MATHEMATICAL PROPERTIES OF STIFFNESS MATRICES
CE 131 — Matrix Structural Analysis
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This handout gives an explanation of some of the mathematical properties of stiffness matrices, in the context of structural behavior.

The stiffness matrix, $[K]$, relates the forces, $\{f\}$, applied at a set of coordinates on a structure to the displacements, $\{v\}$, at the same set of coordinates.

$$[K]\{v\} = \{f\} \tag{1}$$

Coordinates are defined by the locations and directions of the forces, $\{f\}$, and displacements, $\{v\}$. Let's consider a structure with two coordinates:

The stiffness matrix for the two coordinates of this structure can be written,

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2}$$

This stiffness matrix represents a set of two equations with two unknowns.

$$K_{11}v_1 + K_{12}v_2 = f_1 \tag{3}$$
$$K_{21}v_1 + K_{22}v_2 = f_2 \tag{4}$$

The stiffness matrix is *symmetric*, $[K] = [K]^T$, or,

$$K_{ij} = K_{ji} \tag{5}$$

This is a statement of Maxwell's Reciprocity Theorem, which has been described previously. In short, Maxwell's Reciprocity Theorem says that the deflection v_i at coordinate i due to a unit force f_j at coordinate j, is equal to the deflection v_j at coordinate j due to a unit force f_i at coordinate i.

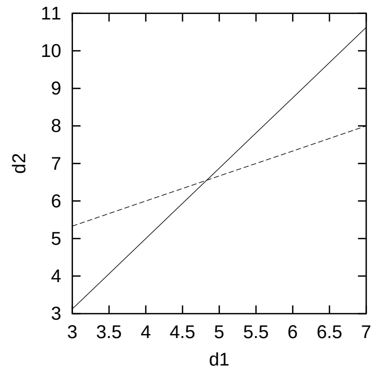
Solving equations (3) and (4) for v_2 in terms of v_1 gives the two equations

$$v_2 = -(K_{11}/K_{12})v_1 + (1/K_{12})f_1 \tag{6}$$
$$v_2 = -(K_{21}/K_{22})v_1 + (1/K_{22})f_2 \tag{7}$$

Given numerical values for $\{f\}$ and $[K]$, these two equations may be plotted. The intersection of these two lines is the solution, $\{v\} = [K]^{-1}\{f\}$. For example,

$$[K] = \left[\begin{array}{rr} 1.5 & -0.8 \\ -0.8 & 1.2 \end{array} \right]$$

$$\{f\} = \left\{ \begin{array}{c} 2 \\ 4 \end{array} \right\}$$



$$\{v\} = \left\{ \begin{array}{c} 4.8276 \\ 6.5517 \end{array} \right\}$$

Note that if $(K_{11}/K_{12}) = (K_{21}/K_{22})$, then the lines are parallel. There is no unique point at which they cross, and there is no solution $\{v\} = [K]^{-1}\{f\}$. In other words if $(K_{11}/K_{12}) = (K_{21}/K_{22})$, then the matrix $[K]$ can not be inverted. The following statements are equivalent:

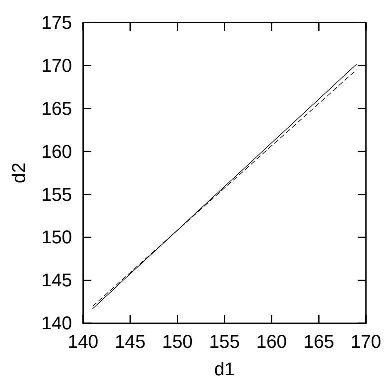
$$K_{11}/K_{12} = K_{21}/K_{22} \quad \Leftrightarrow \quad K_{11}K_{22} - K_{12}K_{21} = 0 \quad \Leftrightarrow \quad \det([K]) = 0. \tag{8}$$

A stiffness matrix $[K]$ is called *ill-conditioned* or *nearly singular* if its determinant, $\det([K])$, is "close to zero". In computer applications, "close to zero" means close to the smallest number a computer can represent.

One way in which this can happen in our 2-by-2 example, is if the diagonal elements of $[K]$ are equal to one another and the off diagonal elements are the negatives of the diagonal elements. Consider an ill-conditioned 2-by-2 stiffness matrix. The v_2 vs. v_1 lines for the deflections are almost parallel.

$$[K] = \left[\begin{array}{rr} 1.22 & -1.2 \\ -1.2 & 1.22 \end{array} \right]$$

$$\{f\} = \left\{ \begin{array}{c} 2 \\ 4 \end{array} \right\}$$



$$\{v\} = \left\{ \begin{array}{c} 149.59 \\ 150.41 \end{array} \right\}$$

In contrast to the previous example it's hard to tell exactly where these two lines cross. There are large ranges for the values of v_1 and v_2 that are "close" to the solution. This is why numerical computation with poorly conditioned matrices can lose precision.

