

LARGE DEFORMATION ANALYSIS OF PLANAR TRUSSES

CE 131L. Matrix Structural Analysis

Henri P. Gavin

Fall, 2009

- Large deformation analysis is more accurate than linear, small deformation analysis.
- It can be useful in predicting the buckling potential of structures.
- In linear small deformation analysis, equilibrium is computed for the undeformed, original configuration of the truss. In large deformation analysis, on the other hand, equilibrium is computed about the deformed shape of the truss.
- This leads to a “chicken-and-egg” problem. One needs to know the deformed shape in order to apply loads and carry out the analysis, but one does not know the deformed shape until the analysis is complete. The solution to this problem, as in many non-linear problems, is to proceed incrementally, and apply deformations in small steps until equilibrium is achieved about the deformed configuration. There are some very efficient ways to do this step-by-step loading, as will be described later in this document.
- We will assume that the element stiffness matrix may be separated into an elastic part, \mathbf{k}_E as found earlier, plus a *geometric* part, \mathbf{k}_G that accounts for the effects of large deformation. The elastic stiffness matrix for a bar element, in local coordinates, as found earlier, is

$$\mathbf{k}_E = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We will start by finding the analog of this stiffness matrix that accounts for large deformation.

- **Large deformation for a bar in local coordinates**

Recall that in local coordinates, the four deflections at the bar ends are u_1 , u_2 , u_3 , and u_4 ; the four forces at the bar ends are q_1 , q_2 , q_3 , and q_4 , and the element stiffness matrix in local coordinates is \mathbf{k} . Define the angle α as the counter-clockwise inclination of the deformed bar with respect to coordinate 1, i.e., the bar's original un-deformed position. Taking equilibrium at joints 1 and 2,

$$\begin{aligned} q_1 &= -T \cos \alpha \\ q_2 &= -T \sin \alpha \\ q_3 &= T \cos \alpha \\ q_4 &= T \sin \alpha \end{aligned} \tag{1}$$

The angle α may be determined from the geometry of the deformed shape, i.e., the displacements u_1 , u_2 , u_3 , and u_4 .

$$\sin \alpha = \frac{1}{L}(u_4 - u_2),$$

and

$$\cos \alpha = \frac{1}{L}(L_o + u_3 - u_1) = \frac{1}{L}(u_3 - u_1) + \frac{L_o}{L} \approx 1,$$

where L_o is the original length of the bar and L is the deformed length of the bar.

$$L = L_o + \Delta = L_o + \frac{TL_o}{EA} = \left(1 + \frac{T}{EA}\right) L_o.$$

Here we will use the approximation that $L_o \approx L$. Note that so far we have made one assumption and two approximations in this “exact” analysis: $\mathbf{k} = \mathbf{k}_E + \mathbf{k}_G$, $\cos \alpha \approx 1$, and $L_o \approx L$. Substituting the expressions for $\sin \alpha$ and $\cos \alpha$ into the equilibrium equations (1) for \mathbf{q} , and writing the expressions in matrix form,

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} -T \\ 0 \\ T \\ 0 \end{Bmatrix} + \frac{T}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}. \tag{2}$$

If the bar deforms only along its original direction, then $u_2 = u_4 = 0$; $u_3 - u_1 = \Delta$; and $T = \frac{EA}{L}\Delta = \frac{EA}{L}(u_3 - u_1) = -q_1 = q_3$. Therefore the vector $\{-T \ 0 \ T \ 0\}^T$, in equation (2) may be found from the elastic stiffness

matrix, and

$$\mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{T}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad (3)$$

where the first matrix is the elastic stiffness matrix, which we have already seen, and the second matrix is the *geometric element stiffness matrix*, \mathbf{k}_G .

- **Coordinate Transformation: local to global**

The coordinate transformation process is identical to the process carried out before for the elastic element stiffness matrix. The coordinate transformation matrix, \mathbf{T} , is

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix},$$

where s and c are the sine and cosine of the counter-clockwise angle θ from global element coordinate number 1 to the bar. Here we are making a *third approximation*: the deformed inclination of the bar is approximately the same as the original inclination of the bar, or $|\alpha| \ll 1$. The element stiffness matrix in global coordinates is found by applying the coordinate transformation operation.

$$\begin{aligned} \mathbf{K} &= \mathbf{T}^T (\mathbf{k}_E + \mathbf{k}_G) \mathbf{T} \\ &= \mathbf{T}^T \mathbf{k}_E \mathbf{T} + \mathbf{T}^T \mathbf{k}_G \mathbf{T}. \end{aligned} \quad (4)$$

From these expressions,

$$\mathbf{K}_E = \mathbf{T}^T \mathbf{k}_E \mathbf{T} = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \quad (5)$$

$$\mathbf{K}_G = \mathbf{T}^T \mathbf{k}_G \mathbf{T} = \frac{T}{L} \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix} \quad (6)$$

It is not hard to confirm these two expressions for \mathbf{K}_E and \mathbf{K}_G , and you should feel encouraged to do so.

