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SOME HYSTERESIS EFFECTS OF THE BEHAVIOUR OF GEOLOGIC MEDIA

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Abstract—A soil mechanics oriented theory of the hysteretic behaviour of materials under alternating loading is presented. Special attention is focused on the behaviour within the yield locus. The single hysteresis branches are described by piecewisely path independent laws with a variable compliance matrix depending on a scalar strain parameter. This parameter is chosen so that the polar symmetry of the unloading/reloading stress–strain curve in a uniaxial stress cycle is assured. The stress–strain constitutive laws are given either in incremental terms or in terms of finite quantities. The constitutive relations are valid piecewisely within the appropriately formulated stress reversal loci. Discrete and hierarchic memory of the cyclic behaviour is discussed and the rules for continuation, reactivation or generation of subsequent laws are formulated.

Finally simple examples are discussed.

1. INTRODUCTION

The engineering interest for an adequate mathematical description of hysteresis effects of rock and soil behaviour arises from the fairly pronounced difference in unloading and reloading response of these materials [1, 2]. In the solid mechanics approach both types of behaviour, within the yield locus, are commonly considered to be elastic, and are usually identified with that of unloading [3, 4]. This is due to that the main interest usually lies in the determination of irrecoverable strain. On the other hand many problems in foundation engineering, e.g. settlement of structures, are reloading problems by their very nature or by an appropriate design. Accuracy of the design thus depends in these cases on adequacy of mathematical description of reloading. It concerns, for instance use of precompression of the soft soil prior to foundation or deep excavations, and essentially every problem related to overconsolidated soils.

Another stimulation for the development of such a theory comes from the analysis of foundation response to repeated loading. Cycling of rock or soil masses caused by an earthquake, wave induced motion of oil platforms or foundation for reciprocating machines are typical problems where such a theory is required. In related topics such as liquefaction and cyclic mobility, the modelling of hysteresis is equally important.

In this paper, the characterisation of unloading and reloading of inviscid geological materials is dealt with aiming at the formulation of a numerically operative stiffness matrix together with a set of limiting conditions. In this view two types of behaviour will be recognized: elastoplastic loading and hysteretic unloading/reloading. The former type should be intended as in the classical theory. Therefore the basic relations governing the elastoplastic loading, i.e. yield condition, flow rule, consistency equation, hardening and softening relations and the elastic–plastic loading criterion, will be all fulfilled. Such an approach permits to use for the loading phase, the conceptual achievements and laboratory data of soil and rock plasticity (Schofield and Wroth [3], Rowe [5]).

The unloading-reloading hysteresis will be understood in what follows as occurring within a current yield locus. This yield locus is a convex, closed surface in the principal stress space and depends on the preconsolidation pressure as in the now classical model of Cam Clay [3].

In this range the material response for a typical undimensional stress cycle may be fairly accurately described by a closed stress–strain loop as depicted in Fig. 1(a) for deviatoric

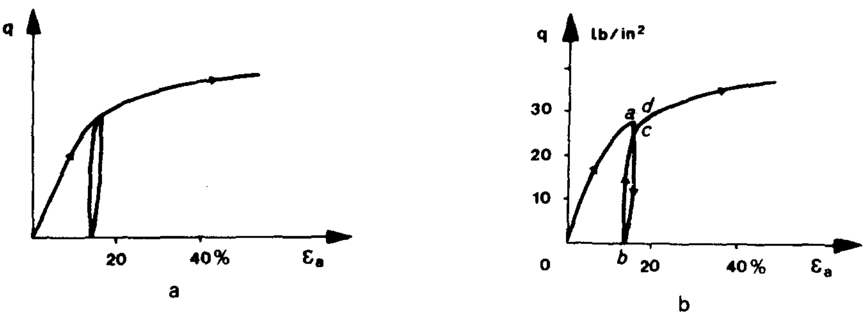

Fig. 1.

components for a clay. In reality the actual hysteresis loop (Fig. 1b) is usually accompanied by a sort of creep effect manifested by a drift in strain of the new yield point from *a* to *d*. Therefore the effective loop is completed at a stress level *c*. This discrepancy is usually associated with the number of cycles performed. In many practical cases it is negligible and will be henceforth disregarded.

Similarly are not considered here such phenomena as cyclic creep or cyclic hardening and softening. It is believed that they might be taken into account in further developments of the present theory. Closer look at these problems may be found in papers by Mróz[6, 7], Dafalias and Popov[8, 9]. Eisenberg and Phillips[10], Valanis[11], Prévost[12] and Cuellar *et al.*[13], both from the general plasticity and soil mechanics points of view.

A peculiarity of the cyclic behaviour of geological media, is the dependence of the unloading/reloading behaviour on the amount of the plastic deformation experienced. This fact is of evidence in the softening behaviour of rocks and is pronounced even in the hardening range. For instance, in unconfined compression, the slope of subsequent loading curves remarkably decreases with the increase of plastic strain. This clearly influences also reloading behaviour, see, e.g. Bieniawski[14]. To account for this fact the elastoplastic coupling concept may be employed, Hueckel, Maier[15, 16]. Apart from this phenomenon, the hysteretic unloading/reloading behaviour will be considered independent of the elastoplastic loading and *vice versa*.

In what follows the term stress will mean effective stress in the sense of soil mechanics; the strain will be understood within the small strain theory.

2. UNIDIMENSIONAL CYCLE

In this section the fundamental features of the hysteresis of a geological medium will be analysed. Figure 2(a) shows an idealized single unloading/reloading loop. The previous loading history is immaterial for the following considerations. Consider first the unloading from an initial stress point *A* to a point *C*. The corresponding stress–strain curve *ABC*, Fig. 2(a), is characterized by the initial compliance modulus C_0 and final secant modulus C_f at point *C*. It is possible to note from abundant experimental evidence, see for example[1–3, 17–20], that the reloading portion of a complete unloading/reloading stress–strain loop enjoys a geometric similarity to the unloading portion. In particular, this similarity can be reasonably idealized assuming that there is a polar symmetry with respect to the point situated in the middle of the segment connecting the loop extremities. The reloading curve *CDA* in Fig. 2(a) can thus be found by constructing the polar symmetry image of the unloading curve with respect to the point *K*. Therefore, the initial compliance modulus of the curve *CDA* is again C_0 and at the point *A* the compliance modulus is again C_f now referred to the new origin at point *C*. If the analytical expression of the unloading curve is formulated with respect to a system of coordinates whose origin is at *A*, the corresponding expression for the reloading curve can be simply obtained by translation of the origin to the point *C* and a rigid rotation of the system of an angle π.

The above discussed properties are relevant also when more than a single unloading/reloading cycle is considered. For instance, if the stress path *ABCD* is followed by unloading from a point *D* back to point *C*, the initial compliance modulus at point *D* is again equal to C_0, no matter what the stress level was achieved at *D*. On the other hand, the compliance modulus at *C*