How can this arise in the stiffness matrix of a structural system? Consider the three-spring system shown below ($K \gg k$) :

The stiffness matrix for this system is

$$\begin{bmatrix} K+k & -K \\ -K & K+k \end{bmatrix}, \tag{9}$$

which is very close to

$$\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}. \tag{10}$$

If $K \gg k$ the determinant of this stiffness matrix is close to zero. How big does K/k need to be before the solution loses accuracy? It depends on the desired precision of the computation.

	Single Precision	Double Precision
significant figures	8	15
K/k	10^n	10^n
remaining accuracy	$8-n$	$15-n$

As a rule of thumb, we would like to have at least 4 or 5 significant figures of accuracy in our results, so (assuming single precision) if we want to make an element "very stiff" as compared to the other elements, we should make it no stiffer than 1000 times the stiffness of the other elements.

Eigen-values

The mathematical meaning of the *eigen-values* and *eigen-vectors* of a symmetric stiffness matrix $[K]$ can be interpreted geometrically.

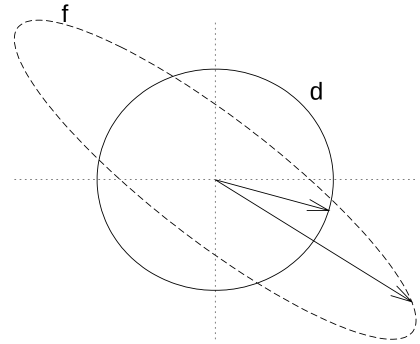
Consider a set of displacement vectors consistent with the reactions (or constraints) of the structure. For a given stiffness matrix, the equation $\{f\} = [K]\{v\}$ will produce a corresponding set of force vectors (in equilibrium). For each displacement vector, there is one, and only one, force vector. Let's suppose that our set of displacement vectors is "spheroidal." In other words, for each displacement vector $\{v\}$ in our set,

$$v_1^2 + v_2^2 + v_3^2 + \cdots + v_N^2 = 1. \tag{11}$$

If $N = 2$ then the tips of the $\{v\}$ vectors trace out a unit circle. A "spheroidal" set of $\{v\}$ vectors produces an "ellipsoidal" set of $\{f\}$ vectors. This ellipsoid has principle axes with principle lengths. If $N = 2$ then the tips of the $\{f\}$ vectors trace out an ellipse. For example,

$$[K] = \begin{bmatrix} 1.5 & -0.8 \\ -0.8 & 1.2 \end{bmatrix}$$

$\lambda_1 = 0.536$ $\lambda_2 = 2.164$



Now let's look at the eigen-value equation along with the force-displacement equation. The eigen-value equation is $[K]\{x\}_i = \lambda_i\{x\}_i$, where λ_i is the i^{th} eigen-value and $\{x\}_i$ is the i^{th} eigen-vector of $[K]$.

$$[K]\,\{x\} = \lambda\{x\} \tag{12}$$
$$[K]\,\{v\} = \{f\} \tag{13}$$

The vectors $\{x\}$ and $\lambda\{x\}$ point in the same direction (because λ is a scalar). Therefore, if $\{v\}$ points in the same direction as $\{f\}$, then $\{v\}$ is an eigen-vector of $[K]$. All of the eigen-values, λ_i, of a symmetric matrix are real numbers. The principle axis directions of the force-ellipsoid are the eigen-vectors of $[K]$, and the lengths of these principle axes are the eigen-values of $[K]$. These principle directions are perpendicular (*orthogonal*) to one another. In the example above, if the radius of the circle '*d*' is 1, then the principle axes of the ellipse '*f*' have lengths equal to λ_1 and λ_2. One way to compute the eigen-values is to find the roots of the *characteristic polynomial* of $[K]$. The characteristic polynomial is defined by

$$\det([K] - \lambda\,[I]_N) = 0, \tag{14}$$

where $[I]_N$ is the N-by-N identity matrix.

If a structure is *stable* (internally and externally), then its stiffness matrix is invertible. Otherwise, the structure is free to move or deflect without deforming. If a structure is free to move in this way, then applied forces can produce infinite or undetermined displacements.

As we saw earlier, a structure has an invertible stiffness matrix if and only if $\det([K]) \neq 0$. The determinant of a matrix is the product of its eigenvalues,

$$\det([K]) = (\lambda_1)(\lambda_2)\cdots(\lambda_N), \tag{15}$$

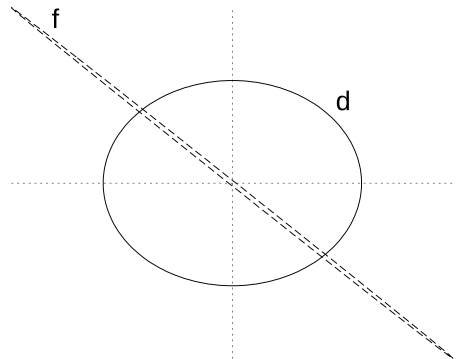
therefore, no eigen-value of an invertible matrix can be zero. All of the eigen-values of a *positive definite matrix* are positive numbers. Hence, the determinant of a positive definite stiffness matrix is also a positive number and a positive definite stiffness matrix is invertible.

