

BEAM ELEMENT

STIFFNESS AND MASS MATRICES

CE 283 — Structural Dynamics

Henri Gavin
Fall, 2007

1 Preliminaries

1.1 Potential Energy and Stiffness

Consider a system comprising an assemblage of linear springs, with stiffnesses k_i , each with an individual stretch, d_i . The total potential energy in the assemblage is

$$V = \frac{1}{2} \sum_i k_i d_i^2$$

If displacements of the assemblage of springs is denoted by a vector \mathbf{r} , not necessarily equal to the stretches in each spring, then the elastic potential energy may also be written

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{2} \mathbf{r}^T \mathbf{K} \mathbf{r} \\ &= \frac{1}{2} \sum_{i=1}^n r_i f_i \\ &= \frac{1}{2} \sum_{i=1}^n r_i \sum_{j=1}^n K_{ij} r_j \end{aligned}$$

where \mathbf{K} is the stiffness matrix with respect to the coordinates \mathbf{r} . The stiffness matrix \mathbf{K} relates the elastic forces f_i to the collocated displacements, r_i .

$$\begin{aligned} f_1 &= K_{11}r_1 + \cdots + K_{1n}r_n \\ f_i &= K_{i1}r_1 + \cdots + K_{in}r_n \\ f_n &= K_{n1}r_1 + \cdots + K_{nn}r_n \end{aligned}$$

A point force f_i acting on an elastic body is the gradient of the elastic potential energy V with respect to the collocated displacement r_i

$$f_i = \frac{\partial}{\partial r_i} V$$

The i, j term of the stiffness matrix may therefore be found from the potential energy function $V(\mathbf{r})$,

$$K_{ij} = \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} V(\mathbf{r}) \quad (1)$$

1.2 Kinetic Energy and Mass

Consider a system comprising an assemblage of point masses, m_i , each with an individual velocity, v_i . The total kinetic energy in the assemblage is

$$T = \frac{1}{2} \sum_i m_i v_i^2$$

If displacements of the assemblage of masses is denoted by a vector \mathbf{r} , not necessarily equal to the velocity coordinates, above, then the kinetic energy may also be written

$$\begin{aligned} T(\dot{\mathbf{r}}) &= \frac{1}{2} \dot{\mathbf{r}}^T \mathbf{M} \dot{\mathbf{r}} \\ &= \frac{1}{2} \sum_{i=1}^n \dot{r}_i \sum_{j=1}^n M_{ij} \dot{r}_j \end{aligned}$$

where \mathbf{M} is the constant mass matrix with respect to the coordinates \mathbf{r} . The mass matrix \mathbf{M} relates the inertial forces f_i to the collocated accelerations, \ddot{r}_i .

$$\begin{aligned} f_1 &= M_{11}\ddot{r}_1 + \cdots + M_{1n}\ddot{r}_n \\ f_i &= M_{i1}\ddot{r}_1 + \cdots + M_{in}\ddot{r}_n \\ f_n &= M_{n1}\ddot{r}_1 + \cdots + M_{nn}\ddot{r}_n \end{aligned}$$

The impulse-momentum relationship states that

$$\begin{aligned} \int f dt &= \delta(m\dot{r}) \\ f &= \frac{d}{dt}(m\dot{r}) \\ f &= \frac{d}{dt} \left(\frac{\partial}{\partial \dot{r}} \frac{1}{2} m \dot{r}^2 \right) \\ f &= \frac{d}{dt} \left(\frac{\partial}{\partial \dot{r}} T \right), \end{aligned}$$

where T is the kinetic energy.