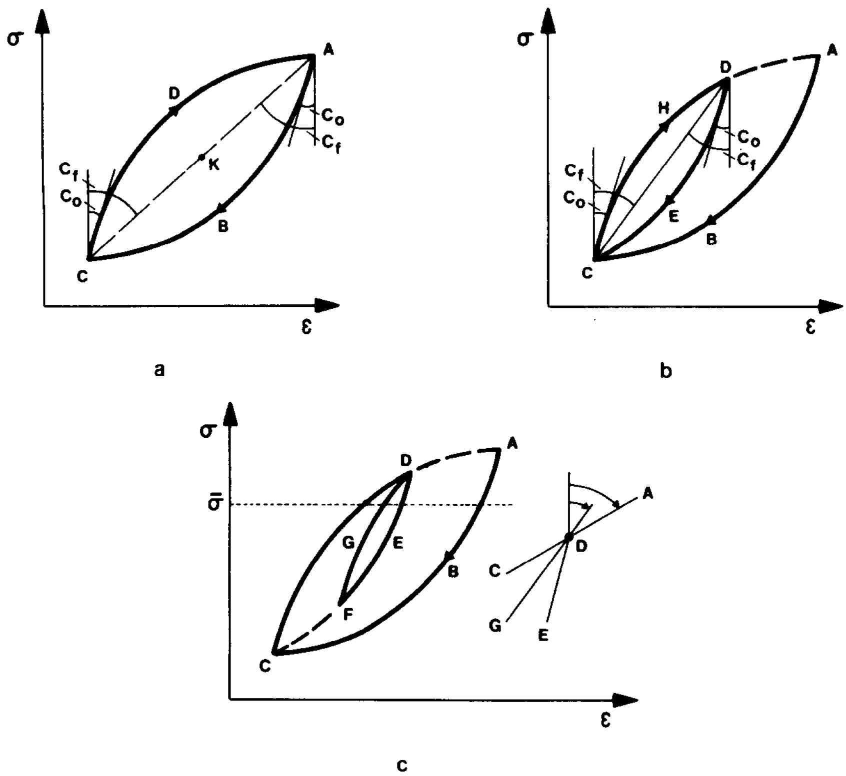

Fig. 2.

after the unloading path *DEC* does depend on the compliance pertinent to point *D* as shown in Fig. 2(b). The curve *DEC* may be conceived as the polar symmetric to *CD* with respect to the point in the middle of the segment *CD*. However, if the stress path *ABCD* is followed by unloading from point *D* to a point *F* between *D* and *C* and then by reloading until point *A*, it is possible to note from Fig. 2(c) that the geometric properties stated above are respected also by the loop *DEFGD*. Moreover, the tangent to the reloading curve during the cycle *ABCDEFGDA* is not continuous at *D*. In fact, just after *D* has been reached, the tangent to the further portion of reloading curve becomes the same as the tangent to the original reloading curve *CDA* in Fig. 2(a). This means that the effect of the unloading/reloading loop *DEFGD* is forgotten after point *D* is reached again.

Summing up the following conclusions may be drawn from the preceding considerations.

(1) Material behaviour has a piecewise character between subsequent stress reversals.

(2) Local response, i.e. strain and local compliance modulus at a generic stress point is history dependent. In fact, as shown in Fig. 2(c), at the stress level $\bar{\sigma}$ four different strain values correspond to the same stress, each associated with different stress paths. Moreover to any stress–strain point that in the past experienced a stress reversal, more than one local compliance modulus is attributed, as depicted in Fig. 2(c) at point *D*. Here and in the following, stress reversal will be intended in terms of rates.

(3) The history pertinent to closed stress loops is forgotten after the loop completion. However an arbitrary stress–strain point, which in the not-forgotten history became a stress reversal point, enjoys full strain reversibility over any further stress circuit. Such points are therefore history independent.

(4) If each of the three loops of Figs. 2(a, b, c), i.e. *ABCDA*, *DECHD* and *DEFGD* is considered with respect to its origin, *A*, *D* and *D* respectively, a common constitutive stress–strain law can be conceived. For the sake of simplicity a linear law will be employed, all non-linear effects between consecutive reversal points being expressed through the dependence of *C* on a parameter. The compliance modulus just after a stress reversal is always equal to C_0. Thus the strain part linearly related through C_0 to the stress difference between the current point say *M*, Fig. 3, and the origin will be defined as elastic, ϵ^e. The difference between the

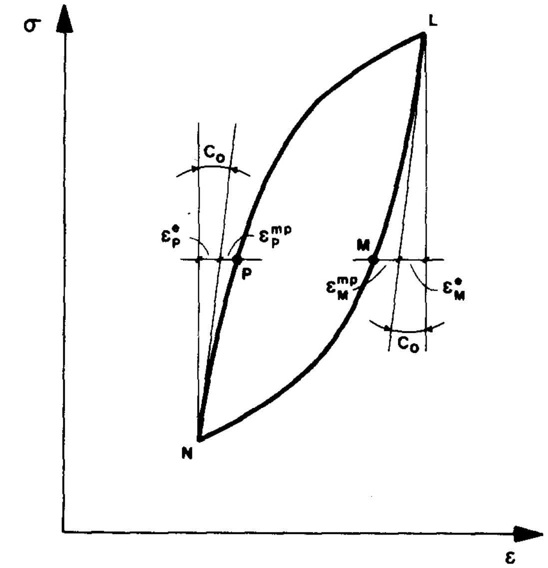

Fig. 3.

actual strain with respect to the origin L and the elastic strain part will be defined as microplastic strain ϵ^{mp}.

Thus:

$$^{\Delta L}\epsilon^{e} = C_0\, ^{\Delta L}\sigma \tag{2.1}$$

$$^{\Delta L}\epsilon^{mp} = (C - C_0)\, ^{\Delta L}\sigma \tag{2.2}$$

where the meaning of superscript ΔL is as follows

$$^{\Delta L}y = y - \hat{y}, \qquad \hat{y} = y_L. \tag{2.3}$$

Consider a generic point N at which a stress reversal occurs. Point N plays now the role of the new origin for further stress–strain law. Equations (2.1) and (2.2) for a current point P between N and L are still valid, referring the strain to point N, once σ_N is substituted to σ_L.

At point L the elastic and microplastic deformations will be then equal, except for the sign, to the corresponding strains calculated at N simply applying formulae (2.1) and (2.2). Therefore at the end of the cycle LNL there will be no residual strain (relative to the situation before the cycle). This property holds obviously for any point because it may potentially become a stress reversal point, as point F in Fig. 2(c). In this view the total strain difference which is referred to the last reversal point and is the sum of the elastic and the microplastic parts will be called henceforth paraelastic strain difference

$$^{\Delta L}\epsilon^{pe} = {}^{\Delta L}\epsilon^{e} + {}^{\Delta L}\epsilon^{mp}. \tag{2.4}$$

The denomination paraelastic is justified by the fact that the strain reversibility is conditioned, i.e. the strain is recoverable on a closed cycle only if the cycle starts from a stress reversal point. In fact, if the closed stress cycle is performed from a point M to N and back, the final state of the sample will be represented by the point P that is different from M. Therefore in this cycle, strain is not recoverable.

(5) The described material has a discrete, limited memory organized in two levels. In the first (operative) level memory it remembers the last stress reversal point and the one before the last. The former enters the constitutive law in that the strain is always referred to the strain relative to it and it is linked by a linear relation to the stress difference between the actual stress and the stress at this reversal point. The latter stress reversal point acts as a limiting point at which the current law ceases to hold. The second level (back) memory is not used directly at

current process. It stores the necessary (discrete) information relevant to all the stress reversal points which were origin of the non-completed loops. The back memory is organised in a hierarchic structure which preserves the sequence of the reversals.

The contents of both memory levels change. It may happen at a new reversal point or when a reversal point of a uncompleted loop is again reached. In the first case the one before the last stress reversal point is shifted from the 1st level to the 2nd level of the memory and the new born reversal point enters the 1st level. In the latter case the whole content of the 1st level of memory is cancelled (forgotten) and is replaced by the pair of stress reversal points which are pertinent to the reentry point recalled from the back memory (2nd level). Summing up, there are three operations upon the memory content, i.e. recording of new born reversals, between-level switches (both ways), forgetting of reversals of completed loops. An analogy can be made here to the elastoplastic process. In rough words, the point D (Fig. 2(c)) can be conceived as a kind of yield point, the cycle DFD as an elastic unloading/reloading cycle and the loading path CDA as an elastic–plastic work hardening process. In other words, the material enjoys a fading memory as classical elastic–plastic materials, but, in contrast, in the present case the memory is not continuous but discrete. This means that only the reversal points and not the whole stress path is memorized. It should be emphasized that the idea of discrete material memory has been first used for cyclic behaviour by Mróz and Lind [21] and has also been independently used by Dafalias and Popov [9] in a different context.