- **Matrix Assembly**

The assembly of the structural stiffness matrix \mathbf{K}_s proceeds exactly as with the elastic stiffness matrix.

- **Bar Forces**

To compute the internal bar forces large deformation considerations need to be included. Let's look at the deformed bar in local coordinates again. The deformed bar is inclined at an angle α from the orientation of the undeformed bar. The bar tension is $T = \frac{EA}{L}\Delta$, where now we need to find the stretch in the bar, Δ , from the four end displacements, u_1 , u_2 , u_3 , and u_4 . The stretched length of the bar is

$$L = \sqrt{(L_o + u_3 - u_1)^2 + (u_4 - u_2)^2} \approx L_o + (u_3 - u_1) + \frac{1}{2L_o}(u_4 - u_2)^2,$$

where the approximation is typically accurate to within 0.1% for strains up to 20%. So,

$$\Delta \approx (u_3 - u_1) + \frac{1}{2L_o}(u_4 - u_2)^2, \quad (7)$$

where the first term in parenthesis is the stretch considering only small deformations, and the second term is the contribution of the large deformation effects. Invoking the approximation $L_o \approx L$, the tension in a bar, including large-deformation effects, is

$$T = \frac{EA}{L_o}(u_3 - u_1) + \frac{EA}{L_o} \frac{1}{2L_o}(u_4 - u_2)^2, \quad (8)$$

Using approximation 3 again, $|\alpha| \ll 1$, the bar tensions can be found from the bar displacements in the global coordinate directions, v_1 , v_2 , v_3 , v_4 .

$$T = \frac{EA}{L_o} ((v_3 - v_1)c + (v_4 - v_2)s) + \frac{EA}{L_o} \frac{1}{2L_o} ((v_1 - v_3)s + (v_4 - v_2)c)^2, \quad (9)$$

Again, you should feel encouraged to confirm these equations.

- **Solving Nonlinear Problems using Newton-Raphson and Broyden Methods**

Recall the truncated Taylor series expansion of a nonlinear function $\mathbf{f}(\mathbf{x})$,

$$\mathbf{f} - \mathbf{f}_o = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right] (\mathbf{x} - \mathbf{x}_o) + \text{h.o.t. .}$$

If we are trying to solve a system of nonlinear equations $\mathbf{f}(\mathbf{x}_o) = \mathbf{f}_o$ for the vector of unknowns \mathbf{x}_o , we may proceed in an incremental fashion, by first evaluating $\mathbf{f}^i = \mathbf{f}(\mathbf{x}^i)$ for some vector \mathbf{x}^i . If we know the *Jacobian matrix* $[\partial \mathbf{f} / \partial \mathbf{x}]$ evaluated at the vector $\mathbf{x} = \mathbf{x}^i$, then the next trial value of the unknown vector should be

$$\mathbf{x}^{i+1} = \mathbf{x}^i - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\mathbf{x}=\mathbf{x}^i}^{-1} (\mathbf{f}^i - \mathbf{f}_o)$$

This is the essence of the Newton-Raphson method for solving sets of nonlinear equations. In the problem of large-deformation analysis of structures, the system of non-linear equations represents the equilibrium equations for the structural system, $\mathbf{p} = \mathbf{K}_s(\mathbf{d}) \mathbf{d}$, where the structural stiffness matrix \mathbf{K}_s includes geometric stiffness effects, which depends on the bar tensions, T , which, in turn depend on the displacements of the structure, equation (9). \mathbf{K}_s is a matrix that depends on the unknown displacements, \mathbf{d} , and we may express \mathbf{K}_s as a function of \mathbf{d} . The equilibrium of the structure is

$$\mathbf{p} = \mathbf{K}_s(\mathbf{d}) \mathbf{d} .$$

Note that the Jacobian matrix of this problem is $[\partial \mathbf{p} / \partial \mathbf{d}]$, which is not equal to $\mathbf{K}_s(\mathbf{d})$. We do not actually have an expression for the Jacobian $[\partial \mathbf{p} / \partial \mathbf{d}]$, but we can approximate it using a numerical technique. This is, in fact, a common difficulty encountered when implementing the Newton Raphson method. There are many ways to approximate an unknown Jacobian, and two approaches will be described here. In the first approach, we use the structural stiffness matrix, \mathbf{K}_s , as if it were the Jacobian matrix. In this approach, the iterations proceed as follows:

