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STRUCTURAL ELEMENT STIFFNESS MATRICES AND MASS MATRICES

1 Preliminaries

This document describes the formulation of stiffness and mass matrices for structural elements such as truss bars, beams, plates, and cables(?). The formulation of each element involves the determination of gradients of potential and kinetic energy functions with respect to a set of coordinates defining the displacements at the ends, or nodes, of the elements. The potential and kinetic energy of the functions are therefore written in terms of these nodal displacements (i.e., generalized coordinates). To do so, the distribution of strains and velocities within the element must be written in terms of nodal coordinates as well. Both of these distributions may be derived from the distribution of internal displacements within the solid element.

1.1 Displacements

Figure 1. Displacements within a solid continuum.

$$u_i(\mathbf{x},t) = \sum_{n=1}^{N} \psi_{in}(x_1,x_2,x_3)\ \bar{u}_n(t) \tag{1}$$

$$= \mathbf{\Psi}_i(\mathbf{x})\ \bar{\mathbf{u}}(t) \tag{2}$$

$$\mathbf{u}(\mathbf{x},t) = [\mathbf{\Psi}(\mathbf{x})]_{3\times N}\ \bar{\mathbf{u}}(t) \tag{3}$$

Engineering strain, axial strain ϵ_{ii}, shear strain γ_{ij}.

$$\epsilon_{ii}(\mathbf{x},t) = \frac{\partial u_i(\mathbf{x},t)}{\partial x_i} \tag{4}$$

$$\gamma_{ij}(\mathbf{x},t) = \frac{\partial u_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial u_j(\mathbf{x},t)}{\partial x_i} \quad (5)$$

$$(6)$$

Displacement gradient

$$\frac{\partial u_i(\mathbf{x})}{\partial x_j} = \sum_{n=1}^{N} \frac{\partial}{\partial x_j} \psi_{in}(x_1,x_2,x_3)\ \bar{u}_n(t) \quad (7)$$

$$u_{i,j}(\mathbf{x}) = \sum_{n=1}^{N} \psi_{in,j}(\mathbf{x})\ \bar{u}_n(t) \quad (8)$$

Strain-displacement relations

$$\epsilon_{ii}(\mathbf{x},t) = \sum_{n=1}^{N} \psi_{in,i}(\mathbf{x})\ \bar{u}_n(t) \quad (9)$$

$$\gamma_{ij}(\mathbf{x},t) = \sum_{n=1}^{N} \left(\psi_{in,j}(\mathbf{x}) + \psi_{jn,i}(\mathbf{x})\right)\ \bar{u}_n(t) \quad (10)$$

Strain vector

$$\boldsymbol{\epsilon}^T(\mathbf{x},t) = \{\ \epsilon_{11}\ \ \epsilon_{22}\ \ \epsilon_{33}\ \ \gamma_{12}\ \ \gamma_{23}\ \ \gamma_{13}\ \} \quad (11)$$

$$\boldsymbol{\epsilon}(\mathbf{x},t) = [\ \mathbf{B}(\mathbf{x})\ ]_{6\times N}\ \bar{\mathbf{u}}(t) \quad (12)$$

Stress vector

$$\boldsymbol{\sigma}^T(\mathbf{x},t) = \{\ \sigma_{11}\ \ \sigma_{22}\ \ \sigma_{33}\ \ \tau_{12}\ \ \tau_{23}\ \ \tau_{13}\ \} \quad (13)$$

Stress-strain relationship (isotropic elastic solid)

$$\left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{cccccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & & \\ & & & & \frac{1-2\nu}{2} & \\ & & & & & \frac{1-2\nu}{2} \end{array}\right] \left\{\begin{array}{c} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{array}\right\} \quad (14)$$

$$\boldsymbol{\sigma} = [\ \mathbf{S}(E,\nu)\ ]_{6\times 6}\ \boldsymbol{\epsilon} \quad (15)$$

1.2 Potential Energy and Stiffness

Consider a system comprising an assemblage of linear springs, with stiffness k_i, each with an individual stretch, d_i. The total potential energy in the assemblage is

$$V = \frac{1}{2} \sum_i k_i d_i^2$$

If displacements of the assemblage of springs is denoted by a vector $\mathbf{u}$, not necessarily equal to the stretches in each spring, then the elastic potential energy may also be written

$$\begin{aligned} V(\mathbf{u}) &= \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} \\ &= \frac{1}{2} \sum_{i=1}^{n} u_i f_i \\ &= \frac{1}{2} \sum_{i=1}^{n} u_i \sum_{j=1}^{n} K_{ij} u_j \end{aligned}$$

where $\mathbf{K}$ is the stiffness matrix with respect to the coordinates $\mathbf{u}$. The stiffness matrix $\mathbf{K}$ relates the elastic forces f_i to the collocated displacements, u_i.

$$\begin{aligned} f_1 &= K_{11} u_1 + \cdots + K_{1j} u_j + \cdots + K_{1N} u_N \\ f_i &= K_{i1} u_1 + \cdots + K_{ij} u_j + \cdots + K_{iN} u_N \\ f_N &= K_{N1} u_1 + \cdots + K_{Nj} u_j + \cdots + K_{NN} u_N \end{aligned}$$

A point force f_i acting on an elastic body is the gradient of the elastic potential energy V with respect to the collocated displacement u_i

$$f_i = \frac{\partial}{\partial u_i} V$$

The i, j term of the stiffness matrix may therefore be found from the potential energy function $V(\mathbf{u})$,

$$K_{ij} = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} V(\mathbf{u}) \tag{16}$$

1.3 Strain Energy and Stiffness in Linear Elastic Continua

$$\begin{aligned} V(\bar{\mathbf{u}}) &= \frac{1}{2} \int_\Omega \boldsymbol{\sigma}(\mathbf{x}, t)^T \boldsymbol{\epsilon}(\mathbf{x}, t) \; d\Omega \qquad (17) \\ &= \frac{1}{2} \int_\Omega \boldsymbol{\epsilon}(\mathbf{x}, t)^T \; \mathbf{S}(E, \nu) \; \boldsymbol{\epsilon}(\mathbf{x}, t) \; d\Omega \\ &= \frac{1}{2} \int_\Omega \bar{\mathbf{u}}(t)^T \mathbf{B}(\mathbf{x})^T \; \mathbf{S}(E, \nu) \; \mathbf{B}(\mathbf{x}) \bar{\mathbf{u}}(t) \; d\Omega \\ &= \frac{1}{2} \; \bar{\mathbf{u}}(t)^T \; \int_\Omega \Big[\mathbf{B}(\mathbf{x})^T \; \mathbf{S}(E, \nu) \; \mathbf{B}(\mathbf{x}) \Big]_{N \times N} d\Omega \; \bar{\mathbf{u}}(t) \qquad (18) \end{aligned}$$

Elastic element stiffness matrix

$$\bar{\mathbf{f}}_E = \frac{\partial V}{\partial \bar{\mathbf{u}}} = \bar{\mathbf{K}}_E \ \bar{\mathbf{u}}$$

$$\bar{\mathbf{K}}_E = \int_\Omega \left[\mathbf{B}(\mathbf{x})^T \ \mathbf{S}(E, \nu) \ \mathbf{B}(\mathbf{x}) \right]_{N\times N} d\Omega \tag{19}$$

1.4 Geometric Strain

Figure 2. Axial strain due to transverse displacement.

δx: axial deformation due to transverse displacement du_y without displacement in the x direction ($du_x = 0$).

$$(dx + \delta x) \left(\cos \left(\arctan \left(\frac{du_y}{dx} \right) \right) \right) = dx \tag{20}$$

$$\left(1 + \frac{\delta x}{dx} \right) \left(\cos \left(\arctan \left(\frac{du_y}{dx} \right) \right) \right) = 1 \tag{21}$$

$$\frac{\delta x}{dx} = \csc \left(\arctan \left(\frac{du_y}{dx} \right) \right) - 1 \tag{22}$$

$$\epsilon_{xx} = \frac{\delta x}{dx} \approx \frac{1}{2} \left(\frac{du_y}{dx} \right)^2 \tag{23}$$

The approximation is accurate to within -1.0% for $du_y/dx < 0.20$, and to within -0.1% for $du_y/dx < 0.07$.