(6) Due to the symmetric behaviour in unloading and reloading the constitutive law between two stress reversal points will be written as:

$$^{\Delta L}\epsilon^{pe} = C\,^{\Delta L}\sigma. \tag{2.5}$$

The parameter C will vary always with the same law depending only on $^{\Delta L}\epsilon^{pe}$ or $^{\Delta L}\sigma$, and not on the total state. Every branch of a loop is then reproducible from the general law. It should be emphasized, however, that this reproducibility concerns only the hysteretic unloading/reloading behaviour here called "paraelastic", and not the elastic–plastic one that is completely independent of it.

The physical meaning of microplastic deformations is not straightforward. In metals they can be due to the change in microstructure under loading as suggested by Krempl [22]. In soils, at least in granular soils, there is not apparent change of the microstructure. In fact, at usual working stress levels soil particles, say sand grains, can be considered practically incompressible and no breakage of the particles occurs. The deformation of a soil specimen is therefore essentially due to the rearrangement of the packing caused by interparticle sliding. Just as the virgin soil is stressed, some deformations take place. These deformations are mainly irrecoverable because of the very nature of frictional behaviour that governs the interparticle sliding. Nevertheless, since soil particles are slightly deformable and can be considered individually elastic, when unloading occurs the deformation is partially recovered. The elastic deformation induces also a contemporary "microplastic" deformation, as it happens for a sequence of elastic strings and frictional blocks or, better, for a regular packing of elastic spheres, as shown by Deresiewicz [19]. Microplastic deformations can be considered to be linked to this kind of behaviour.

3. THE PARAELASTIC STRESS–STRAIN RELATION

In this section a generalization to multiaxial stress conditions of the behaviour of the material described in Section 2 will be presented. To make the presentation concise a vectorial notation will be adopted, so that the independent components of the stress and strain tensors and of their rates will be listed in column vectors. An underlined symbol will mean vector or matrix, a superimposed tilde transpose and a dot increment or rate. The constitutive rules will consist of two parts: stress–strain equations, valid piecewisely for single path portions, and conditions delimiting such portions and ruling the consequent changes of the stress–strain relation. In this section the constitutive equation will be dealt with in detail.

The constitutive equation (2.5) will be generalized in the following form:

$$^{\Delta L}\underline{\epsilon}^{pe} = \underline{C}\,^{\Delta L}\underline{\sigma} \tag{3.1}$$

where $\underset{\sim}{C}$ is the paraelastic compliance matrix. As a consequence of the assumed isotropy of the material the matrix $\underset{\sim}{C}$ is symmetric, so that it is possible to define a stress potential $V({}^{\Delta L}\underset{\sim}{\sigma}, \underset{\sim}{C})$ such that the strain difference ${}^{\Delta L}\underset{\sim}{\epsilon}^{pe}$ can be alternatively defined through the tensorially linear law, for $\underset{\sim}{C}$ being constant:

$$ {}^{\Delta L}\underset{\sim}{\epsilon}^{pe} = \left.\frac{\partial V({}^{\Delta L}\underset{\sim}{\sigma}, \underset{\sim}{C})}{\partial \underset{\sim}{\sigma}}\right|_{\underset{\sim}{C}=\text{cons.}} \tag{3.2} $$

In order to guarantee that eqn (3.1) enjoys uniqueness of solution for a given ${}^{\Delta L}\underset{\sim}{\sigma}$ the function V must be convex, i.e. the matrix $\underset{\sim}{C}$ must be positive definite. The matrix $\underset{\sim}{C}$ does not depend explicitly neither on ${}^{\Delta L}\underset{\sim}{\sigma}$ nor on ${}^{\Delta L}\underset{\sim}{\epsilon}^{pe}$, but it is a function of a scalar parameter χ which will be specified later on. Therefore $\underset{\sim}{C}$ has the meaning of the matrix of (secant) compliances

$$ \underset{\sim}{C} = \underset{\sim}{C}(\chi). \tag{3.3} $$

The intrinsic non-linearity of the hysteretic behaviour is reproduced by means of the variation of the matrix $\underset{\sim}{C}$. The purely elastic part of the strain ${}^{\Delta L}\underset{\sim}{\epsilon}^{e}$, and the microplastic part, ${}^{\Delta L}\underset{\sim}{\epsilon}^{mp}$, will be then defined by:

$$ {}^{\Delta L}\underset{\sim}{\epsilon}^{e} = \underset{\sim}{C}_0 \, {}^{\Delta L}\underset{\sim}{\sigma} \tag{3.4} $$

$$ {}^{\Delta L}\underset{\sim}{\epsilon}^{mp} = [\underset{\sim}{C}(\chi) - \underset{\sim}{C}_0] \, {}^{\Delta L}\underset{\sim}{\sigma} \tag{3.5} $$

where $\underset{\sim}{C}_0$ is a matrix of constant elastic compliances.

The parameter χ may be conceived as a function of the paraelastic strain difference and defined between two consecutive stress reversal points. Starting from a new stress reversal point new χ is generated. To avoid confusion superposed L will indicate that paraelastic strain and stress differences are referred to the Lth stress reversal point. Several reasonable proposals of the explicit expression $\chi^L({}^{\Delta L}\underset{\sim}{\epsilon}^{pe})$ are possible, *a priori*. Since χ^L is a "measure" of the paraelastic strains, the proposal that seems to be the most direct is to assume that χ^L is their Euclidean norm, so that:

$$ \chi^L = \|{}^{\Delta L}\underset{\sim}{\epsilon}^{pe}\| = ({}^{\Delta L}\underset{\sim}{\tilde{\epsilon}}^{pe} \, {}^{\Delta L}\underset{\sim}{\epsilon}^{pe})^{1/2}. \tag{3.6} $$

With this choice, it is possible, to see that for a uniaxial case χ^L is insensitive to a change in sign of ${}^{\Delta L}\underset{\sim}{\epsilon}^{pe}$, so that the polar symmetry of a single uniaxial loop is automatically ensured. Also, in this case χ^L coincides with the paraelastic strain difference itself, as confirmed experimentally by Hardin and Drnevich[20]. These authors introduced the concept of "strain amplitude" of a cycle that for simple shear fits the definition of χ^L employed in the present work. Therefore in the following we shall refer to χ^L as to the strain amplitude parameter.

In a detailed analysis it was established[20], that the shear (secant) modulus in hollow cylinder simple shear test on soils strongly depends on what is called here parameter χ^L, effective mean principal stress and void ratio, other factors being recognized as less-important or unimportant. The account of dependence on the effective mean principal stress may be taken through the elastoplastic coupling, as pursued elsewhere[15]. We shall consider only materials at fixed initial void ratio. Therefore the only variable to which the compliance variation may be accounted for within the present theory is the parameter χ^L. It is assumed here that such variation is linear in χ^L. This assumption seems to comply within a reasonable accuracy with fundamental results of Hardin and Drnevich[20], and results on over ten soils listed by these authors. Therefore, a reasonable hypothesis for $\underset{\sim}{C}$ is:

$$ \underset{\sim}{C} = \underset{\sim}{C}_0(\underset{\sim}{I} + \chi^L \underset{\sim}{\Omega}) \tag{3.7} $$

where $\underset{\sim}{C}_0$ is the initial compliance matrix, when $\chi^L = 0$, i.e. at the stress reversal point, $\underset{\sim}{I}$ is the identity matrix and $\underset{\sim}{\Omega}$ is a matrix of constant softening moduli. To ensure the positive definiteness of $\underset{\sim}{C}$ for any (always positive) value of χ^L, it is necessary that $\underset{\sim}{C}_0$ is positive definite and it is sufficient that the matrix $\underline{C}_0\underline{\Omega}$ is at least positive semi-definite as it will be assumed in what follows. C_0 and $\underline{\Omega}$ should be obviously symmetric.