The i, j term of the constant mass matrix may therefore be found from the kinetic energy function T ,

$$M_{ij} = \frac{\partial}{\partial \ddot{r}_i} \frac{\partial}{\partial t} \frac{\partial}{\partial \dot{r}_j} T(\dot{\mathbf{r}}) = \frac{\partial}{\partial \dot{r}_i} \frac{\partial}{\partial \dot{r}_j} T(\dot{\mathbf{r}}) \quad (2)$$

2 Beam Displacements

Consider the geometry of a deformed beam. The functions $u_x(x)$ and $u_y(x)$ describe the

translation of points along the neutral axis of the beam as a function of the location along the un-stretched neutral axis.

We will describe the deformation of the beam as a function of the end displacements (u_1, u_2, u_4, u_5) and the end rotations (u_3, u_6) . In a dynamic context, these end displacements will change with time.

$$\begin{aligned} u_x(x, t) &= \sum_{k=1}^6 \psi_{xk}(x) u_k(t) \\ u_y(x, t) &= \sum_{k=1}^6 \psi_{yk}(x) u_k(t) \end{aligned}$$

The functions $\psi_{xk}(x)$ and $\psi_{yk}(x)$ satisfy the boundary conditions at the end of the beam and the differential equation describing bending of a Bernouli-Euler beam.. For extension of the neutral axis,

$$\begin{aligned} \psi_{x1}(x) &= 1 - \frac{x}{L} \\ \psi_{x4}(x) &= \frac{x}{L} \end{aligned}$$

and $\psi_{x2} = \psi_{x3} = \psi_{x5} = \psi_{x6} = 0$ along the neutral axis. For bending of the neutral axis,

$$\begin{aligned} \psi_{y2}(x) &= 1 - 3 \left(\frac{x}{L}\right)^2 + 2 \left(\frac{x}{L}\right)^3 \\ \psi_{y3}(x) &= \left(\frac{x}{L} - 2 \left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) L \\ \psi_{y5}(x) &= 3 \left(\frac{x}{L}\right)^2 - 2 \left(\frac{x}{L}\right)^3 \\ \psi_{y6}(x) &= \left(-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) L \end{aligned}$$

and $\psi_{y1} = \psi_{y4} = 0$.

These expressions are analytical solutions for the displacements of Bernouli-Euler beams loaded only with concentrated point loads and concentrated point moments at their ends.

3 Potential Energy and Stiffness

In extension, the elastic potential energy in a beam is the strain energy related to the uniform extensional strain, ϵ_{xx} . If the strain is small, then

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \sum_{k=1}^6 \frac{\partial}{\partial x} \psi_{xk}(x) u_k = \sum_{k=1}^6 \psi'_{xk}(x) u_k$$

The elastic strain energy is for extension

$$\begin{aligned} V &= \frac{1}{2} \int_{x=0}^L EA (\epsilon_{xx})^2 dx \\ &= \frac{1}{2} \int_{x=0}^L EA \left(\sum_{k=1}^6 \psi'_{xk}(x) u_k \right)^2 dx \end{aligned}$$

and the ij stiffness coefficient (for indices 1 and 4) is

$$\begin{aligned} K_{ij} &= \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \frac{1}{2} \int_{x=0}^L EA \left(\sum_{k=1}^6 \psi'_{xk}(x) u_k \right)^2 dx \\ &= \int_{x=0}^L EA \psi'_{xi}(x) \psi'_{xj}(x) dx. \end{aligned} \quad (3)$$

In bending, the elastic potential energy in a Bernouli-Euler beam is the strain energy related to the curvature, ϕ_z . The strain energy related to shearing strain is neglected, for the time being.

$$\phi_z = \frac{\partial^2 u_y}{\partial x^2} = \sum_{k=1}^6 \frac{\partial^2}{\partial x^2} \psi_{yk}(x) u_k = \sum_{k=1}^6 \psi''_{yk}(x) u_k$$