Large deflection strain-displacement equations:

$$\epsilon_{ii} = \frac{\partial u_i}{\partial x_i} + \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} \right)^2 + \frac{1}{2} \left(\frac{\partial u_k}{\partial x_i} \right)^2 \tag{24}$$

$$= u_{i,i} + \frac{1}{2} u_{j,i}^2 + \frac{1}{2} u_{k,i}^2 \tag{25}$$

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_i} \tag{26}$$

$$= u_{i,j} + u_{j,i} + u_{i,j} u_{j,j} + u_{j,i} u_{i,i} \tag{27}$$

1.5 Kinetic Energy and Mass

The impulse-momentum relationship states that

$$\begin{aligned}
\int f\ dt &= \delta(m\dot{u}) \\
f &= \frac{d}{dt}(m\dot{u}) \\
f &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{u}}\frac{1}{2}m\dot{u}^2\right) \\
f &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{u}}T\right) \ ,
\end{aligned}$$

where T is the kinetic energy.

Consider a system comprising an assemblage of point masses, m_i, each with an individual velocity, v_i. The total kinetic energy in the assemblage is

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

If displacements of the assemblage of masses are defined by a generalized coordinate vector $\mathbf{u}$, not necessarily equal to the velocity coordinates, above, then the kinetic energy may also be written

$$\begin{aligned}
T(\dot{\mathbf{u}}) &= \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} \\
&= \frac{1}{2}\sum_{i=1}^{n}\dot{u}_i\sum_{j=1}^{n}M_{ij}\dot{u}_j
\end{aligned}$$

where $\mathbf{M}$ is the constant mass matrix with respect to the generalized coordinates $\mathbf{u}$. The mass matrix $\mathbf{M}$ relates the inertial forces f_i to the collocated accelerations, $\ddot{u}_i$.

$$\begin{aligned}
f_1 &= M_{11}\ddot{u}_1 + \cdots + M_{1j}\ddot{u}_j + \cdots + M_{1N}\ddot{u}_N \\
f_i &= M_{i1}\ddot{u}_1 + \cdots + M_{ij}\ddot{u}_j + \cdots + M_{iN}\ddot{u}_N \\
f_N &= M_{N1}\ddot{u}_1 + \cdots + M_{Nj}\ddot{u}_j + \cdots + M_{NN}\ddot{u}_N
\end{aligned}$$

The i, j term of the constant mass matrix may therefore be found from the kinetic energy function T,

$$M_{ij} = \frac{\partial}{\partial \ddot{u}_i}\frac{\partial}{\partial t}\frac{\partial}{\partial \dot{u}_j}T(\dot{\mathbf{u}}) = \frac{\partial}{\partial \dot{u}_i}\frac{\partial}{\partial \dot{u}_j}T(\dot{\mathbf{u}}) \tag{28}$$

1.6 Inertial Energy and Mass in Deforming Continua

$$
\begin{aligned}
T(\bar{\mathbf{u}}, \dot{\bar{\mathbf{u}}}) &= \frac{1}{2} \int_\Omega \rho \; |\dot{\mathbf{u}}(\mathbf{x},t)|^2 \; d\Omega \qquad (29) \\
&= \frac{1}{2} \int_\Omega \rho \; \dot{\mathbf{u}}(\mathbf{x},t)^T \dot{\mathbf{u}}(\mathbf{x},t) \; d\Omega \\
&= \frac{1}{2} \int_\Omega \rho \; \dot{\bar{\mathbf{u}}}(t)^T \boldsymbol{\Psi}(\mathbf{x})^T \; \boldsymbol{\Psi}(\mathbf{x}) \dot{\bar{\mathbf{u}}}(t) \; d\Omega \\
&= \frac{1}{2} \; \dot{\bar{\mathbf{u}}}(t)^T \int_\Omega \rho \; \left[\boldsymbol{\Psi}(\mathbf{x})^T \; \boldsymbol{\Psi}(\mathbf{x})\right]_{N\times N} \; d\Omega \; \dot{\bar{\mathbf{u}}}(t) \qquad (30)
\end{aligned}
$$

Consistent mass matrix

$$
\begin{aligned}
\frac{\partial T}{\partial \bar{\mathbf{u}}} &= 0 \qquad (31) \\
\frac{\partial T}{\partial \dot{\bar{\mathbf{u}}}} &= \int_\Omega \rho \; \left[\boldsymbol{\Psi}(\mathbf{x})^T \; \boldsymbol{\Psi}(\mathbf{x})\right]_{N\times N} \; d\Omega \; \dot{\bar{\mathbf{u}}}(t) \qquad (32) \\
\bar{\mathbf{f}}_I = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{\mathbf{u}}}}\right) &= \int_\Omega \rho \; \left[\boldsymbol{\Psi}(\mathbf{x})^T \; \boldsymbol{\Psi}(\mathbf{x})\right]_{N\times N} \; d\Omega \; \ddot{\bar{\mathbf{u}}}(t) \qquad (33) \\
\bar{\mathbf{M}} &= \int_\Omega \rho \; \left[\boldsymbol{\Psi}(\mathbf{x})^T \; \boldsymbol{\Psi}(\mathbf{x})\right]_{N\times N} \; d\Omega \qquad (34)
\end{aligned}
$$

2 Bar Element Matrices

2D prismatic homogeneous isotropic truss bar.

Uniform uni-axial stress, σ_{xx}, only.
Uniform uni-axial strain, ϵ_{xx}.
$\sigma_{xx} = E\epsilon_{xx}$

2.1 Bar Displacements

Figure 3. Truss bar element coordinates and displacements.

$$u_x(x,t) = \left(1-\frac{x}{L}\right)\bar{u}_1(t) + \left(\frac{x}{L}\right)\bar{u}_3(t) \tag{35}$$
$$= \psi_{x1}(x)\ \bar{u}_1(t) + \psi_{x3}(x)\ \bar{u}_3(t) \tag{36}$$
$$u_y(x,t) = \left(1-\frac{x}{L}\right)\bar{u}_2(t) + \left(\frac{x}{L}\right)\bar{u}_4(t) \tag{37}$$
$$= \psi_{y2}(x)\ \bar{u}_2(t) + \psi_{y4}(x)\ \bar{u}_4(t) \tag{38}$$

$$\mathbf{\Psi}(x) = \left[\begin{array}{c|c|c|c} 1-\frac{x}{L} & 0 & \frac{x}{L} & 0 \\ \hline 0 & 1-\frac{x}{L} & 0 & \frac{x}{L} \end{array}\right] \tag{39}$$

$$\begin{bmatrix} u_x(x,t) \\ u_y(x,t) \end{bmatrix} = \mathbf{\Psi}(x)\ \bar{\mathbf{u}}(t) \tag{40}$$

2.2 Bar Strain Energy and Elastic Stiffness Matrix

Strain-displacement relation

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 \tag{41}$$
$$= \psi_{x1,x}\bar{u}_1 + \psi_{x3,x}\bar{u}_3 + \frac{1}{2}\left\{\psi_{y2,x} + \psi_{y4,x}\right\}^2 \tag{42}$$
$$= \left(-\frac{1}{L}\right)\bar{u}_1 + \left(\frac{1}{L}\right)\bar{u}_3 + \frac{1}{2}\left\{\left(-\frac{1}{L}\right)\bar{u}_2 + \left(\frac{1}{L}\right)\bar{u}_4\right\}^2 \tag{43}$$
$$= \left[\begin{array}{cccc} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \end{array}\right]\bar{\mathbf{u}} + \frac{1}{2}\left\{\left(-\frac{1}{L}\right)\bar{u}_2 + \left(\frac{1}{L}\right)\bar{u}_4\right\}^2 \tag{44}$$
$$= \mathbf{B}\ \bar{\mathbf{u}} + \frac{1}{2}\left\{\left(-\frac{1}{L}\right)\bar{u}_2 + \left(\frac{1}{L}\right)\bar{u}_4\right\}^2 \tag{45}$$

$$\mathbf{B} = \left[\begin{array}{cccc} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \end{array}\right]. \tag{46}$$