The constitutive eqn (3.1) is given in the implicit form, since $\underline{C} = \underline{C}[\chi^L({}^{\Delta L}\underline{\epsilon}^{pe})]$. However, the parameter χ may be alternatively expressed in terms of the stress difference. In fact, by the very definition of χ^L and of ${}^{\Delta L}\underline{\epsilon}^{pe}$

$$(\chi^L)^2 = {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}\underline{C}\,{}^{\Delta L}\underline{\sigma}. \tag{3.8}$$

By the assumption (3.7), eqn (3.8) reduces to:

$$(\chi^L)^2(1 - {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma}) - 2\chi^{L\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma} - {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{C}_0\,{}^{\Delta L}\underline{\sigma} = 0 \tag{3.9}$$

Since the two last terms in l.h.s. of eqn (3.9) are certainly non-positive, the condition for the solution of this equation is that:

$$0 < {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma} < 1. \tag{3.10}$$

It is easy to see that under this condition the solution for χ^L is unique and finite and is given by:

$$\chi^L = \{{}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma} + [({}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma})^2 + {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{C}_0\,{}^{\Delta L}\underline{\sigma}$$
$$\times (1 - {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma})]^{1/2}\} \cdot (1 - {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}_0\underline{\Omega}\underline{\Omega}\underline{C}_0\,{}^{\Delta L}\underline{\sigma})^{-1} \tag{3.11}$$

When the term constrained by (3.10) tends to 1, χ^L tends to infinity and consequently so does $\underline{C}$. Therefore, the paraelastic deformation also tends to infinity. Once χ^L is known, $\underline{C}$ is uniquely determined through definition (3.7) and paraelastic strain is determined for a given stress difference simply through eqn (3.1). The direct problem is then formally solved.

The inverse constitutive law is formulated in the explicit form. For given paraelastic strain ${}^{\Delta L}\underline{\epsilon}^{pe}$, χ^L is straight-forwardly determined through definition (3.8) and then $\underline{C}$ is known. Matrix $\underline{C}$ can be always inverted, since it is a positive definite matrix. Thus, formally

$$^{\Delta L}\underline{\sigma} = \underline{C}^{-1}[\chi^L({}^{\Delta L}\underline{\epsilon}^{pe})]\,{}^{\Delta L}\underline{\epsilon}^{pe}. \tag{3.12}$$

In the derivation of all the equations of this section it has been tactily admitted that no stress reversal has occurred between the Lth stress reversal point and the point represented by ${}^{\Delta L}\underline{\sigma}$ and ${}^{\Delta L}\underline{\epsilon}^{pe}$. Moreover, a proper meaning of the term stress reversal has not yet been defined although, in contrast with the uniaxial case where the definition of stress reversal is selfevident, in multiaxial conditions it is not straightforward. Several such definitions; which do not violate the condition of invariancy with respect to the system of stress coordinates, are possible. It is most straightforward to employ a condition for stress reversal based on the stress increment orientation. This requires however to reformulate the constitutive equations in terms of rates more than in total stress or strain differences. Such a structure of the constitutive equation is consistent with the elastic–plastic one and allows to write the global constitutive equation for this type of material behaviour in an incremental form.

4. INCREMENTAL FORM OF THE CONSTITUTIVE RELATION

From the very definition of the paraelastic strain difference as expressed in Section 3 the strain increment should be considered as its total differential in the variables ${}^{\Delta L}\underline{\sigma}$ and χ^L so that over an infinitesimal time interval the paraelastic strain rate is given by:

$$\dot{\underline{\epsilon}}^{pe} = \underline{C}(\chi^L)\dot{\underline{\sigma}} + \underline{C}_\chi\,{}^{\Delta L}\underline{\sigma}\dot{\chi} \tag{4.1a}$$

$$= \underline{C}_0\dot{\underline{\sigma}} + (\underline{C} - \underline{C}_0)\dot{\underline{\sigma}} + \underline{C}_\chi\,{}^{\Delta L}\underline{\sigma}\dot{\chi} \tag{4.1b}$$

where:

$$\underline{C}_\chi = \frac{\partial \underline{C}}{\partial \chi^L}. \tag{4.2}$$

The first term in eqn (4.1b) is the reversible strain rate while the two other are irreversible over the incremental stress cycle originated at a point which is not a stress reversal point, Fig. 4. In the latter case no irreversible strain rate occurs.

The rate of the strain amplitude parameter χ^L, has the following form:

$$\dot{\chi} = \frac{1}{\chi^L}({}^{\Delta L}\tilde{\underline{\epsilon}}^{pe}\dot{\underline{\epsilon}}^{pe}). \tag{4.3}$$

Thus eqn (4.1) gives the paraelastic strain rate in an implicit form. The explicit form of eqn (4.1) can be easily found writing eqn (4.3) as follows

$${}^{\Delta L}\tilde{\underline{\epsilon}}^{pe}\,\dot{\underline{\epsilon}}^{pe} = {}^{\Delta L}\tilde{\underline{\epsilon}}^{pe}\,\underline{C}\dot{\underline{\sigma}} + \frac{1}{\chi^L}\,{}^{\Delta L}\tilde{\underline{\epsilon}}^{pe}\underline{C}_\chi\,{}^{\Delta L}\underline{\sigma}\,{}^{\Delta L}\underline{\epsilon}^{pe}\,\dot{\underline{\epsilon}}^{pe}. \tag{4.4}$$

Therefore

$$\frac{1}{\chi^L}\,{}^{\Delta L}\underline{\epsilon}^{pe}\,\dot{\underline{\epsilon}}^{pe}(\chi^L - \chi^{L*}) = {}^{\Delta L}\tilde{\underline{\epsilon}}^{pe}\,\underline{C}\dot{\underline{\sigma}}, \tag{4.5}$$

where χ^{L*} is defined as

$$\chi^{L*} \equiv {}^{\Delta L}\tilde{\underline{\epsilon}}^{pe}\underline{C}_\chi\,{}^{\Delta L}\underline{\sigma} = {}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}\underline{C}_\chi\,{}^{\Delta L}\underline{\sigma}. \tag{4.6}$$

If $\chi^L \neq \chi^{L*}$ it is possible to derive $\dot{\chi}$ as

$$\dot{\chi} = \frac{{}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}\underline{C}\dot{\underline{\sigma}}}{\chi^L - \chi^{L*}} \tag{4.7}$$

and finally from eqns (4.1) and (4.7) the direct incremental stress-parelastic strain relation as

$$\dot{\underline{\epsilon}}^{pe} = \underline{B}\dot{\underline{\sigma}} \equiv \left(\underline{I} + \frac{\underline{A}}{\chi^L - \chi^{L*}}\right)\underline{C}\dot{\underline{\sigma}} \tag{4.8}$$

where $\underline{I}$ is the identity matrix and matrix $\underline{A}$ is defined as

$$\underline{A} \equiv \underline{C}_\chi\,{}^{\Delta L}\underline{\sigma}\,{}^{\Delta L}\tilde{\underline{\sigma}}\underline{C}. \tag{4.9}$$

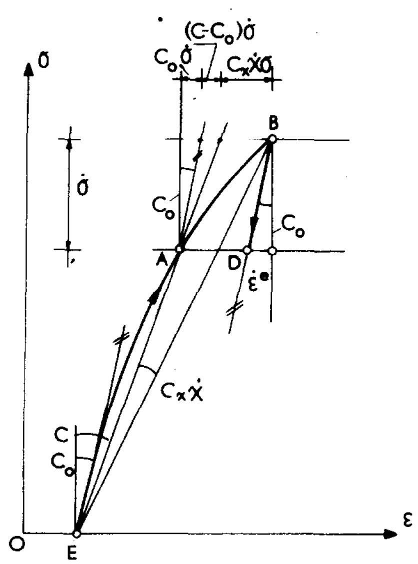

Fig. 4.