The elastic strain energy for pure bending is

$$\begin{aligned} V &= \frac{1}{2} \int_{x=0}^L EI (\phi_z)^2 dx \\ &= \frac{1}{2} \int_{x=0}^L EI \left(\sum_{k=1}^6 \psi''_{yk}(x) u_k \right)^2 dx \end{aligned}$$

and the ij stiffness coefficient (for indices 2,3,5 and 6) is

$$\begin{aligned} K_{ij} &= \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \frac{1}{2} \int_{x=0}^L EI \left(\sum_{k=1}^6 \psi''_{yk}(x) u_k \right)^2 dx \\ &= \int_{x=0}^L EI \psi''_{yi}(x) \psi''_{yj}(x) dx. \end{aligned} \quad (4)$$

4 Kinetic Energy and Mass

The kinetic energy of a particle within a beam is the mass of the particle, $\rho A dx$, times its velocity, \dot{u} , squared. For velocities along the direction of the neutral axis,

$$\dot{u}_x(x) = \sum_{k=1}^6 \psi_{xk}(x) \dot{u}_k ,$$

the kinetic energy for extension of the neutral axis is

$$\begin{aligned} T &= \frac{1}{2} \int_{x=0}^L \rho A (\dot{u}_x)^2 dx \\ &= \frac{1}{2} \int_{x=0}^L \rho A \left(\sum_{k=1}^6 \psi_{xk}(x) \dot{u}_k \right)^2 dx \end{aligned}$$

and the ij mass coefficient (for indices 1 and 4) is

$$\begin{aligned} M_{ij} &= \frac{\partial}{\partial \dot{u}_i} \frac{\partial}{\partial \dot{u}_j} \frac{1}{2} \int_{x=0}^L \rho A \left(\sum_{k=1}^6 \psi_{xk}(x) \dot{u}_k \right)^2 dx \\ &= \int_{x=0}^L \rho A \psi_{xi}(x) \psi_{xj}(x) dx. \end{aligned} \quad (5)$$

For velocities transverse to the neutral axis,

$$\dot{u}_y(x) = \sum_{k=1}^6 \psi_{yk}(x) \dot{u}_k ,$$

the kinetic energy for velocity across the neutral axis is

$$\begin{aligned} T &= \frac{1}{2} \int_{x=0}^L \rho A (\dot{u}_y)^2 dx \\ &= \frac{1}{2} \int_{x=0}^L \rho A \left(\sum_{k=1}^6 \psi_{yk}(x) \dot{u}_k \right)^2 dx \end{aligned}$$

and the ij mass coefficient (for indices 2,3,5 and 6) is

$$\begin{aligned} M_{ij} &= \frac{\partial}{\partial \dot{u}_i} \frac{\partial}{\partial \dot{u}_j} \frac{1}{2} \int_{x=0}^L \rho A \left(\sum_{k=1}^6 \psi_{yk}(x) \dot{u}_k \right)^2 dx \\ &= \int_{x=0}^L \rho A \psi_{yi}(x) \psi_{yj}(x) dx. \end{aligned} \quad (6)$$

5 Beam Element Stiffness and Mass Matrices

For prismatic beams, substituting the expressions for the functions ψ_{xk} and ψ_{yk} into equations (3) - (6), results in element stiffness matrices \mathbf{K} and \mathbf{M} ,

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix} \quad (7)$$

$$\mathbf{M} = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ & 156 & 22L & 0 & 54 & -13L \\ & & 4L^2 & 0 & 13L & -3L^2 \\ & & & 140 & 0 & 0 \\ & \text{SYM} & & & 156 & -22L \\ & & & & & 4L^2 \end{bmatrix} \quad (8)$$

6 Beam Displacements including Shear Deformation

Consider again the geometry of a deformed beam. When shear deformations are included sections that are originally perpendicular to the neutral axis may not be perpendicular to the neutral axis after deformation.