Strain energy and elastic stiffness

$$V = \frac{1}{2}\int_\Omega \epsilon_{xx}\ E\ \epsilon_{xx}\ d\Omega \tag{47}$$
$$\bar{\mathbf{K}}_E = \int_{x=0}^{L}\left[\mathbf{B}^T\ E\ \mathbf{B}\right]\ Adx \tag{48}$$
$$= EA\int_{x=0}^{L}\begin{bmatrix} 1/L^2 & 0 & -1/L^2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/L^2 & 0 & 1/L^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} dx \tag{49}$$
$$= \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

2.3 Bar Kinetic Energy and Mass Matrix

$$\begin{aligned}
T &= \dot{\mathbf{u}}^T \int_\Omega \rho \left[\mathbf{\Psi}(x)^T \ \mathbf{\Psi}(x)\right]_{N\times N} \ d\Omega \ \dot{\mathbf{u}}(t) && (51)\\
\bar{\mathbf{M}} &= \int_{x=0}^{L} \rho \left[\mathbf{\Psi}(x)^T \ \mathbf{\Psi}(x)\right] \ Adx && (52)\\
&= \rho A \int_{x=0}^{L} \begin{bmatrix} (1-\frac{x}{L})^2 & 0 & (1-\frac{x}{L})(\frac{x}{L}) & 0 \\ 0 & (1-\frac{x}{L})^2 & 0 & (1-\frac{x}{L})(\frac{x}{L}) \\ (\frac{x}{L})(1-\frac{x}{L}) & 0 & (\frac{x}{L})^2 & 0 \\ 0 & (\frac{x}{L})(1-\frac{x}{L}) & 0 & (\frac{x}{L})^2 \end{bmatrix} dx && (53)\\
&= \frac{1}{6}\rho AL \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} && (54)
\end{aligned}$$

2.4 Bar Stiffness Matrix with Geometric Strain Effects

$$\begin{aligned}
V &= \frac{1}{2}\int_0^L \epsilon_{xx} \ E \ \epsilon_{xx} \ Adx && (55)\\
&= \frac{EA}{2}\int_0^L \left(\frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2\right)^2 \ dx && (56)\\
&= \frac{EA}{2}\int_0^L \left(\left(\frac{\partial u_x}{\partial x}\right)^2 + \frac{\partial u_x}{\partial x}\left(\frac{\partial u_y}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial u_y}{\partial x}\right)^4\right) \ dx && (57)
\end{aligned}$$

Substitute

$$\begin{aligned}
\frac{\partial u_x}{\partial x} &= -\frac{1}{L}\bar{u}_1 + \frac{1}{L}\bar{u}_3 && (58)\\
\frac{\partial u_y}{\partial x} &= -\frac{1}{L}\bar{u}_2 + \frac{1}{L}\bar{u}_4 && (59)
\end{aligned}$$

to obtain

$$V = \frac{EA}{2L}\left((\bar{u}_3-\bar{u}_1)^2 + \frac{1}{L}(\bar{u}_3-\bar{u}_1)(\bar{u}_4-\bar{u}_2)^2\right) \qquad (60)$$

So,

$$\begin{aligned}
\frac{\partial V}{\partial \bar{\mathbf{u}}} &= \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} + \frac{EA(\bar{u}_3-\bar{u}_1)}{L^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{bmatrix} && (61)\\
&= \bar{\mathbf{K}}_E \ \bar{\mathbf{u}} + \frac{N}{L}\bar{\mathbf{K}}_G \ \bar{\mathbf{u}} && (62)
\end{aligned}$$

3 Bernoulli-Euler Beam Element Matrices

2D prismatic homogeneous isotropic beam element, neglecting shear deformation and rotatory inertia.

3.1 Bernoulli-Euler Beam Coordinates and Internal Displacements

Consider the geometry of a deformed beam. The functions $u_x(x)$ and $u_y(x)$ describe the translation of points along the neutral axis of the beam as a function of the location along the un-stretched neutral axis.

Figure 4. Beam element coordinates and displacements.

We will describe the deformation of the beam as a function of the end displacements $(\bar{u}_1, \bar{u}_2, \bar{u}_4, \bar{u}_5)$ and the end rotations $(\bar{u}_3, \bar{u}_6)$. In a dynamic context, these end displacements will change with time.

$$
\begin{aligned}
u_x(x,t) &= \sum_{n=1}^{6} \psi_{xn}(x)\ \bar{u}_n(t) \\
u_y(x,t) &= \sum_{n=1}^{6} \psi_{yn}(x)\ \bar{u}_n(t)
\end{aligned}
$$

The functions $\psi_{xn}(x)$ and $\psi_{yn}(x)$ satisfy the boundary conditions at the end of the beam and the differential equation describing bending of a Bernoulli-Euler beam loaded statically at the nodal coordinates. In such beams the effects of shear deformation and rotatory inertia are neglected. For extension of the neutral axis,

$$
\begin{aligned}
\psi_{x1}(x) &= 1 - \frac{x}{L} \\
\psi_{x4}(x) &= \frac{x}{L}
\end{aligned}
$$

and $\psi_{x2} = \psi_{x3} = \psi_{x5} = \psi_{x6} = 0$ along the neutral axis. For bending of the neutral axis,

$$
\psi_{y2}(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3
$$

$$
\begin{aligned}
\psi_{y3}(x) &= \left(\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 \right) L \\
\psi_{y5}(x) &= 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \\
\psi_{y6}(x) &= \left(-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 \right) L
\end{aligned}
$$

and $\psi_{y1} = \psi_{y4} = 0$.

$$
\boldsymbol{\Psi}(x) = \left[\begin{array}{c|c|c|c|c|c} 1-\frac{x}{L} & 0 & 0 & \frac{x}{L} & 0 & 0 \\ \hline 0 & 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & \left(\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) L & 0 & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 & \left(-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) L \end{array} \right] \quad (63)
$$

$$
\left[\begin{array}{c} u_x(x,t) \\ u_y(x,t) \end{array} \right] = \boldsymbol{\Psi}(x)\ \bar{\mathbf{u}}(t) \quad (64)
$$

These expressions are analytical solutions for the displacements of Bernoulli-Euler beams loaded only with concentrated point loads and concentrated point moments at their ends. Internal bending moments are linear within beams loaded only at their ends, and the beam displacements may be expressed with cubic polynomials.

3.2 Bernoulli-Euler Beam Strain Energy and Elastic Stiffness Matrix

In extension, the elastic potential energy in a beam is the strain energy related to the uniform extensional strain, ϵ_{xx}. If the strain is small, then the extensional strain within the cross section is equal to an extension of the neutral axis, $(\partial u_x / \partial x)$, plus the bending strain, $-(\partial^2 u_y / \partial x^2) y$.

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u_x}{\partial x} - \frac{\partial^2 u_y}{\partial x^2}\ y \\
&= \sum_{n=1}^{6} \frac{\partial}{\partial x} \psi_{xn}(x)\ \bar{u}_n - \sum_{n=1}^{6} \frac{\partial^2}{\partial x^2} \psi_{yn}(x)\ y\ \bar{u}_n \quad (65) \\
&= \sum_{n=1}^{6} \psi'_{xn}(x)\ \bar{u}_n - \sum_{n=1}^{6} \psi''_{yn}(x)\ y\ \bar{u}_n \\
&= \sum_{n=1}^{6} B_n(x,y)\ \bar{u}_n \\
&= \mathbf{B}(x,y)\ \bar{\mathbf{u}} \quad (66)
\end{aligned}
$$

where

$$
\mathbf{B}(x,y) = \left[-\frac{1}{L},\ \ \frac{6y}{L^2} - \frac{12xy}{L^3},\ \ \frac{4y}{L} - \frac{6xy}{L^3},\ \ \frac{1}{L},\ \ \frac{-6y}{L^2} + \frac{12xy}{L^3},\ \ \frac{2y}{L} - \frac{6xy}{L^2} \right] . \quad (67)
$$

The elastic stiffness matrix can be found directly from the strain energy of axial strains ϵ_{xx}.

$$
\begin{aligned}
V &= \frac{1}{2} \int_{\Omega} \epsilon_{xx}\ E\ \epsilon_{xx}\ d\Omega \quad (68) \\
\bar{\mathbf{K}}_E &= \int_{x=0}^{L} \int_{A} \left[\mathbf{B}(x,y)^T\ E\ \mathbf{B}(x,y) \right]\ dA\ dx. \quad (69)
\end{aligned}
$$

Note that this integral involves terms such as $\int_A y^2 dA$ and $\int_A y dA$ in which the origin of the coordinate axis is placed at the centroid of the section. The integral $\int_A y^2 dA$ is the bending moment of inertia for the cross section, I, and the integral $\int_A y dA$ is zero.