1. Initialize the displacements and bar tensions to be zero.
2. Assemble the structural stiffness matrix, for the initial configuration with zero displacements and zero bar tensions, call this matrix $\mathbf{K}_s^0 = \mathbf{K}_s(\mathbf{0})$.
3. Find the first approximation to the displacements, \mathbf{d}^0 , by solving $\mathbf{p} = \mathbf{K}_s^0 \mathbf{d}^0$.
4. With these displacements, find the bar forces, including large-deformation effects, using equation (9).
5. Re-compute the structural stiffness matrix using this set of bar forces for deflections $\mathbf{d} = \mathbf{d}^0$, $\mathbf{K}_s^1 = \mathbf{K}_s(\mathbf{d}^0)$.
6. Go back to step 3, and continue to iterate until $\|\mathbf{p} - \mathbf{K}_s^{i-1} \mathbf{d}^i\| < \epsilon$ where ϵ is the convergence tolerance for the equilibrium error, or until you get tired of iterating.

In principle, this approach will always converge to the correct solution, for both stiffening systems and softening systems, as seen below. However, this approach can be quite slow to converge, as illustrated below. In the second approach, the Jacobian matrix, which is also called the *tangent stiffness matrix*¹ is approximated by a *secant stiffness matrix*, for which we will use the symbol $\bar{\mathbf{K}}_s$. The secant stiffness matrix will be calculated using a technique attributed to C.G. Broyden.² In each iteration with the secant stiffness approach, we find incremental displacements, $\delta \mathbf{d}$, and add those displacements to the previously computed displacements. The procedure using Broyden's secant stiffness matrix is as follows:

1. Initialize the displacements and bar tensions to be zero.
2. Assemble the structural stiffness matrix, for the initial configuration with zero displacements and zero bar tensions, call this matrix $\mathbf{K}_s^0 = \mathbf{K}_s(\mathbf{0})$. This matrix will also be the initial secant stiffness matrix, $\bar{\mathbf{K}}_s^0 = \mathbf{K}_s^0$.³
3. Find the first incremental displacements, $\delta \mathbf{d}^0$, by solving $\mathbf{p} = \mathbf{K}_s^0 \delta \mathbf{d}^0$.

¹Can you guess why?

²Press, W.H., et. al, *Numerical Recipes in C*, Cambridge, 1992, section 9.7

Broyden, C.G., *Mathematics of Computation*, vol. 19, 1965, pp. 577–593.

³This is a good idea. Can you see why?

4. Add these displacements to the initial displacements, \mathbf{d}^0 , (which are zero),

$$\mathbf{d}^1 = \mathbf{d}^0 + \delta\mathbf{d}^0 = \delta\mathbf{d}^0.$$
5. Initialize the iteration counter, $i = 1$.
6. With the displacements \mathbf{d}^i find the bar forces, including large-deformation effects, using equation (9).
7. Compute the structural stiffness matrix, $\mathbf{K}_s^i(\mathbf{d}^i)$ using the set of bar forces for deflections \mathbf{d}^i , and compute the equilibrium error $\mathbf{p} - \mathbf{K}_s^i(\mathbf{d}^i) \mathbf{d}^i$
8. Update the secant stiffness matrix, using Broyden's formula.⁴

$$\bar{\mathbf{K}}_s^i = \bar{\mathbf{K}}_s^{i-1} - \frac{(\mathbf{p} - \mathbf{K}_s^i(\mathbf{d}^i) \mathbf{d}^i) \times \delta\mathbf{d}^{i-1}}{\delta\mathbf{d}^{i-1} \cdot \delta\mathbf{d}^{i-1}}$$

The *times* symbol (\times) represents the vector outer product, which results in a matrix, and the *dot* (\cdot) represents the vector inner product, which results in a scalar.