Any path can be then followed step by step by means of the constitutive relation (4.8) rearranging $\underset{\sim}{B}$ at any step, updating ${}^{\Delta L}\underset{\sim}{\sigma}$, $\underset{\sim}{C}$, C_χ and $(\chi^L - \chi^{L*})$.

The inverse relation that gives the stress rate for a given strain rate may be found straightforwardly. In fact, since $\underset{\sim}{C}$ is symmetric positive definite, from eqns (4.1) and (4.3)

$$\underset{\sim}{C}^{-1}\dot{\underset{\sim}{\epsilon}}^{pe} = \dot{\underset{\sim}{\sigma}} + \frac{1}{\chi^L}\underset{\sim}{C}^{-1}\underset{\sim}{C}_\chi {}^{\Delta L}\underset{\sim}{\sigma}\, {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}\dot{\underset{\sim}{\epsilon}}^{pe}. \tag{4.10}$$

Then

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{B}^{-1}\dot{\underset{\sim}{\epsilon}}^{pe} \equiv \underset{\sim}{C}^{-1}\left(\underset{\sim}{I} - \frac{\underset{\sim}{A}}{\chi^L}\right)\dot{\underset{\sim}{\epsilon}}^{pe}. \tag{4.11}$$

The matrix $\underset{\sim}{I} - (\underset{\sim}{A}/\chi^L)$ is therefore the inverse of $\underset{\sim}{I} + \underset{\sim}{A}/(\chi^L - \chi^{L*})$ as it can be easily verified taking account of the following property of $\underset{\sim}{A}$:

$$\underset{\sim}{A}(\underset{\sim}{A} - \chi^*\underset{\sim}{I}) = \underset{\sim}{0}. \tag{4.12}$$

Some properties of the constitutive matrix $\underset{\sim}{B}$ will be now discussed.

For any kind of tensorially linear stress–strain law in which the compliance matrix depends on a positive scalar it is possible to distinguish two types of nonlinear behaviour. Consider a function $\sigma = \sigma(\epsilon^e)$ for a uniaxial purely elastic process and a stress increment at a given point. The resulting incremental paraelastic strain (4.2) can be greater or smaller than that one would have had if the local compliance would have been constant. By analogy with the terminology of plasticity the former case will be referred to as elastic softening and the latter as elastic hardening, [15]. Generalizing it to the multidimensional case, there will be a response of the elastic softening type if the increase of the function V^L, (3.2), is greater than that one has had if the material has been purely elastic. The opposite occurs for the hardening case. The hysteretic loop branch is in general of the softening type as shown schematically in Figs. 1–3. The softening condition that should be imposed on matrix $\underset{\sim}{C}$ is then

$$\dot{V} = \frac{\partial V}{\partial\, {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}}\dot{\underset{\sim}{\sigma}} + \frac{\partial V}{\partial \chi}\dot{\chi} \geq \dot{V}^e = \frac{\partial V}{\partial\, {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}}\dot{\underset{\sim}{\sigma}}. \tag{4.13}$$

Therefore

$$\frac{\partial V}{\partial \chi}\dot{\chi} \geq 0 \tag{4.14}$$

and by definition (4.3)

$$\frac{1}{2}{}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}_\chi {}^{\Delta L}\underset{\sim}{\sigma}\,\dot{\chi} \geq 0. \tag{4.15}$$

In the present theory the matrix $\underset{\sim}{C}_\chi$ is equal to the matrix $\underset{\sim}{C}_0\underset{\sim}{\Omega}$ assumed positive semi-definite (Section 3). Therefore the condition for elastic softening is that

$$\dot{\chi} \geq 0. \tag{4.16}$$

The matrix $\underset{\sim}{B}$ defined through eqn (4.8) is a non-symmetric matrix, the positive definiteness of which may be proved under certain conditions. It may be shown that the matrix $\underline{AC}$ is positive semi-definite and that the scalar $(\chi^L - \chi^{L*})$ is never negative (see appendix), what are sufficient conditions for positive definiteness of $\underset{\sim}{B}$. The inequality $\chi^L - \chi^{L*} \geq 0$ is proved under the condition that the eigenvectors of matrices $\underset{\sim}{C}$ and $\underset{\sim}{C}_\chi$ coincide. It implies, in mechanical terms, that no anisotropy is induced in the material due to hysteretic behaviour. This hypothesis, certainly not strictly verified by experimental evidence is common to many theories of soil behaviour and will not be discussed here.

The main consequence of the positive definiteness of the matrix $\underset{\sim}{B}$ is that the uniqueness of the paraelastic strain rate for a given stress rate is guaranteed and it is also ensured for the inverse rate relation.

5. CONDITIONS FOR CHANGE OF CONSTITUTIVE LAWS. MATERIAL MEMORY RULES

The stress–strain relations derived in the preceding sections are valid for loading path protions along with neither a stress reversal nor a reentry in a previous stress reversal point occur. Suitable multidimensional (invariant to coordinate choice) criteria for such occurrence are required in order to delimit the validity of the current law. Moreover, a consequent rule of changes of the memory content and between-level switches should be established.

The criterion for stress reversal adopted in this paper is based on the stress rate orientation. A locus in the stress space is conceived, which delineates between stress rates which give rise to the continuation of the current law (say reloading) or to the stress reversal (say unloading or *vice versa*). For the former case the stress rate vector is directed outwards the locus whilst in the latter it is oriented inwards. The tangent stress rate vector will correspond to a neutral loading.

The continuation condition will be connected here with the condition that the strain amplitude parameter non-decreases, $\dot{\chi} \geq 0$. Rewriting it in terms of the stress rate, having taken account of positiveness of $(\chi^L - \chi^{L*})$ (see A11) one has

$$(\chi^L - \chi^{L*})\dot{\chi} = {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}\underset{\sim}{C}\dot{\underset{\sim}{\sigma}} \geq 0. \tag{5.1}$$

The stress rate that would violate this condition is inadmissible within the current law and is thus thought to give rise to a stress reversal. Consequently a change in the constitutive law occurs in that the current reference stress point $\hat{\underset{\sim}{\sigma}}$ is henceforth the stress point at which condition (5.1) is violated. The left hand side of the inequality (5.1) may be conceived as the scalar product of the stress rate vector and the gradient vector of a surface $\bar{W}^L({}^{\Delta L}\underset{\sim}{\sigma}) = 0$, pertinent to the actual stress state

$$\bar{W}^L = W^L - W_0{}^L = 0. \tag{5.2}$$

The surface $\bar{W}^L = 0$, referred to as the stress reversal locus, is an equipotential surface at a constant χ^L of the potential function (quadratic form)

$$W^L = {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}\underset{\sim}{C}\,{}^{\Delta L}\underset{\sim}{\sigma} \tag{5.3}$$

$W_0{}^L$ is the value of (5.3) at ${}^{\Delta L}\underset{\sim}{\sigma}$.

Therefore a loading process portion started from $\underset{\sim}{\sigma}^L$, i.e. the stress state at Lth stress reversal, is accompanied by a growth of the ellipsoides $\bar{W}^L$ parametrized by the value $W_0{}^L$ pertinent to ${}^{\Delta L}\underset{\sim}{\sigma}$ (as shown schematically in Fig. 5a), which at each instant furnishes the current criterion.