The functions $u_x(x)$ and $u_y(x)$ describe the translation of points along the neutral axis of the beam as a function of the location along the un-stretched neutral axis. If the beam is not slender (length/depth < 2), then shear strains will contribute significantly to the strain energy within the beam. The deformed shape of slender beams is different from the deformed shape of stocky beams. When shear deformations are significant, the following shape functions satisfy the Timoshenko beam equation for transverse displacements.

$$\begin{aligned}\psi_{y2}(x) &= 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 + \left(1 - \frac{x}{L}\right)\Phi \\ \psi_{y3}(x) &= \left(\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)\Phi\right)L \\ \psi_{y5}(x) &= 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 + \frac{x}{L}\Phi \\ \psi_{y6}(x) &= \left(-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)\Phi\right)L\end{aligned}$$

The term Φ gives the relative importance of the shear deformations to the bending deformations,

$$\Phi = \frac{12EI}{G(A/\alpha)L^2} = 24\alpha(1 + \nu)\left(\frac{r}{L}\right)^2,$$

where r is the “radius of gyration” of the cross section, $r = \sqrt{I/A}$, ν is Poisson’s ratio, and α is a dimensionless reduction factor for the cross section to account for the non-uniform distribution of shear stresses in the cross section. To neglect shear deformation, set $\Phi = 0$.

7 Beam Element Stiffness and Mass Matrices, including Shear Deformation

For prismatic beams, substituting the previous expressions for the functions ψ_{xk} and ψ_{yk} into equations (3) - (6), results in new element stiffness matrices \mathbf{K} and \mathbf{M} ,

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12}{1+\Phi} \frac{EI}{L^3} & \frac{6}{1+\Phi} \frac{EI}{L^2} & 0 & -\frac{12}{1+\Phi} \frac{EI}{L^3} & \frac{6}{1+\Phi} \frac{EI}{L^2} \\ & & \frac{4+\Phi}{1+\Phi} \frac{EI}{L} & 0 & -\frac{6}{1+\Phi} \frac{EI}{L^2} & \frac{2-\Phi}{1+\Phi} \frac{EI}{L} \\ & \text{SYM} & & \frac{EA}{L} & 0 & 0 \\ & & & & \frac{12}{1+\Phi} \frac{EI}{L^3} & -\frac{6}{1+\Phi} \frac{EI}{L^2} \\ & & & & & \frac{4+\Phi}{1+\Phi} \frac{EI}{L} \end{bmatrix} \quad (9)$$

$$\mathbf{M} = \frac{\rho AL}{840} \begin{bmatrix} 280 & 0 & 0 \\ & 312 + 588\Phi + 280\Phi^2 & (44 + 77\Phi + 35\Phi^2)L \\ & & (8 + 14\Phi + 7\Phi^2)L^2 \\ & \text{SYM} & \\ & & & 140 & 0 & 0 \\ & & & & 0 & 108 + 252\Phi + 175\Phi^2 & -(26 + 63\Phi + 35\Phi^2)L \\ & & & & & & (26 + 63\Phi + 35\Phi^2)L & -(6 + 14\Phi + 7\Phi^2)L^2 \\ & & & & & & & & 280 & 0 & 0 \\ & & & & & & & & & & & 312 + 588\Phi + 280\Phi^2 & -(44 + 77\Phi + 35\Phi^2)L \\ & & & & & & & & & & & & (8 + 14\Phi + 7\Phi^2)L^2 \end{bmatrix} \quad (10)$$

8 Beam Displacements outside of the Neutral Axis

Consider again the geometry of a deformed beam.

The functions $u_x(x, y)$ and $u_y(x, y)$ now describe the translation of points anywhere within the beam, as a function of the location within the beam. We will again describe these displacements in terms of a set of shape functions, $\psi_{xk}(x, y)$ and $\psi_{yk}(x, y)$, and the end displacements u_1, \dots, u_6 .

