; x + \frac{1}{EI_y}\left(\frac{1}{6} q_3 \; x^3 + \frac{1}{2} q_5 \; x^2\right) - u_5 \; x + C_2 \; .$$

Now inserting the end condition $h_z(0) = u_3$ gives $C_2 = u_3$ and noting that $-q_3/(GA_{sz}) = -\Phi_z L^2 q_3/(12EI_y)$ the deformed shape may be written

$$h_z(x) = \frac{1}{EI_y}\left(\frac{1}{6} q_3 \; x^3 + \frac{1}{2} q_5 \; x^2 - \frac{1}{12}\Phi_z L^2 q_3 \; x\right) - u_5 \; x + u_3 \; .$$

Finally, using the coefficients from the local stiffness matrix, the shape function for a frame element bending and shear in the local $x - z$ plane is

$$\begin{aligned} h_z(x) \;=\; \tfrac{1}{L^3}\tfrac{1}{1+\Phi_z} \; \{ \; & [2x^3 - 3Lx^2 - \Phi_z L^2 x + L^3(1+\Phi_z)] \; u_3 + \\ & [-Lx^3 + L^2(2+\Phi_z/2)x^2 - L^3(1+\Phi_z/2)x] \; u_5 + \\ & [-2x^3 + 3Lx^2 + \Phi_z L^2 x] \; u_9 + \\ & [-Lx^3 + L^2(1-\Phi_z/2)x^2 + \Phi_z L^3 x/2] \; u_{11} \; \} \; . \end{aligned} \tag{25}$$

Note that this expression is equivalent to equation (24) except for the fact that $\Phi_z$ replaces $\Phi_y$ and that the signs of the $u_5$ and $u_6$ shape functions are reversed, as are the signs of the $u_{11}$ and $u_{12}$ shape functions. This is consistent with the right-hand coordinate system.

For axial displacements, the shape function is the same as for a truss,

$$h_x(x) = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_7 \; . \tag{26}$$

Likewise, for torsional displacements, the shape function is analogous to the axial displacement shape function

$$h_{\theta x}(x) = \left(1 - \frac{x}{L}\right) u_4 + \frac{x}{L} u_{10} \; . \tag{27}$$

For axial displacements, the shape functions are the same as for a truss,

### Formulation of the geometric stiffness matrix from the cubic shape functions

For the elements of the geometric stiffness matrix in rows 2,6,8, and 12, the potential energy function is

$$U_{\mathrm{G}y} = \frac{1}{2}\, T \int_0^L h_y'(x)\; h_y'(x)] \; dx \; ,$$

where $h_y(x)$ is given by equation (24). Likewise, for rows 3,5,9, and 11, the potential energy function is

$$U_{\mathrm{G}z} = \frac{1}{2}\, T \int_0^L h_z'(x)\; h_z'(x) \; dx \; ,$$

where $h_z(x)$ is given by equation (25). For rows 1 and 7, the potential energy function is

$$U_{\mathrm{G}x} = \frac{1}{2}\, T \int_0^L h_x'(x)\; h_x'(x) \; dx \; ,$$

where $h_x(x)$ is given by equation (26). In the torsion of non-circular sections, torsional displacements result in axial deformation (warping) of the cross-section. In such cases, the work of the axial tension, $T$, moving through the warping displacements provides the potential energy function for rows 4 and 10,

$$U_{\mathrm{G}\theta} = \frac{1}{2}\, T \frac{I_x}{A_x} \int_0^L h_{\theta x}'(x)\; h_{\theta x}'(x) \; dx \; ,$$

where $h_{\theta x}(x)$ is given by equation (27).

The geometric stiffness coefficients may then be found by forming the Hessian of the appropriate potential energy function,

$$k_{ij} = \frac{\partial^2 \, U_{\mathrm{G}}}{\partial u_i \, \partial u_j}.$$

The three-dimensional geometric stiffness matrix for frame elements in local coordinates including axial, bending, shear and torsional warping effects is:

$$\mathbf{k}_{\mathrm{G}} \;\; = \frac{T}{L} \left[ \begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{6/5+2\Phi_y+{\Phi_y}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{6/5+2\Phi_z+{\Phi_z}^2}{(1+\Phi_z)^2} & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & \frac{I_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 & \frac{2L^2/15+L^2\Phi_z/6+L^2{\Phi_z}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{2L^2/15+L^2\Phi_y/6+L^2{\Phi_y}^2/12}{(1+\Phi_y)^2} \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-6/5-2\Phi_y-{\Phi_y}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{-6/5-2\Phi_z-{\Phi_z}^2}{(1+\Phi_z)^2} & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & -\frac{I_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 & \frac{-L^2/30-L^2\Phi_z/6-L^2{\Phi_z}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L^2/30-L^2\Phi_y/6-L^2{\Phi_y}^2/12}{(1+\Phi_y)^2}
\end{array} \right.$$

$$\left. \begin{array}{cccccc}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-6/5-2\Phi_y-{\Phi_y}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{-6/5-2\Phi_z-{\Phi_z}^2}{(1+\Phi_z)^2} & 0 & \frac{-L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & -\frac{I_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 & \frac{-L^2/30-L^2\Phi_z/6-L^2{\Phi_z}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{-L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L^2/30-L^2\Phi_y/6-L^2{\Phi_y}^2/12}{(1+\Phi_y)^2} \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{6/5+2\Phi_y+{\Phi_y}^2}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{-L/10}{(1+\Phi_y)^2} \\
0 & 0 & \frac{6/5+2\Phi_z+{\Phi_z}^2}{(1+\Phi_z)^2} & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 \\
0 & 0 & 0 & \frac{I_x}{A_x} & 0 & 0 \\
0 & 0 & \frac{L/10}{(1+\Phi_z)^2} & 0 & \frac{2L^2/15+L^2\Phi_z/6+L^2{\Phi_z}^2/12}{(1+\Phi_z)^2} & 0 \\
0 & \frac{-L/10}{(1+\Phi_y)^2} & 0 & 0 & 0 & \frac{2L^2/15+L^2\Phi_y/6+L^2{\Phi_y}^2/12}{(1+\Phi_y)^2}
\end{array} \right] ,$$

where

$$T = EA(u_7 - u_1)/L \ ,$$

$$\Phi_y = \frac{12EI_z}{GA_{sy}L^2} \qquad \text{and} \qquad \Phi_z = \frac{12EI_y}{GA_{sz}L^2} \ .$$