Based on (5.1) (5.3), a loading function is introduced

$$\mathscr{L}(\bar{W}^L) = \frac{\partial \bar{W}^L}{\partial\, {}^{\Delta L}\underset{\sim}{\sigma}}\,\dot{\underset{\sim}{\sigma}}. \tag{5.4}$$

If for given vectors $\underset{\sim}{\sigma}$, $\underset{\sim}{\epsilon}^{pe}$ with $\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^L$

$$\mathscr{L}(\bar{W}^L) \geq 0, \tag{5.5}$$

the governing paraelastic rate relation remains unaltered, i.e. so does $\hat{\underset{\sim}{\sigma}}$,

$$\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^L. \tag{5.6}$$

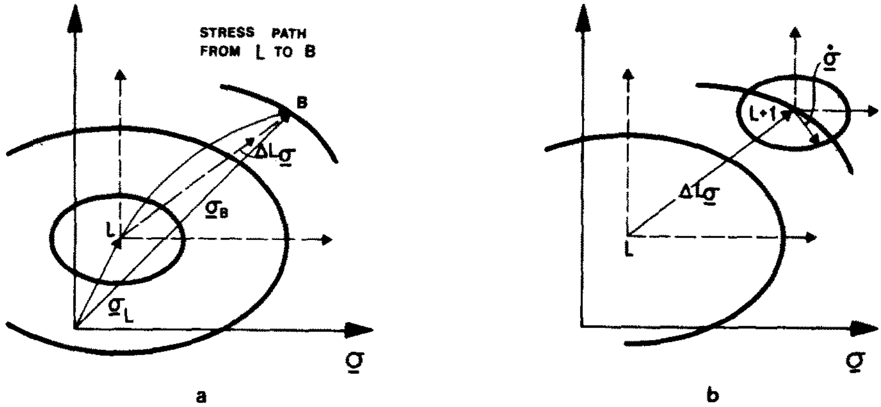

Fig. 5.

If on the contrary at the point $\underset{\sim}{\sigma}$, $\underset{\sim}{\epsilon}^{pe}$ with $\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^L$

$$\mathscr{L}(\bar{W}^L) < 0 \tag{5.7}$$

such stress point $\underset{\sim}{\sigma}$ becomes the new $(L+1)$th stress reversal point (Fig. 5b)

$$\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma} \equiv \underset{\sim}{\sigma}^{L+1}. \tag{5.8}$$

Moreover the surface

$$\bar{W}^{L,L+1} = 0, \tag{5.9}$$

of the family given by the eqn (5.3) and passing through the point $\underset{\sim}{\sigma} \equiv \underset{\sim}{\sigma}^{L+1}$ becomes a dead locus which defines the domain of validity of the new incremental law.

Consequently, a new family of active loci

$$\bar{W}^{L+1} = W^{L+1} - W_0^{L+1} = 0 \tag{5.10}$$

is generated from the point $\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^{L+1}$.

More generally, after several reversals the range of the validity of the current law is defined as internal to the common domain formed by the corresponding dead loci, therefore satisfying the set of inequalities

$$\bar{W}I,^{I+1} \leq 0, \qquad I = 1 \ldots L \tag{5.11}$$

where $I+1$ identifies the dead member of the family $\bar{W}^I = 0$ with the origin at $\underset{\sim}{\sigma}^I$ and passing through the $(I+1)$th stress reversal point.

If the stress path reaches again and crosses a dead locus say $\bar{W}^{K,K+1} = 0$, $K < L$, such locus is reactivated.

Therefore

$$\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^K \text{ if } \bar{W}^{K,K+1} = 0 \text{ and } \mathscr{L}(\bar{W}^{K,K+1}) = \frac{\partial \bar{W}^{K,K+1}}{\partial\, {}^{\Delta K}\underset{\sim}{\sigma}} \dot{\underset{\sim}{\sigma}} > 0. \tag{5.12}$$

Consequently, all the loci from $(K+1)$th to the last dead locus, i.e. $(L-1)$th (numbers referred to the locus origins) and the last active locus are forgotten. Therefore in the current set of the L stress reversal points in (5.11) are not included those loci which have been forgotten.

The current value $\hat{\underset{\sim}{\sigma}}$ given by (5.6), (5.8) or (5.12) together with the corresponding $\underset{\sim}{\epsilon}^{pe}$ and the set of inequalities (5.11) constitute the content of the 1st level memory (operative). The 2nd level (back) memory contains all $\underset{\sim}{\sigma}^K$, $K = 1, \ldots, L$, which were not forgotten in the preceding history and do not enter the 1st level. Figure 6 represents an example of a stress path

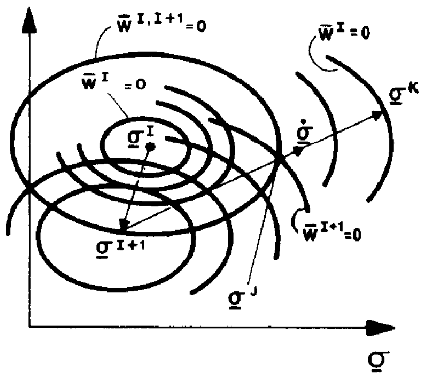

Fig. 6.

experiencing the situations mentioned above. The stress path starts from the point $\underline{\sigma}^I$ where a family of active loci $\bar{W}^I = 0$ is generated, the current member of which is determined by eqn (5.2). At the point $\underline{\sigma}^{I+1}$ a stress reversal occurs, i.e. $(\partial \bar{W}^I/\partial\, ^{\Delta I}\underline{\tilde{\sigma}})|_{\sigma^{I+1}}\, \dot{\underline{\sigma}} < 0$. The curve $\bar{W}^{I,I+1} = 0$, passing through the point $\underline{\sigma}^{I+1}$ becomes the border of the domain of the new constitutive law. This new law is then active until the stress path reaches the border, say at $\underline{\sigma}^J$. At this point the law relative to $\underline{\sigma}^{I+1}$ is forgotten, and the law pertinent to $\underline{\sigma}^I$ is reactivated, since $(\partial \bar{W}^{I,I+1}/\partial^{\Delta I}\underline{\tilde{\sigma}})|_{\sigma_J}\dot{\underline{\sigma}} > 0$. At the current point $\underline{\sigma}^K$ the border is the active locus given by equation $\bar{W}^I = 0$.

The changes of the $\hat{\sigma}$ for a generic stress state $\underline{\sigma}$ after the lth stress reversal point (not-forgotten) may be put in more formal structure in the following form

$$\underline{\hat{\sigma}} = \sum_{K=1}^{L} [\underline{\sigma}^K + (\underline{\sigma} - \underline{\sigma}^K)\vartheta\{\mathscr{L}^K\}]\xi_K\zeta_K \tag{5.13}$$

where

$$\begin{aligned} \vartheta\{y\} &= 1 \qquad \text{if } y < 0 \\ \vartheta\{y\} &= 0 \qquad \text{if } y \geq 0 \end{aligned} \tag{5.14}$$

while $\mathscr{L}^K$ is the joint loading function

$$\mathscr{L}^K = \begin{cases} \left(\dfrac{\partial \bar{W}^{K,K+1}}{\partial\, ^{\Delta K}\underline{\tilde{\sigma}}}\right)\dot{\underline{\sigma}}, & 1 \leq K \leq L-1 \\[2ex] \left(\dfrac{\partial \bar{W}^{K}}{\partial\, ^{\Delta K}\underline{\tilde{\sigma}}}\right)\dot{\underline{\sigma}}, & K = L. \end{cases} \tag{5.15}$$

The function ξ_K is non-zero when the kth locus is reached, in such a way

$$\xi_K = \begin{cases} 1 - \vartheta\{\bar{W}^{K,K+1}\} & 1 \leq K \leq L-1 \\ 1 - \vartheta\{\bar{W}^K\} \quad , & K = L \end{cases}. \tag{5.16}$$

The function ζ_K represents the effect of forgetting stress reversal points.

$$\zeta_K = \prod_{n=1}^{K-1} (1 - \xi_n) \tag{5.17}$$

where Π is the multiplication symbol over the values $n = 1, \ldots, K-1$.