It is also important to recognize that the elastic strain energy may be evaluated separately for extension effects and bending effects. For extension, the elastic strain energy is

$$\begin{aligned} V &= \frac{1}{2}\int_{x=0}^{L} EA\,(\epsilon_{xx})^2\,dx \\ &= \frac{1}{2}\int_{x=0}^{L} EA\left(\sum_{n=1}^{6}\psi'_{xn}(x)\,\bar{u}_n\right)^2\,dx \end{aligned}$$

and the ij stiffness coefficient (for indices 1 and 4) is

$$\begin{aligned} \bar{K}_{ij} &= \frac{\partial}{\partial\bar{u}_i}\frac{\partial}{\partial\bar{u}_j}\frac{1}{2}\int_{x=0}^{L} EA\left(\sum_{n=1}^{6}\psi'_{xn}(x)\,\bar{u}_n\right)^2\,dx \\ &= \int_{x=0}^{L} EA\,\psi'_{xi}(x)\,\psi'_{xj}(x)\,dx. \end{aligned} \tag{70}$$

In bending, the elastic potential energy in a Bernoulli-Euler beam is the strain energy related to the curvature, κ_z.

$$\kappa_z = \frac{\partial^2 u_y}{\partial x^2} = \sum_{n=1}^{6}\frac{\partial^2}{\partial x^2}\psi_{yn}(x)\,\bar{u}_n = \sum_{n=1}^{6}\psi''_{yn}(x)\,\bar{u}_n$$

The elastic strain energy for pure bending is

$$\begin{aligned} V &= \frac{1}{2}\int_{x=0}^{L} EI\,(\kappa_z)^2\,dx \\ &= \frac{1}{2}\int_{x=0}^{L} EI\left(\sum_{n=1}^{6}\psi''_{yn}(x)\,\bar{u}_n\right)^2\,dx \end{aligned}$$

and the ij stiffness coefficient (for indices 2,3,5 and 6) is

$$\begin{aligned} \bar{K}_{ij} &= \frac{\partial}{\partial\bar{u}_i}\frac{\partial}{\partial\bar{u}_j}\frac{1}{2}\int_{x=0}^{L} EI\left(\sum_{n=1}^{6}\psi''_{yn}(x)\,\bar{u}_n\right)^2\,dx \\ &= \int_{x=0}^{L} EI\,\psi''_{yi}(x)\,\psi''_{yj}(x)\,dx. \end{aligned} \tag{71}$$

3.3 Bernoulli-Euler Beam Kinetic Energy and Mass Matrix

The kinetic energy of a particle within a beam is half the mass of the particle, $\rho A dx$, times its velocity, $\dot{u}$, squared. For velocities along the direction of the neutral axis,

$$\dot{u}_x(x) = \sum_{n=1}^{6} \psi_{xn}(x)\ \dot{\bar{u}}_n\ ,$$

The kinetic energy function and the mass matrix may be by substituting equation (63) into equations (30) and (34).

$$T = \dot{\bar{\mathbf{u}}}^T \int_\Omega \rho\ \left[\boldsymbol{\Psi}(x)^T\ \boldsymbol{\Psi}(x)\right]_{N\times N}\ d\Omega\ \dot{\bar{\mathbf{u}}}(t) \tag{72}$$

$$\bar{\mathbf{M}} = \int_{x=0}^{L} \rho\ \left[\boldsymbol{\Psi}(x)^T\ \boldsymbol{\Psi}(x)\right]\ A dx \tag{73}$$

It is important to recognize that kinetic energy and mass associated with extensional velocities may be determined separately from those associated with transverse velocities. The kinetic energy for extension of the neutral axis is

$$\begin{aligned} T &= \frac{1}{2}\int_{x=0}^{L} \rho A\ (\dot{u}_x)^2\ dx \\ &= \frac{1}{2}\int_{x=0}^{L} \rho A \left(\sum_{n=1}^{6} \psi_{xn}(x)\ \dot{\bar{u}}_n\right)^2\ dx \end{aligned}$$

and the ij mass coefficient (for indices 1 and 4) is

$$\begin{aligned} \bar{M}_{ij} &= \frac{\partial}{\partial \dot{\bar{u}}_i}\frac{\partial}{\partial \dot{\bar{u}}_j}\frac{1}{2}\int_{x=0}^{L} \rho A \left(\sum_{n=1}^{6} \psi_{xn}(x)\ \dot{\bar{u}}_n\right)^2\ dx \\ &= \int_{x=0}^{L} \rho A\ \psi_{xi}(x)\ \psi_{xj}(x)\ dx. \end{aligned} \tag{74}$$

For velocities transverse to the neutral axis,

$$\dot{u}_y(x) = \sum_{n=1}^{6} \psi_{yn}(x)\ \dot{\bar{u}}_n\ ,$$

the kinetic energy for velocity across the neutral axis is

$$\begin{aligned} T &= \frac{1}{2}\int_{x=0}^{L} \rho A\ (\dot{u}_y)^2\ dx \\ &= \frac{1}{2}\int_{x=0}^{L} \rho A \left(\sum_{n=1}^{6} \psi_{yn}(x)\ \dot{\bar{u}}_n\right)^2\ dx \end{aligned}$$

and the ij mass coefficient (for indices 2,3,5 and 6) is

$$\begin{aligned} \bar{M}_{ij} &= \frac{\partial}{\partial \dot{\bar{u}}_i}\frac{\partial}{\partial \dot{\bar{u}}_j}\frac{1}{2}\int_{x=0}^{L} \rho A \left(\sum_{n=1}^{6} \psi_{yn}(x)\ \dot{\bar{u}}_n\right)^2\ dx \\ &= \int_{x=0}^{L} \rho A\ \psi_{yi}(x)\ \psi_{yj}(x)\ dx. \end{aligned} \tag{75}$$

3.4 Bernoulli-Euler Stiffness Matrix with Geometric Strain Effects

The axial strain in a Bernoulli-Euler beam including the geometric strain is

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} - \frac{\partial^2 u_y}{\partial x^2}\; y + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 \tag{76}$$

The potential energy with geometric strain effects is

$$\begin{aligned} V &= \frac{1}{2}\int_{x=0}^{L}\int_A \epsilon_{xx}\; E\; \epsilon_{xx}\; dA\; dx && (77)\\ &= \frac{1}{2}\int_0^L E\int_A \left(\frac{\partial u_x}{\partial x} - \frac{\partial^2 u_y}{\partial x^2}\; y + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2\right)^2 \; dx && (78)\\ &= \frac{1}{2}\int_0^L E\int_A \left(u_{x,x}^2 - 2u_{x,x}u_{y,xx}y + u_{x,x}u_{y,x}^2 + u_{y,xx}^2 y^2 - u_{y,xx}u_{y,x}^2\; y + \frac{1}{4}u_{y,x}^4\right) dAdx && (79)\end{aligned}$$

Note that $\int_A y dA = 0$ and $\int_A y^2 dA = I$ and neglect $u_{y,x}^4$ so that

$$V = \frac{1}{2}\int_0^L EA\left(u_{x,x}^2\right) dx + \frac{1}{2}\int_0^L EI\left(u_{y,xx}^2\right) dx + \int_0^L EA\left(u_{x,x}u_{y,x}^2\right) dx\;. \tag{80}$$

Substitute

$$u_{y,x} = \sum_{n=1}^{6} \psi'_{yn}(x)\; \bar{u}_n \tag{81}$$

$$u_{y,xx} = \sum_{n=1}^{6} \psi''_{yn}(x)\; \bar{u}_n \tag{82}$$

$$u_{x,x} = \sum_{n=1}^{6} \psi'_{xn}\; \bar{u}_n = \frac{N}{EA} \tag{83}$$

and differentiate with respect to $\bar{u}_i$ and $\bar{u}_j$ to obtain,

$$\bar{K}_{ij} = EA\int_0^L \psi'_{xi}\psi'_{xj}\; dx + EI\int_0^L \psi''_{yi}(x)\psi''_{yj}(x)\; dx + N\int_0^L \psi'_{yi}(x)\psi'_{yj}(x)\; dx \tag{84}$$

so that,

$$\bar{\mathbf{K}} = \bar{\mathbf{K}}_E + \frac{N}{L}\bar{\mathbf{K}}_G \tag{85}$$