9. Using this secant stiffness matrix, find the next set of incremental displacements, $\delta\mathbf{d}^i$.

$$\bar{\mathbf{K}}_s^i \delta\mathbf{d}^i = \mathbf{p} - \mathbf{K}_s^i(\mathbf{d}^i) \mathbf{d}^i$$

10. Add the incremental displacements, $\delta\mathbf{d}^i$ to the current displacements, \mathbf{d}^i ,

$$\mathbf{d}^{i+1} = \mathbf{d}^i + \delta\mathbf{d}^i$$

11. Increment the iteration counter, $i = i + 1$, return to step 6, and continue to iterate until the equilibrium error is sufficiently small, $\|\mathbf{p} - \mathbf{K}_s^i(\mathbf{d}^i) \mathbf{d}^i\| < \epsilon$.

Note that the secant stiffness matrix as computed by Broyden's method is not symmetric. This could potentially lead to problems. The following diagram illustrates Newton-Raphson iterations using Broyden's secant stiffness approach. It is not hard to see how this approach can be substantially more efficient than the first approach.

⁴See Press, W.H., et. al, *Numerical Recipes in C*, Cambridge, 1992, section 9.7, for details

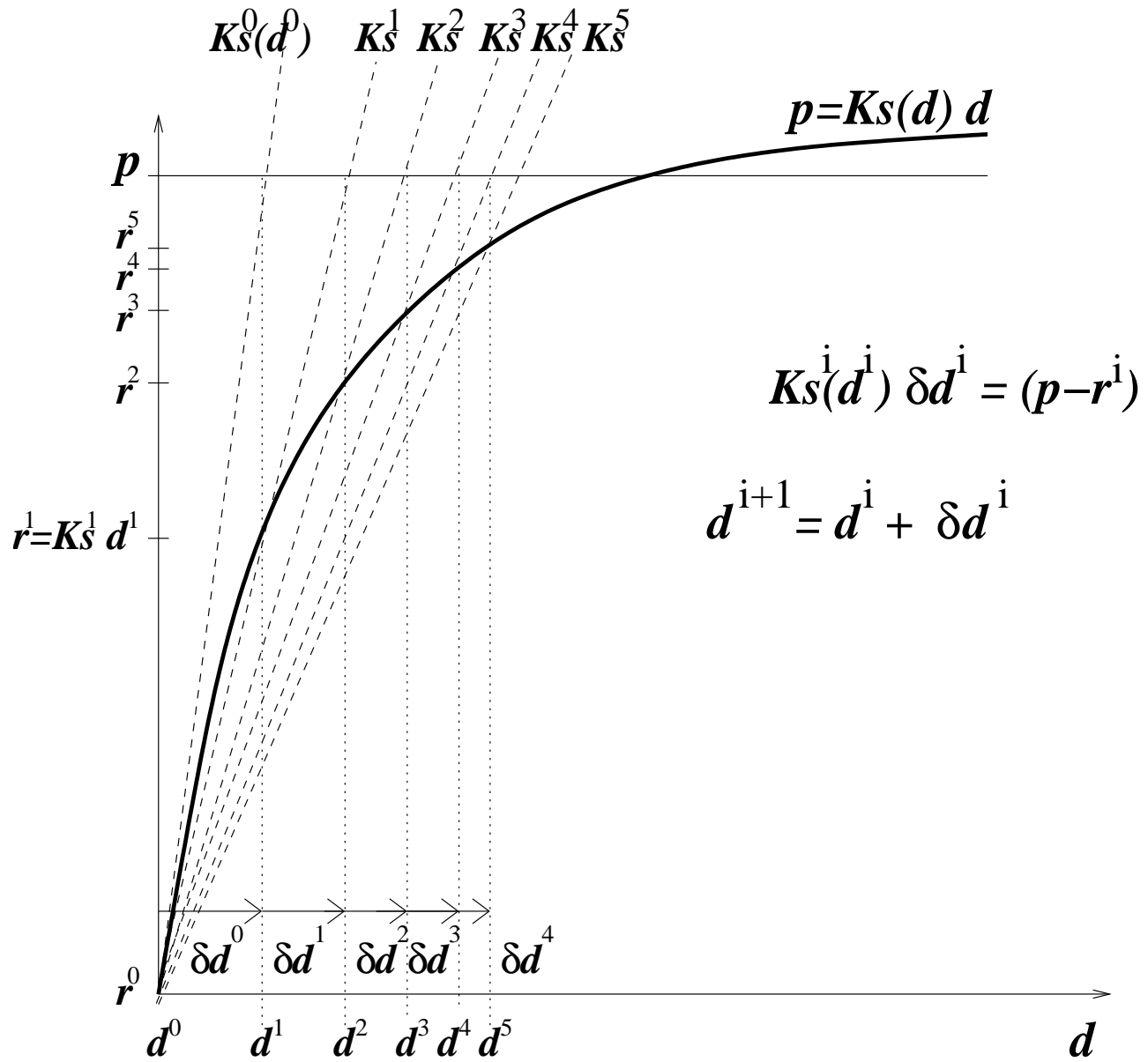


Figure 1: Modified Newton Raphson Method.