For a given stress for any stress rate $\dot{\underline{\sigma}}$ it is possible to calculate through (5.16) the corresponding value of $\underline{\hat{\sigma}}$. For any current stress $\underline{\sigma}$ surely $\bar{W}^L(\underline{\sigma}) = 0$. The domain of validity of the current constitutive law will be limited by other $L-1$ inequalities of the type (5.2). Suppose first that all these inequalities are strict, i.e. the stress point does not lie on any dead locus (see

Fig. 7(a)). Then by virtue of eqns (5.17) and (5.19). $\xi_K = 0$ for any value of K between 1 and $L-1$, and $\xi_L = 1$. $\zeta_L = 1$ for any K. Suppose that $\dot{\underline{\sigma}}$ is directed outward the domain so that $\mathscr{L}^L > 0$. In this case $\vartheta^L = 0$ and finally $\hat{\underline{\sigma}} = \underline{\sigma}^L$, i.e. the loading continues with respect to the last origin $\underline{\sigma}^L$. If $\dot{\underline{\sigma}}$ is directed inwards an "unloading" occurs and the current $\underline{\sigma}$ becomes the new origin. In fact in this case case $\mathscr{L}^L < 0$, $\vartheta^L = 1$ and $\hat{\underline{\sigma}} = \underline{\sigma} \equiv \underline{\sigma}^{L+1}$. Suppose now that for some K $\bar{W}^{K,K+1} = 0$, i.e. the stress point does lie on dead loci (and what is more at the corner formed by some of them), see Fig. 7(b). For all these K, $\xi_K = 1$, but only $\zeta_J = 1$, J being the "oldest" stress reversal point for which $\bar{W}^{K,K+1}(\underline{\sigma}) = 0$. For all "younger" K, $\zeta_K = 0$, since in the corresponding $\overset{K}{\Pi}$ enters $1 - \xi_J = 0$. If $\mathscr{L}^J \geq 0$, $\hat{\underline{\sigma}} = \underline{\sigma}^J$, otherwise $\hat{\underline{\sigma}} = \underline{\sigma} \equiv \underline{\sigma}^{J+1}$. In both cases all $L-J$ dead loci "younger" than $\bar{W}^{J,J+1} = 0$ are forgotten.

The number of the dead stress reversal loci stored in the material memory is theoretically not limited for a generic, complex stress path. In numerical applications this number is reduced by successive forgetting of some loci and by the restrictive form of most practical stress path. In particular, a stabilization of the number of loci involved may occur. if for instance a repeated loading circuit is considered, e.g. see the second example in Section 6. On the other hand a numerical device for limiting the current number of loci may be applied for specific loading in the same spirit of what has been done in[31].

As mentioned before if the Mth dead locus is reactivated say at $\underline{\sigma} = \underline{\sigma}^*$ the incremental stress-paraelastic strain law is henceforth again the one partaining to the Mth stress reversal point, so that

$$\dot{\underline{\epsilon}}^{pe} = \underline{B}({}^{\Delta M}\underline{\sigma}, \chi^M)\dot{\underline{\sigma}}. \tag{5.18}$$

The above rate equation may be integrated along the path in order to obtain relation between stress and paraelastic strain differences. So that

$${}^{\Delta M}\underline{\epsilon}^{pe} = \underline{C}(\chi^M)\,{}^{\Delta M}\underline{\sigma} + \overline{{}^{\Delta M}\underline{\epsilon}^{pe}} \tag{5.19}$$

where the first term of the right hand side of eqn (5.19) is the paraelastic strain that the material would have experienced if no other stress reversals have been occurred after the Mth one.

The second, constant term of (5.19) is due to the complex history of the material from the Mth stress reversal.

$$\overline{{}^{\Delta M}\underline{\epsilon}^{pe}} = \int_{t_{\sigma^M}}^{t_{\sigma^*}} \dot{\underline{\epsilon}}^{pe}\,dt - \underline{C}(\chi^M)(\underline{\sigma}^* - \underline{\sigma}M). \tag{5.20}$$

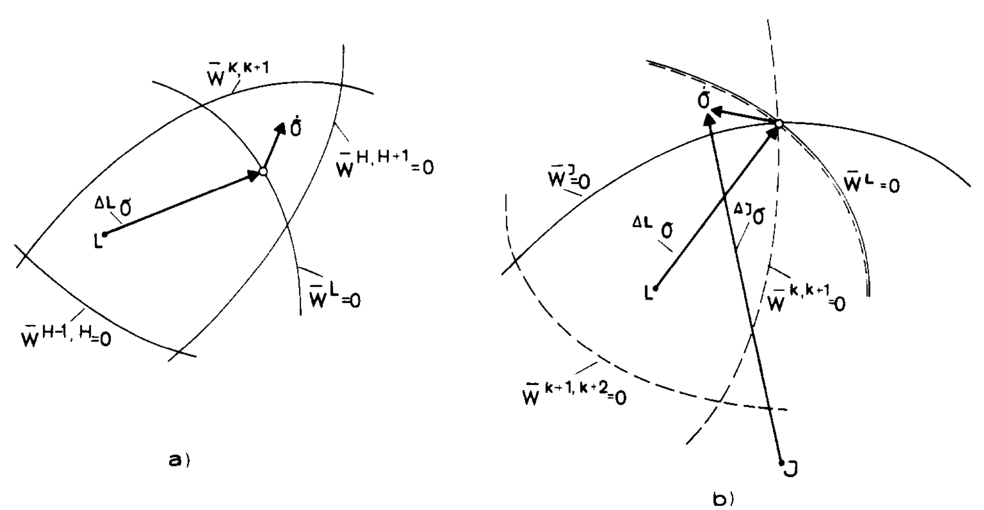

Fig. 7.

In the case of radial (or straight) stress path originated at M and in particular in the unidimensional case clearly $\overline{{}^{\Delta M}\underline{\epsilon}^{pe}} = 0$. Note that $\overline{{}^{\Delta M}\underline{\epsilon}^{pe}}$ does not depend on the actual values of $\underline{\sigma}$ and thus enters the memory in the discrete form.

The idea of the generation of a new constitutive law at subsequent stress reversal points together with "forgeting" rules has initially been introduced by Phillips and Greenstreet [23, 24]. This concept referred to the path dependent process, plastic more than piecewise path independent as in the present theory, was limited to radial paths only. Those authors assumed also that the stress strain reloading curve approaches asymptotically the initial one which would have existed provided the unloading had not occurred. This hypothesis is justified only for the considered type of loading, for which, moreover, no such devices as dead loci and relative memory rules are required, whilst they are necessary in cases of more complex loading paths, see also Mròz and Lind [21].

6. EXAMPLES

It is now possible to solve some sample problems to illustrate how the theory works.

For the sake of simplicity only simple drained triaxial compression tests $\sigma_2 = \sigma_3$ will be considered. As usual in soil mechanics, the variables p and q are employed defined as

$$p = \frac{\sigma_1 + 2\sigma_3}{3} \qquad q = \sigma_1 - \sigma_3.$$

In the considered test p is the first invariant of the effective stress tensor and q is proportional to the square root of the second invariant of the deviatoric stress tensor. The volumetric strain v and the deviatoric strain defined as

$$v = \epsilon_1 + 2\epsilon_3; \qquad \epsilon = \frac{2}{3}(\epsilon_1 - \epsilon_3)$$

are the corresponding invariants of the strain tensor. Equation (3.1) is then given by

$$\begin{Bmatrix} {}^{\Delta L}v^{pe} \\ {}^{\Delta L}\epsilon^{pe} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} {}^{\Delta L}p \\ {}^{\Delta L}q \end{Bmatrix}.$$

For the assumed isotropy of the material $C_{12} = C_{21} = 0$. Consider first the p constant test