3.5 Bernoulli-Euler Beam Element Stiffness and Mass Matrices

For prismatic homogeneous isotropic beams, substituting the expressions for the functions ψ_{xn} and ψ_{yn} into equations (70) - (75), or substituting equation (67) into equation (69) and (63) to equation (73) results in element stiffness matrices $\bar{\mathbf{K}}_E$, $\bar{\mathbf{M}}$, and $\bar{\mathbf{K}}_G$.

$$\bar{\mathbf{K}}_E = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix} \tag{86}$$

$$\bar{\mathbf{M}} = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ & 156 & 22L & 0 & 54 & -13L \\ & & 4L^2 & 0 & 13L & -3L^2 \\ & & & 140 & 0 & 0 \\ & \text{SYM} & & & 156 & -22L \\ & & & & & 4L^2 \end{bmatrix} \tag{87}$$

$$\bar{\mathbf{K}}_G = \frac{N}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ & & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ & & & 0 & 0 & 0 \\ & \text{SYM} & & & \frac{6}{5} & -\frac{L}{10} \\ & & & & & -\frac{2L^2}{15} \end{bmatrix} \tag{88}$$

4 Timoshenko Beam Element Matrices

2D prismatic homogeneous isotropic beam element, including shear deformation and rotatory inertia

Consider again the geometry of a deformed beam. When shear deformations are included sections that are originally perpendicular to the neutral axis may not be perpendicular to the neutral axis after deformation. The functions $u_x(x)$ and $u_y(x)$ describe the translation of

Figure 5. Deformation of beam element including shear deformation.

points along the neutral axis of the beam as a function of the location along the un-stretched neutral axis. If the beam is not slender (length/depth < 5), then shear strains will contribute significantly to the strain energy within the beam. The deformed shape of slender beams is different from the deformed shape of stocky beams.

The beam carries a bending moment $M(x)$ related to axial strain ϵ_{xx} and a shear force, S related to shear strain γ_{xy}. The potential energy has a bending strain component and a shear strain component.

$$\begin{aligned}
V &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma}^T \boldsymbol{\epsilon}\, d\Omega \\
&= \frac{1}{2}\int_\Omega \sigma_{xx}\epsilon_{xx}\, d\Omega + \frac{1}{2}\int_\Omega \tau_{xy}\gamma_{xy}\, d\Omega \\
&= \frac{1}{2}\int_0^L \int_A \frac{M(x)y}{I}\frac{M(x)y}{EI}\, dA\, dx + \frac{1}{2}\int_0^L \int_A \frac{SQ(y)}{Ib(y)}\frac{SQ(y)}{GIb(y)}\, dA\, dx \\
&= \frac{1}{2}\int_0^L \frac{M(x)^2}{EI^2}\int_A y^2\, dA\, dx + \frac{1}{2}\int_0^L \frac{S^2}{GI^2}\int_A \frac{Q(y)^2}{b(y)^2}\, dA\, dx \\
&= \frac{1}{2}\int_0^L \frac{M(x)^2}{EI}\, dx + \frac{1}{2}\int_0^L \frac{S^2}{G(A/\alpha)}\, dx
\end{aligned} \tag{89}$$

where the *shear area coefficient* α reduces the cross section area to account for the non-uniform distribution of shear stresses in the cross section,

$$\alpha = \frac{A}{I^2}\int_A \frac{Q(y)^2}{b(y)^2}\, dA\ .$$

For solid rectangular sections $\alpha = 6/5$ and for solid circular sections $\alpha = 10/9$.

4.1 Timoshenko Beam Coordinates and Internal Displacements (including shear deformation effects)

The transverse deformation of a beam with shear and bending strains may be separated into a portion related to shear deformation and a portion related to bending deformation,

$$u_y(x,t) = u_{(b)y}(x) + u_{(s)y}(x) \tag{90}$$

where

$$\begin{aligned} EI u''_{(b)y}(x) &= M(x) \quad (91) \\ G(A/\alpha) u'_{(s)y}(x) &= S(x) \quad (92) \end{aligned}$$

It can be shown that the following shape functions satisfy the Timoshenko beam equations (equations (90), (91) and (92)) for transverse displacements.

$$\begin{aligned}
\psi_{y2}(x) &= \frac{1}{1+\Phi}\left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 + \left(1 - \frac{x}{L}\right)\Phi\right] \\
\psi_{y3}(x) &= \frac{L}{1+\Phi}\left[\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)\Phi\right] \\
\psi_{y5}(x) &= \frac{1}{1+\Phi}\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 + \frac{x}{L}\Phi\right] \\
\psi_{y6}(x) &= \frac{L}{1+\Phi}\left[-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)\Phi\right]
\end{aligned}$$

The term Φ gives the relative importance of the shear deformations to the bending deformations,

$$\Phi = \frac{12EI}{G(A/\alpha)L^2} = 24\alpha(1+\nu)\left(\frac{r}{L}\right)^2 , \tag{93}$$

where r is the "radius of gyration" of the cross section, $r = \sqrt{I/A}$, ν is Poisson's ratio. Shear deformation effects are significant for beams which have a length-to-depth ratio less than 5. To neglect shear deformation, set $\Phi = 0$. These displacement functions are exact for frame elements with constant shear forces S and linearly varying bending moment distributions, $M(x)$, in which the strain energy has both a shear stress component and a normal stress component,

$$V = \frac{1}{2}\int_0^L EI\left(\sum_{n=1}^{6} \psi''_{(b)yn}(x)\bar{u}_n\right)^2 dx + \int_0^L G(A/\alpha)\left(\sum_{n=1}^{6} \psi'_{(s)yn}(x)\bar{u}_n\right)^2 dx \tag{94}$$

where the bending and shear components of the shape functions, $\psi_{(b)yn}(x)$ and $\psi_{(s)yn}(x)$ are:

$$
\begin{aligned}
\psi_{(b)y2}(x) &= \frac{1}{1+\Phi}\left[1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3\right] \\
\psi_{(s)y2}(x) &= \frac{\Phi}{1+\Phi}\left[1-\frac{x}{L}\right] \\
\psi_{(b)y3}(x) &= \frac{L}{1+\Phi}\left[\frac{x}{L}-2\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3+\frac{1}{2}\left(2\frac{x}{L}-\left(\frac{x}{L}\right)^2\right)\Phi\right] \\
\psi_{(s)y3}(x) &= -\frac{L\Phi}{1+\Phi}\left[\frac{1}{2}\frac{x}{L}\right] \\
\psi_{(b)y5}(x) &= \frac{1}{1+\Phi}\left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right] \\
\psi_{(s)y5}(x) &= \frac{\Phi}{1+\Phi}\left[\frac{x}{L}\right] \\
\psi_{(b)y6}(x) &= \frac{L}{1+\Phi}\left[-\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3+\frac{1}{2}\left(\left(\frac{x}{L}\right)^2\right)\Phi\right] \\
\psi_{(s)y6}(x) &= -\frac{L}{1+\Phi}\left[\frac{1}{2}\frac{x}{L}\Phi\right]
\end{aligned}
$$

4.2 Timoshenko Beam Element Stiffness Matrices

The geometric stiffness matrix for a Timoshenko beam element may be derived as was done with the Bernoulli-Euler beam element from the potential energy of linear and geometric strain,

$$
\begin{aligned}
\bar{K}_{ij} = \quad & EA \int_0^L \psi'_{xi}(x)\psi'_{xj}(x)\, dx \\
+ \quad & EI \int_0^L \psi''_{(b)yi}(x)\psi''_{(b)yj}(x)\, dx \\
+ \quad & G(A/\alpha) \int_0^L \psi'_{(s)yi}(x)\psi'_{(s)yj}(x)\, dx \\
+ \quad & N \int_0^L \psi'_{yi}(x)\psi'_{yj}(x)\, dx
\end{aligned} \tag{95}
$$

where the displacement shape functions $\psi(x)$ are provided in section 4.1.