$$u_x(x, y, t) = \sum_{k=1}^6 \psi_{xk}(x, y) u_k(t)$$

$$u_y(x, y, t) = \sum_{k=1}^6 \psi_{yk}(x, y) u_k(t)$$

The shape functions for transverse displacements $\psi_{yk}(x, y)$ are the same as the shape functions $\psi_{yk}(x)$ used previously. The shape functions for axial displacements along the neutral axis, $\psi_{x1}(x, y)$ and $\psi_{x4}(x, y)$ are also the same as the shape functions $\psi_{x1}(x)$ and $\psi_{x4}(x)$ used previously. To account for axial displacements outside of the neutral axis, four new shape functions are defined

$$\psi_{x2}(x, y) = 6 \left(\frac{x}{L} - \left(\frac{x}{L} \right)^2 \right) y$$

$$\psi_{x3}(x, y) = \left(-1 + 4 \frac{x}{L} - 3 \left(\frac{x}{L} \right)^2 \right) Ly$$

$$\psi_{x5}(x, y) = 6 \left(-\frac{x}{L} + \left(\frac{x}{L} \right)^2 \right) y$$

$$\psi_{x6}(x, y) = \left(2 \frac{x}{L} - 3 \left(\frac{x}{L} \right)^2 \right) Ly$$

Because ψ_{yk} , ψ_{x1} and ψ_{x4} are unchanged, the stiffness matrix is also unchanged. The kinetic energy of the beam, including axial and transverse effects is now,

$$T = \frac{1}{2} \int_{x=0}^L \int_{y=-h/2}^{h/2} \rho b(y) \left(\sum_{k=1}^6 \psi_{xk}(x, y) \dot{u}_k \right)^2 dy dx + \frac{1}{2} \int_{x=0}^L \rho A \left(\sum_{k=1}^6 \psi_{yk}(x) \dot{u}_k \right)^2 dx$$

and the mass matrix coefficients are found from

$$M_{ij} = \frac{\partial}{\partial \dot{r}_i} \frac{\partial}{\partial \dot{r}_j} T(\dot{\mathbf{r}})$$

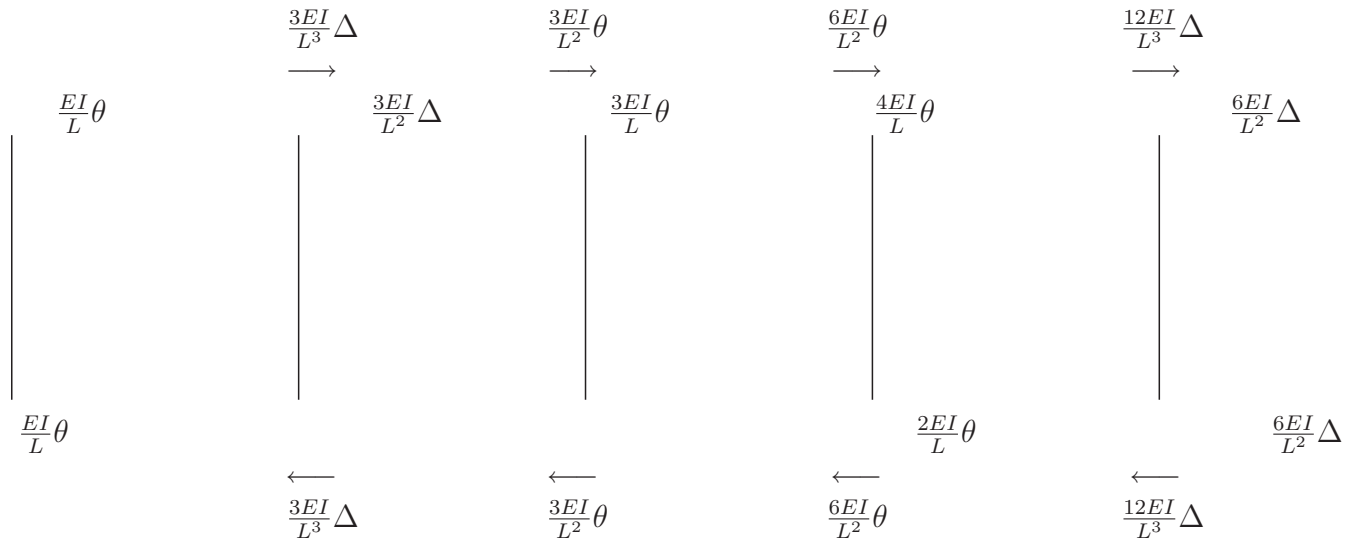
9 Beam Element Mass Matrix, including Rotatory Inertia