If a structure is *not stable* (internally or externally), then it's stiffness matrix will have one or more eigen-value equal to zero. In this case, the stiffness matrix is said to be *singular.*

We can use the geometric illustration above to interpret what a zero eigen-value means. If a stiffness matrix $[K]$ has one eigen-value equal to zero, then the force ellipsoid will have a principle axis of zero length; it will have one less dimension than the displacement spheroid. In other words, for $N = 2$, the force ellipse will become a line. In this case, two displacement vectors map to the same point on the force line (which is actually a degenerate ellipse). So, inversely, for a given force vector and a singular stiffness matrix, there is more than one displacement vector, there is not a unique displacement for a given force, and $[K]$ can not be inverted.

$$[K] = \begin{bmatrix} 1.22 & -1.2 \\ -1.2 & 1.22 \end{bmatrix}$$

$\lambda_1 = 0.02$ $\lambda_2 = 2.42$



A matrix is called *stiff* if the ratio of the largest to smallest eigen-value, $\lambda_{\max}/\lambda_{\min}$ is much greater than one.

Solving $[K]\{v\} = \{f\}$ for $\{v\}$ without finding $[K]^{-1}$

An efficient, simple, and accurate way to solve $[K]\{v\} = \{f\}$ for $\{v\}$ (where $[K]$ is symmetric and invertible) is by a method called *LDL^T decomposition.* In this method, the stiffness matrix is represented by the product of three matrices

$$[K] = [L][D][L]^T, \tag{16}$$

where $[L]$ is *lower triangular* and $[D]$ is *diagonal.*

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} =$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \begin{bmatrix} D_{11} & 0 & 0 & 0 \\ 0 & D_{22} & 0 & 0 \\ 0 & 0 & D_{33} & 0 \\ 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} 1 & L_{21} & L_{31} & L_{41} \\ 0 & 1 & L_{32} & L_{42} \\ 0 & 0 & 1 & L_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Here, the term *decomposition* means the separation of the matrix $[K]$ into the product of three simple matrices. Once the factors $[D]$ and $[L]$ have been found, the vector $[D][L]^T\{v\}$ is computed with a simple back-substitution starting with the first element.

$$\{f\} = [L][D][L]^T\{v\} = [L]([D][L]^T\{v\}) \tag{18}$$

Once $[D][L]^T\{v\}$ is computed, it is just as easy to find $\{v\}$, starting with the last element, and, after dividing by D_{ii}, the problem is solved. It is interesting to note that the number of positive values of $[D]$ equals the number of positive eigenvalues of $[K]$, and that the number of negative values of $[D]$ equals the number of negative eigenvalues of $[K]$. Since the LDL^T decomposition of a matrix is much faster than its eigen-value decomposition, LDL^T decomposition is an easy way to check if a matrix is positive definite or negative definate.

A complete derivation of the LDL^T decomposition method may be found in section 9.9 of the text book.[1]

Solving ill-conditioned problems using Singular Value Decomposition

The best way to solve for the deflections of a system with a nearly singular stiffness matrix is to use a method called *singular value decomposition* (SVD). The singular value decomposition of a stiffness matrix is the product of three other matrices,

$$[K] = [U][\Sigma][V]^T, \tag{19}$$

where $[\Sigma]$ is a diagonal matrix of the *singular values* of $[K]$, and the matrices $[U]$ and $[V]$ are ortho-normal, $[U]^T = [U]^{-1}$ and $[V]^T = [V]^{-1}$, or

$$[U]^T[U] = [V]^T[V] = [I]_N. \tag{20}$$

The singular values, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0$ are equal to the squares of the eigen-values of $[K]$. If some eigen-values of $[K]$ are nearly zero then the corresponding singular values will also be very small (less than ϵ), $\epsilon > \sigma_{n+1} > \sigma_{n+2} > \cdots > \sigma_N \geq 0$. In this case, we are typically interested only in the first n singular values $(\sigma_1, \cdots, \sigma_n)$ and we can zero-out the rest of the singular values. The diagonal singular value matrix then becomes

$$[\Sigma] = \left[\begin{array}{cc} [\bar{\Sigma}] & [0] \\ [0] & [0] \end{array} \right] \tag{21}$$

where $[\bar{\Sigma}]$ is a diagonal matrix of the first n singular values,

$$[\bar{\Sigma}] = \left[\begin{array}{ccc} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{array} \right]. \tag{22}$$

Having done this, we can solve for the displacements of the system as follows

$$\{v\} = [V]_n [\bar{\Sigma}]^{-1} [U]_n^T \{f\}, \tag{23}$$

where $[U]_n$ and $[V]_n$ are matrices containing the first n columns of $[U]$ and $[V]$ and $[\bar{\Sigma}]^{-1}$ is simply a diagonal matrix of the inverses of the first n (positive) singular values.

The text *Numerical Recipes in C* by Press, Teukolsky, Vetterling, and Flannery (Cambridge University Press, 1995), contains routines for Cholesky decomposition, SVD, and many other useful algorithms. In Matlab the function `chol` returns the Cholesky factor of a matrix and the function `svd` returns the singular value decomposition of a matrix.

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[1] A. Kassimali, *Matrix Analysis of Structures,* Brooks/Cole 1999.