4.3 Timoshenko Beam Element Stiffness and Mass Matrices, (including shear deformation effects but not rotatory inertia)

For prismatic homogeneous isotropic beams, substituting the previous expressions for the functions $\psi_{xn}(x)$ and $\psi_{(b)yn}(x)$, and $\psi_{(s)yn}(x)$ into equation (95) and (73), results in the Timoshenko element elastic stiffness matrices $\bar{\mathbf{K}}_E$, mass matrix $\bar{\mathbf{M}}$, and geometric stiffness matrix $\bar{\mathbf{K}}_G$

$$\bar{\mathbf{K}}_E = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12}{1+\Phi}\frac{EI}{L^3} & \frac{6}{1+\Phi}\frac{EI}{L^2} & 0 & -\frac{12}{1+\Phi}\frac{EI}{L^3} & \frac{6}{1+\Phi}\frac{EI}{L^2} \\ & & \frac{4+\Phi}{1+\Phi}\frac{EI}{L} & 0 & -\frac{6}{1+\Phi}\frac{EI}{L^2} & \frac{2-\Phi}{1+\Phi}\frac{EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{12}{1+\Phi}\frac{EI}{L^3} & -\frac{6}{1+\Phi}\frac{EI}{L^2} \\ & & & & & \frac{4+\Phi}{1+\Phi}\frac{EI}{L} \end{bmatrix} \quad (96)$$

$$\bar{\mathbf{M}} = \frac{\rho AL}{840}\begin{bmatrix} 280 & 0 & 0 & 140 & 0 & 0 \\ & 312+588\Phi+280\Phi^2 & (44+77\Phi+35\Phi^2)L & 0 & 108+252\Phi+175\Phi^2 & -(26+63\Phi+35\Phi^2)L \\ & & (8+14\Phi+7\Phi^2)L^2 & 0 & (26+63\Phi+35\Phi^2)L & -(6+14\Phi+7\Phi^2)L^2 \\ & & & 280 & 0 & 0 \\ & \text{SYM} & & & 312+588\Phi+280\Phi^2 & -(44+77\Phi+35\Phi^2)L \\ & & & & & (8+14\Phi+7\Phi^2)L^2 \end{bmatrix} \quad (97)$$

$$\bar{\mathbf{K}}_G = \frac{N}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{6/5+2\Phi+\Phi^2}{(1+\Phi)^2} & \frac{L/10}{(1+\Phi)^2} & 0 & \frac{-6/5-2\Phi-\Phi^2}{(1+\Phi)^2} & \frac{L/10}{(1+\Phi)^2} \\ & & \frac{2L^2/15+L^2\Phi/6+L^2\Phi^2/12}{(1+\Phi)^2} & 0 & \frac{-L/10}{(1+\Phi)^2} & \frac{-L^2/30-L^2\Phi/6-L^2\Phi^2/12}{(1+\Phi)^2} \\ & & & 0 & 0 & 0 \\ & \text{SYM} & & & \frac{6/5+2\Phi+\Phi^2}{(1+\Phi)^2} & \frac{-L/10}{(1+\Phi)^2} \\ & & & & & \frac{2L^2/15+L^2\Phi/6+L^2\Phi^2/12}{(1+\Phi)^2} \end{bmatrix} \quad (98)$$

4.4 Timoshenko Beam Element Mass Matrix (including rotatory inertia but not shear deformation effects)

Consider again the geometry of a deformed beam with linearly-varying axial beam displacements outside of the neutral axis. The functions $u_x(x, y)$ and $u_y(x, y)$ now describe the

Figure 6. Deformation of beam element showing axial-direction displacements $u_x(x, y, t)$ outside the neutral axis.

translation of points anywhere within the beam, as a function of the location within the beam. We will again describe these displacements in terms of a set of shape functions, $\psi_{xn}(x, y)$ and $\psi_{yn}(x)$, and the end displacements $\bar{u}_1, \cdots, \bar{u}_6$.

$$
\begin{aligned}
u_x(x, y, t) &= \sum_{n=1}^{6} \psi_{xn}(x, y)\ \bar{u}_n(t) \\
u_y(x, t) &= \sum_{n=1}^{6} \psi_{yn}(x)\ \bar{u}_n(t)
\end{aligned}
$$

The shape functions for transverse displacements $\psi_{yn}(x)$ are the same as the shape functions $\psi_{yn}(x)$ used previously. The shape functions for axial displacements along the neutral axis, $\psi_{x1}(x, y)$ and $\psi_{x4}(x, y)$ are also the same as the shape functions $\psi_{x1}(x)$ and $\psi_{x4}(x)$ used previously. To account for axial displacements outside of the neutral axis, four new shape functions are derived from the assumption that plane sections remain plane, $u_x(x, y) = -u'_{(b)y}(x)y$.

$$
\begin{aligned}
\psi_{x2}(x, y) &= -\psi'_{y2}\ y = \ 6\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)\frac{y}{L} \\
\psi_{x3}(x, y) &= -\psi'_{y3}\ y = \ \left(-1 + 4\frac{x}{L} - 3\left(\frac{x}{L}\right)^2\right) y \\
\psi_{x5}(x, y) &= -\psi'_{y5}\ y = \ 6\left(-\frac{x}{L} + \left(\frac{x}{L}\right)^2\right)\frac{y}{L} \\
\psi_{x6}(x, y) &= -\psi'_{y6}\ y = \ \left(2\frac{x}{L} - 3\left(\frac{x}{L}\right)^2\right) y
\end{aligned}
$$

Because ψ_{yn}, ψ_{x1} and ψ_{x4} are unchanged, the stiffness matrix is also unchanged. The kinetic energy of the beam, including axial and transverse effects is now,

$$T = \frac{1}{2}\int_{x=0}^{L}\int_{y=-h/2}^{h/2} \rho b(y) \left(\sum_{n=1}^{6} \psi_{xn}(x,y)\ \dot{\bar{u}}_n\right)^2 \ dy\ dx + \frac{1}{2}\int_{x=0}^{L} \rho A \left(\sum_{n=1}^{6} \psi_{yn}(x)\ \dot{\bar{u}}_n\right)^2 \ dx \quad (99)$$

and the mass matrix coefficients are found from

$$\bar{M}_{ij} = \frac{\partial}{\partial \dot{\bar{u}}_i}\frac{\partial}{\partial \dot{\bar{u}}_j} T(\dot{\bar{\mathbf{u}}})$$

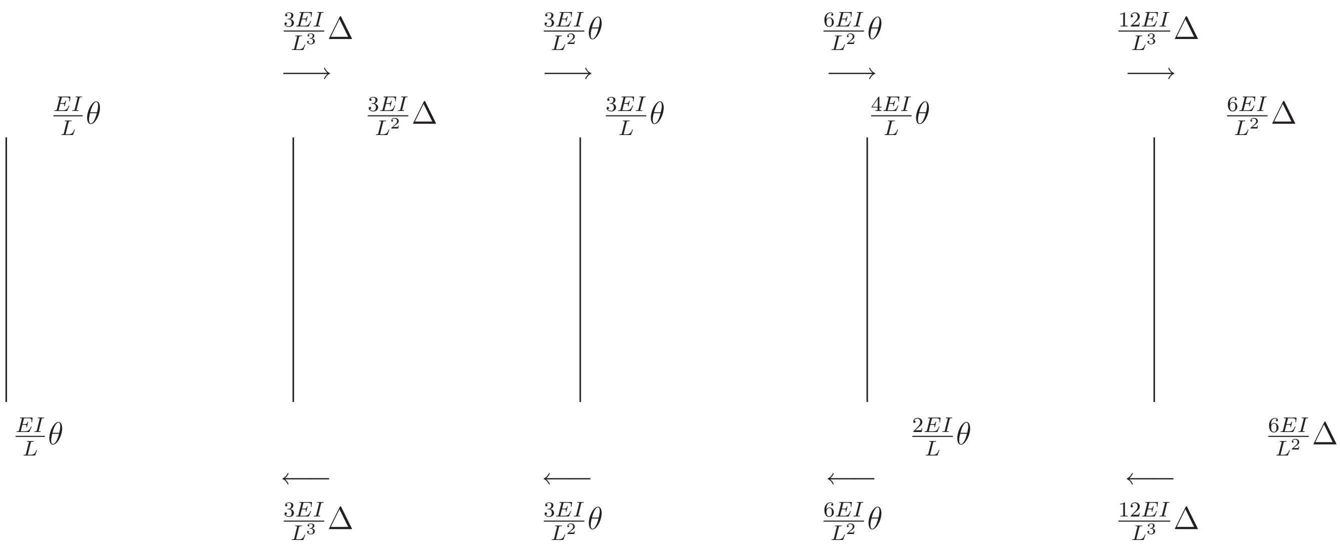
Evaluating equation (28) using the new shape functions ψ_{x2}, ψ_{x3}, ψ_{x5}, and ψ_{x6}, results in a mass matrix incorporating rotatory inertia.