Evaluating equation (2) using the new shape functions ψ_{x2} , ψ_{x3} , ψ_{x5} , and ψ_{x6} , results in a mass matrix incorporating rotatory inertia.

$$\mathbf{M} = \rho AL \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ \frac{13}{35} + \frac{6}{5} \frac{r^2}{L^2} & \frac{11}{210} L + \frac{1}{10} \frac{r^2}{L} & 0 & \frac{9}{70} - \frac{6}{5} \frac{r^2}{L^2} & -\frac{13}{420} L + \frac{1}{10} \frac{r^2}{L} \\ & \frac{1}{105} L^2 + \frac{2}{15} r^2 & 0 & \frac{13}{420} L + \frac{1}{10} \frac{r^2}{L} & 0 \\ \text{SYM} & & \frac{1}{3} & 0 & 0 \\ & & & \frac{13}{35} + \frac{6}{5} \frac{r^2}{L^2} & -\frac{11}{210} L + \frac{1}{10} \frac{r^2}{L} \\ & & & & \frac{1}{105} L^2 + \frac{2}{15} r^2 \end{bmatrix} \quad (11)$$

Beam element mass matrices including the effects of shear deformation on rotatory inertia are more complicated. Refer to p 295 of *Theory of Matrix Structural Analysis*, by J.S. Przemieniecki (Dover Pub., 1985).

10 Beam Element Stiffness Matrix in Local Coordinates, \mathbf{K}



$$\begin{Bmatrix} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & \text{SYM} & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

12 Beam Element Consistent Mass Matrix in Local Coordinates, \mathbf{M}

$$\begin{Bmatrix} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{Bmatrix} = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ & 156 & 22L & 0 & 54 & -13L \\ & & 4L^2 & 0 & 13L & -3L^2 \\ & & & 140 & 0 & 0 \\ \text{SYM} & & & & 156 & -22L \\ & & & & & 4L^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{Bmatrix}$$

13 Beam Element Consistent Mass Matrix in Global Coordinates, \mathbf{M}_e

Geometric relationship between \mathbf{u} and \mathbf{d} : $\mathbf{u} = \mathbf{T} \mathbf{d}$

$$u_1 = d_1 \cos \theta + d_2 \sin \theta \quad u_2 = -d_1 \sin \theta + d_2 \cos \theta \quad u_3 = d_3$$

where

$$\mathbf{T} = \begin{bmatrix} c & s & 0 & & & \\ -s & c & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \quad c = \cos \theta = \frac{x_2 - x_1}{L}$$

$$s = \sin \theta = \frac{y_2 - y_1}{L}$$

The consistent mass matrix in global coordinates is $\mathbf{M}_e = \mathbf{T}^T \mathbf{M} \mathbf{T}$

$$\mathbf{M}_e = \frac{\rho AL}{420} \begin{bmatrix} 140c^2 & -16cs & -22sL & 70c^2 & 16cs & 13sL \\ +15s^2 & & & +54s^2 & & \\ & 140s^2 & 22cL & 16cs & 70s^2 & -13cL \\ & +156c^2 & & & +54c^2 & \\ & & 4L^2 & -13sL & 13cL & -3L^2 \\ & & & 140c^2 & -16cs & 22sL \\ & \text{SYM} & & +156s^2 & & \\ & & & & 140s^2 & -22cL \\ & & & & +156c^2 & \\ & & & & & 4L^2 \end{bmatrix}$$