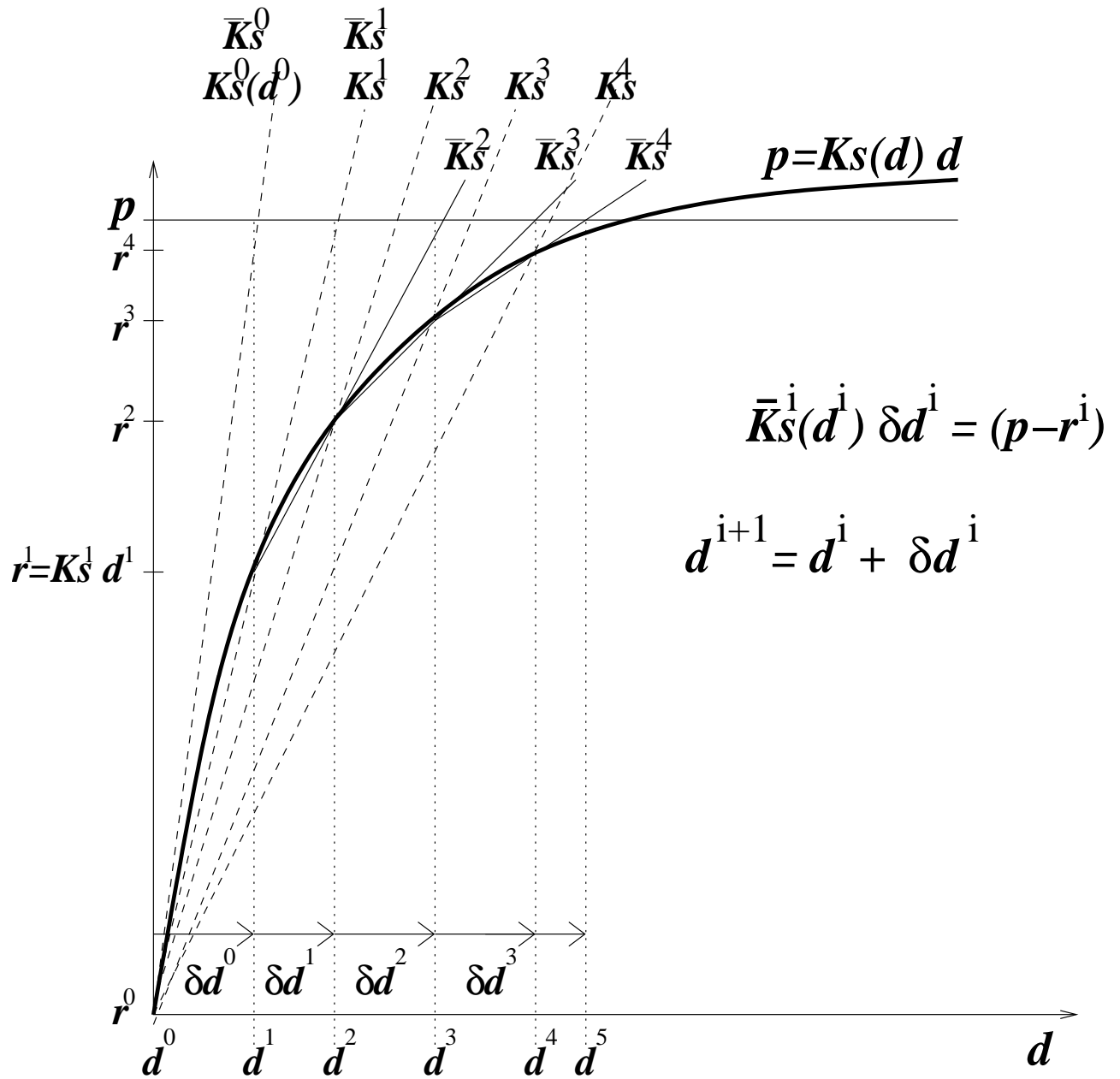


Figure 2: Modified Newton Raphson Method with Broyden updating.