$$\bar{\mathbf{M}} = \rho A L \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ & \frac{13}{35} + \frac{6}{5}\frac{r^2}{L^2} & \frac{11}{210}L + \frac{1}{10}\frac{r^2}{L} & 0 & \frac{9}{70} - \frac{6}{5}\frac{r^2}{L^2} & -\frac{13}{420}L + \frac{1}{10}\frac{r^2}{L} \\ & & \frac{1}{105}L^2 + \frac{2}{15}r^2 & 0 & \frac{13}{420}L + \frac{1}{10}\frac{r^2}{L} & 0 \\ & \text{SYM} & & \frac{1}{3} & 0 & 0 \\ & & & & \frac{13}{35} + \frac{6}{5}\frac{r^2}{L^2} & -\frac{11}{210}L + \frac{1}{10}\frac{r^2}{L} \\ & & & & & \frac{1}{105}L^2 + \frac{2}{15}r^2 \end{bmatrix} \quad (100)$$

Beam element mass matrices including the effects of shear deformation on rotatory inertia are more complicated. Refer to p 295 of *Theory of Matrix Structural Analysis,* by J.S. Przemieniecki (Dover Pub., 1985).

5 Coordinate Transformations for Bars and Beams

5.1 Beam Element Stiffness Matrix in Local Coordinates, $\bar{\mathbf{K}}$



$$
\begin{Bmatrix} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{Bmatrix} =
\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
 & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
 & & & \frac{EA}{L} & 0 & 0 \\
 & \text{SYM} & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
 & & & & & \frac{4EI}{L}
\end{bmatrix}
\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \end{Bmatrix}
$$

$$\bar{\mathbf{f}} = \bar{\mathbf{K}}\ \bar{\mathbf{u}}$$

5.2 Beam Element Stiffness Matrix in Global Coordinates, $\mathbf{K}$

Geometric relationship between $\bar{\mathbf{u}}$ and $\mathbf{u}$: $\bar{\mathbf{u}} = \mathbf{T}\ \mathbf{u}$

$$\bar{u}_1 = u_1 \cos\theta + u_2 \sin\theta \qquad \bar{u}_2 = -u_1 \sin\theta + u_2 \cos\theta \qquad \bar{u}_3 = u_3$$

where

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \qquad c = \cos\theta = \frac{x_2 - x_1}{L} \qquad s = \sin\theta = \frac{y_2 - y_1}{L}$$

The stiffness matrix in global coordinates is $\mathbf{K} = \mathbf{T}^T\ \bar{\mathbf{K}}\ \mathbf{T}$

$$\mathbf{K} = \begin{bmatrix}
\frac{EA}{L}c^2 + \frac{12EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{12EI}{L^3}cs & -\frac{6EI}{L^2}s & -\frac{EA}{L}c^2 - \frac{12EI}{L^3}s^2 & -\frac{EA}{L}cs + \frac{12EI}{L^3}cs & -\frac{6EI}{L^2}s \\
 & \frac{EA}{L}s^2 + \frac{12EI}{L^3}c^2 & \frac{6EI}{L^2}c & -\frac{EA}{L}cs + \frac{12EI}{L^3}cs & -\frac{EA}{L}s^2 - \frac{12EI}{L^3}c^2 & \frac{6EI}{L^2}c \\
 & & \frac{4EI}{L} & \frac{6EI}{L^2}s & -\frac{6EI}{L^2}c & \frac{2EI}{L} \\
 & & & \frac{EA}{L}c^2 + \frac{12EI}{L^3}s^2 & \frac{EA}{L}cs - \frac{12EI}{L^3}cs & \frac{6EI}{L^2}s \\
 & \text{SYM} & & & \frac{EA}{L}s^2 + \frac{12EI}{L^3}c^2 & -\frac{6EI}{L^2}c \\
 & & & & & \frac{4EI}{L}
\end{bmatrix}$$

$$\mathbf{f} = \mathbf{K}\ \mathbf{u}$$

5.3 Beam Element Consistent Mass Matrix in Local Coordinates, $\bar{\mathbf{M}}$

$$\left\{ \begin{array}{c} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{array} \right\} = \frac{\rho A L}{420} \left[\begin{array}{cccccc} 140 & 0 & 0 & 70 & 0 & 0 \\ & 156 & 22L & 0 & 54 & -13L \\ & & 4L^2 & 0 & 13L & -3L^2 \\ & & & 140 & 0 & 0 \\ & \text{SYM} & & & 156 & -22L \\ & & & & & 4L^2 \end{array} \right] \left\{ \begin{array}{c} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{array} \right\}$$

$$\bar{\mathbf{f}} = \bar{\mathbf{M}}\ \ddot{\mathbf{u}}$$

5.4 Beam Element Consistent Mass Matrix in Global Coordinates, $\mathbf{M}$

Geometric relationship between $\bar{\mathbf{u}}$ and $\mathbf{u}$: $\bar{\mathbf{u}} = \mathbf{T}\ \mathbf{u}$

$$\bar{u}_1 = u_1 \cos\theta + u_2 \sin\theta \qquad \bar{u}_2 = -u_1 \sin\theta + u_2 \cos\theta \qquad \bar{u}_3 = u_3$$

where

$$\mathbf{T} = \left[\begin{array}{cccccc} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{array} \right] \qquad c = \cos\theta = \frac{x_2 - x_1}{L} \qquad s = \sin\theta = \frac{y_2 - y_1}{L}$$

The consistent mass matrix in global coordinates is $\mathbf{M} = \mathbf{T}^T\ \bar{\mathbf{M}}\ \mathbf{T}$

$$\mathbf{M} = \frac{\rho A L}{420} \left[\begin{array}{cccccc} 140c^2 + 15s^2 & -16cs & -22sL & 70c^2 + 54s^2 & 16cs & 13sL \\ & 140s^2 + 156c^2 & 22cL & 16cs & 70s^2 + 54c^2 & -13cL \\ & & 4L^2 & -13sL & 13cL & -3L^2 \\ & & & 140c^2 + 156s^2 & -16cs & 22sL \\ & \text{SYM} & & & 140s^2 + 156c^2 & -22cL \\ & & & & & 4L^2 \end{array} \right]$$

$$\mathbf{f} = \mathbf{M}\ \mathbf{u}$$

6 2D Rectangular Plate Plane-Stress Element Matrices

2D, plane-stress, isotropic, homogeneous plate element.

Uniform thickness h.

Approximate element stiffness and mass matrices based on assumed distribution of internal displacements.

For plane-stress elasticity, the stress-strain relationship simplifies to

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{101}$$

or

$$\boldsymbol{\sigma} = \mathbf{S}\, \boldsymbol{\epsilon} \tag{102}$$

6.1 2D Plate Coordinates and Internal Displacements

Consider the geometry of a rectangular plate with edges aligned with a Cartesian coordinate system. The functions $\bar{u}_x(x, y, t)$ and $\bar{u}_y(x, y, t)$ describe the in-plane displacements of the plate as a function of the location within the plate. Internal displacements are assumed to

Figure 7. 2D Plate element coordinates and displacements.

vary linearly within the plate.

$$\begin{aligned} u_x(x, y, t) &= c_1 \frac{x}{a} + c_2 \frac{x}{a}\frac{y}{b} + c_3 \frac{y}{b} + c_4 \\ u_y(x, y, t) &= c_5 \frac{x}{a} + c_6 \frac{x}{a}\frac{y}{b} + c_7 \frac{y}{b} + c_8 \end{aligned}$$

The eight coefficients $c_1, \cdots, c_8$ may be found uniquely from matching the displacement coordinates at the corners.

$$\begin{aligned} u_x(0,0) &= \bar{u}_1 \quad , \quad u_y(0,0) = \bar{u}_2 \\ u_x(0,b) &= \bar{u}_3 \quad , \quad u_y(0,b) = \bar{u}_4 \\ u_x(a,b) &= \bar{u}_5 \quad , \quad u_y(a,b) = \bar{u}_6 \\ u_x(a,0) &= \bar{u}_7 \quad , \quad u_y(a,0) = \bar{u}_8 \end{aligned}$$

resulting in internal plate displacements

$$u_x(x,y,t) = (1-\hat{x})(1-\hat{y})\ \bar{u}_1(t) + (1-\hat{x})\hat{y}\ \bar{u}_3(t) + \hat{x}\hat{y}\ \bar{u}_5(t) + \hat{x}(1-\hat{y})\ \bar{u}_7(t) \quad (103)$$

$$u_y(x,y,t) = (1-\hat{x})(1-\hat{y})\ \bar{u}_2(t) + (1-\hat{x})\hat{y}\ \bar{u}_4(t) + \hat{x}\hat{y}\ \bar{u}_6(t) + \hat{x}(1-\hat{y})\ \bar{u}_8(t) \quad (104)$$

where $\hat{x} = x/a$ and $\hat{y} = y/b$ so that

$$\boldsymbol{\Psi}(\hat{x},\hat{y}) = \left[\begin{array}{c|c|c|c|c|c|c|c} (1-\hat{x})(1-\hat{y}) & 0 & (1-\hat{x})\hat{y} & 0 & \hat{x}\hat{y} & 0 & \hat{x}(1-\hat{y}) & 0 \\ \hline 0 & (1-\hat{x})(1-\hat{y}) & 0 & (1-\hat{x})\hat{y} & 0 & \hat{x}\hat{y} & 0 & \hat{x}(1-\hat{y}) \end{array}\right] \quad (105)$$

and

$$\begin{bmatrix} u_x(x,y,t) \\ u_y(x,y,t) \end{bmatrix} = \boldsymbol{\Psi}(\hat{x},\hat{y})\ \bar{\mathbf{u}}(t) \quad (106)$$

Strain-displacement relations

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial \bar{u}_x}{\partial x} = \frac{1}{a}\frac{\partial u_x}{\partial \hat{x}} \\ \epsilon_{yy} &= \frac{\partial u_y}{\partial y} = \frac{1}{b}\frac{\partial u_y}{\partial \hat{y}} \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{1}{b}\frac{\partial u_x}{\partial \hat{y}} + \frac{1}{a}\frac{\partial u_y}{\partial \hat{x}} \end{aligned}$$

so that

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{a}(1-\hat{y}) & 0 & -\frac{\hat{y}}{a} & 0 & \frac{\hat{y}}{a} & 0 & \frac{1}{a}(1-\hat{y}) & 0 \\ 0 & -\frac{1}{b}(1-\hat{x}) & 0 & -\frac{1}{b}(1-\hat{x}) & 0 & \frac{\hat{x}}{b} & 0 & -\frac{\hat{x}}{b} \\ -\frac{1}{b}(1-\hat{x}) & -\frac{1}{a}(1-\hat{y}) & \frac{1}{b}(1-\hat{x}) & -\frac{\hat{y}}{a} & -\frac{\hat{x}}{b} & \frac{\hat{y}}{a} & -\frac{\hat{x}}{b} & \frac{1}{a}(1-\hat{y}) \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \\ \bar{u}_7 \\ \bar{u}_8 \end{bmatrix} \quad (107)$$

or

$$\boldsymbol{\epsilon}(x,y,t) = \mathbf{B}(x,y)\ \bar{\mathbf{u}}(t)$$

6.2 2D Plate Strain Energy and Elastic Stiffness Matrix

$$V = \frac{1}{2}\int_A \boldsymbol{\sigma}(x,y,t)^T \boldsymbol{\epsilon}(x,y,t)\ h\ dx\ dy \quad (108)$$

$$= \frac{1}{2}\ \bar{\mathbf{u}}(t)^T \int_A \left[\mathbf{B}(x,y)^T\ \mathbf{S}(E,\nu)\ \mathbf{B}(x,y)\right]_{8\times 8} h\ dx\ dy\ \bar{\mathbf{u}}(t) \quad (109)$$

Elastic element stiffness matrix

$$\bar{\mathbf{K}}_E = \int_A \left[\mathbf{B}(x,y)^T\ \mathbf{S}(E,\nu)\ \mathbf{B}(x,y)\right]_{8\times 8} h\ dx\ dy \quad (110)$$

6.3 2D Plate Kinetic Energy and Mass Matrix

$$
\begin{aligned}
T(\dot{\bar{\mathbf{u}}}) &= \frac{1}{2}\int_A \rho\, |\dot{\mathbf{u}}(x,y,t)|^2 \; h \; dx \; dy & (111)\\
&= \frac{1}{2}\, \dot{\bar{\mathbf{u}}}(t)^T \int_A \rho \; \Big[\boldsymbol{\Psi}(x,y)^T \;\; \boldsymbol{\Psi}(x,y)\Big]_{8\times 8} \; h \; dx \; dy \; \dot{\bar{\mathbf{u}}}(t) & (112)
\end{aligned}
$$

Consistent mass matrix

$$
\bar{\mathbf{M}} = \int_A \rho \; \Big[\boldsymbol{\Psi}(x,y)^T \;\; \boldsymbol{\Psi}(x,y)\Big]_{8\times 8} \; h \; dx \; dy \tag{113}
$$

6.4 2D Plate Element Stiffness and Mass Matrix

$$
\bar{\mathbf{K}}_E = \frac{Eh}{12(1-\nu^2)} \cdot
$$

$$
\begin{bmatrix}
4c+\frac{2}{c}(1-\nu) & \frac{3}{2}(1+\nu) & 2c-\frac{2}{c}(1-\nu) & \frac{3}{2}(1-3\nu) & -2c-\frac{1}{c}(1-\nu) & -\frac{3}{2}(1+\nu) & -4c+\frac{1}{c}(1-\nu) & -\frac{3}{2}(1-3\nu) \\
 & \frac{4}{c}+2c(1-\nu) & -\frac{3}{2}(1-3\nu) & -\frac{4}{c}+c(1-\nu) & -\frac{3}{2}(1+\nu) & -\frac{2}{c}-c(1-\nu) & \frac{3}{2}(1-3\nu) & \frac{2}{c}-2c(1-\nu) \\
 & & 4c+\frac{2}{c}(1-\nu) & -\frac{3}{2}(1+\nu) & -4c+\frac{1}{c}(1-\nu) & \frac{3}{2}(1-3\nu) & -2c-\frac{1}{c}(1-\nu) & \frac{3}{2}(1+\nu) \\
 & & & \frac{4}{c}+2c(1-\nu) & -\frac{3}{2}(1-3\nu) & \frac{2}{c}-2c(1-\nu) & \frac{3}{2}(1+\nu) & -\frac{2}{c}-c(1-\nu) \\
 & & & & 4c+\frac{2}{c}(1-\nu) & \frac{3}{2}(1+\nu) & 2c-\frac{2}{c}(1-\nu) & \frac{3}{2}(1-3\nu) \\
 & & & & & \frac{4}{c}+2c(1-\nu) & -\frac{3}{2}(1-3\nu) & -\frac{4}{c}+c(1-\nu) \\
 & & & & & & 4c+\frac{2}{c}(1-\nu) & -\frac{3}{2}(1+\nu) \\
 & & & & & & & \frac{4}{c}+2c(1-\nu)
\end{bmatrix} \tag{114}
$$

where $c = b/a$.

$$
\bar{\mathbf{M}} = \frac{\rho a b h}{36}
\begin{bmatrix}
4 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\
0 & 4 & 0 & 2 & 0 & 1 & 0 & 2 \\
2 & 0 & 4 & 0 & 2 & 0 & 1 & 0 \\
0 & 2 & 0 & 4 & 0 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 & 4 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 & 0 & 4 & 0 & 2 \\
2 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \\
0 & 2 & 0 & 1 & 0 & 2 & 0 & 4
\end{bmatrix} \tag{115}
$$

Note, again, that these element stiffness matrices are approximations based on an assumed distribution of internal displacements.

References

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[3] Przemieniecki, J.S., *Theory of Matrix Structural Analysis,* Dover, 1985. ⋆