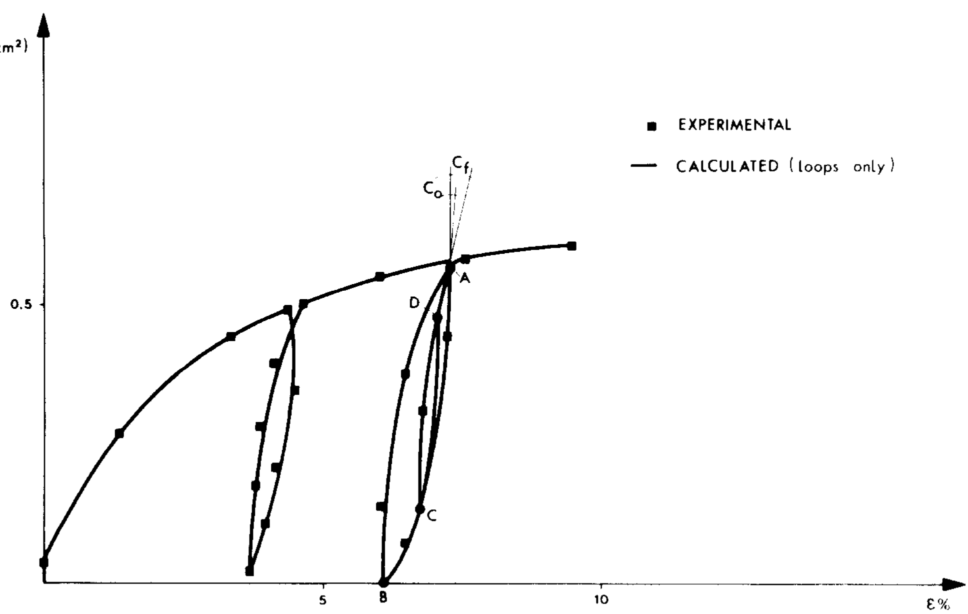

Fig. 8.

ABACDC (Fig. 8) entirely within the yield locus so that only paraelastic strains change. Suppose that A is a stress reversal point.

Along this path only deviatoric strains take place since $\underset{\sim}{C}$ is diagonal. Consider then only the element $C_{22} = L = L_0(1 + \beta\chi)$ where $\beta = \Omega_{22}$ and $L_0 = C_{022}$ are positive scalars, since $\underset{\sim}{C}_0$ must be positive definite and $\underset{\sim}{C}_\chi$ must be positive semi-definite. If β is zero the problem reduces to a trivial linear elastic unloading/reloading.

The variable χ is given by:

$$\chi = \sqrt{({}^{\Delta L}\epsilon^{pe})^2} = |{}^{\Delta L}\epsilon^{pe}|.$$

Since the stress path is extremely simple, the stress–strain law can be directly employed. Thus for paths "down":

$$\epsilon^{pe} = \epsilon_U{}^{pe} + \frac{L_0(q - q_U)}{1 + \beta(q - q_U)L_0}$$

where U should be substituted by A or D where appropriate, for paths "up":

$$\epsilon^{pe} = \epsilon_R{}^{pe} + \frac{L_0(q - q_R)}{1 - \beta(q - q_R)L_0} \quad R \text{ being } B \text{ or } C.$$

For the data in Fig. 8, from Wood[25], it can be established that $L_0 = 0.00571\ \text{cm}^2/\text{kg}, = 225{,}73$, at $p = 1\ \text{kg/cm}^2$ and the resulting stress–strain curves are presented in Fig. 8. Note that due to the form of the quadratic expression for the function $V({}^{\Delta L}\underset{\sim}{\sigma})$ at eqn (3.2) there is no coupling between the deviatoric and volumetric strain differences. Clearly, according to the theory there is no residual strain after the loop completion.

In the second example a complex path in the (p, q) plane will be considered. Such a path passing the points *ABCDEFGHILM* is shown in Fig. 9(a) and the calculated strains are reported in Figs. 9(b–d). The loading history consists of stress reversal (B), reactivation point (C), direction change with the continuation of the law at E, other reversals at F and G and

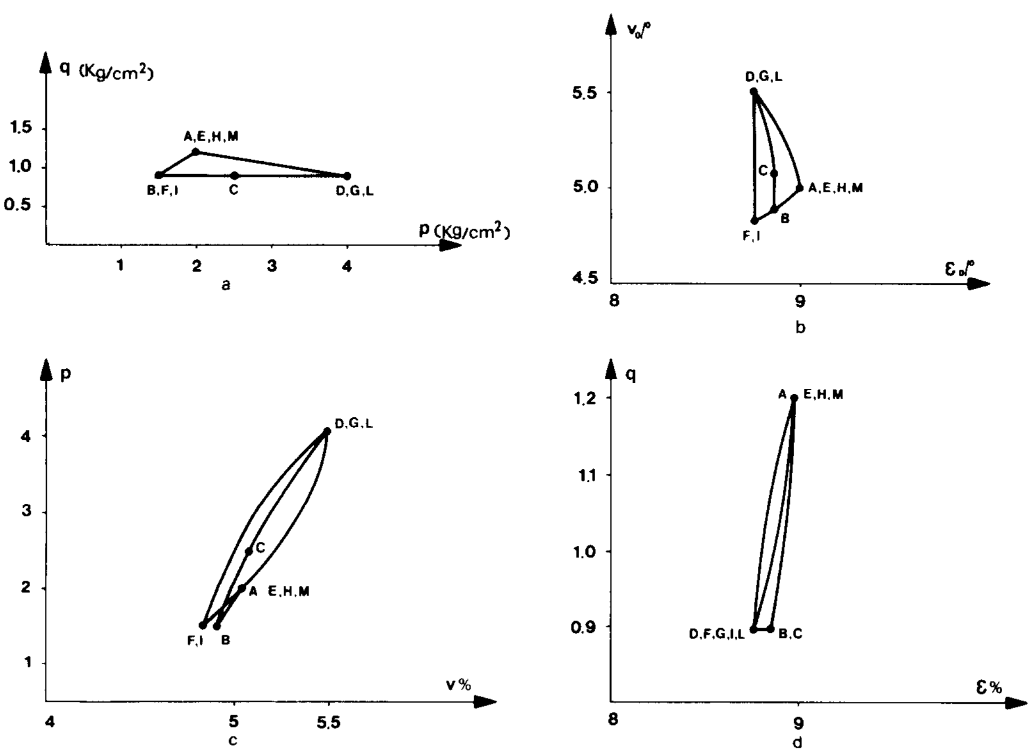

Fig. 9.

again changes with continuity at H and M. The parameters employed here are $C_{011} = 0.00135$ cm^2/kg, $C_{022} = 0.00335$ cm^2/kg, $\Omega_{11} = 144$, $\Omega_{22} = 240$, $q_A = 1.2$ kg/cm^2, $p_A = 2.0$ kg/cm^2, $\epsilon_A = 9\%$, $v_A = 5\%$. and correspond to a soft clay. The complex three point cycle stabilizes at the second circuit as visible from the plots. This results from an adaptation of the domain, formed by the stress reversal loci $\bar{W}^{D,F} = 0$ and $\bar{W}^{F,G} = 0$, at which on the second and subsequent circuits is insensitive to the forgotten dead locus $\bar{W}^{A,B} = 0$ at the point C. In reality such kind of adaptation occurs after several circuits, but the contribution of the first one is the most significant. Note that the "loop traveling" or "creep" does not result here from the constitutive law but is merely the effect of the memory changes in the process.

7. CONCLUSIONS

In this paper the concern has been exclusively with material hysteretic behaviour within the current yield locus. The behaviour at yielding, i.e. the pertinent constitutive law together with conditions for plastic loading and for unloading/reloading has been willingly left apart. To take into account such complex kind of behaviour, recourse may be made to the well developed continuum theory of plastic behaviour of geologic media, see Wroth and Schofield[3], Drucker and Prager[26], Sandler, Di Maggio[4], Mróz[27], Rudnicki and Rice[28], Maier and Hueckel[16], Nova[29]. Such a theory takes account of characteristic phenomena of the plastic behaviour of soils and rocks such as strain softening, hardening, dilatancy, the existence of a critical state and their interaction with material instability and response uniqueness together with the sensitivity of the unloading/reloading response to the plastic strain (elastoplastic coupling)—Bieniawski[14], Hueckel[15], Dafalias[30].

A theory that combines the two ranges of material behaviour, paraelastic and plastic strain-hardening or softening is beyond the scope of this presentation and will be persued elsewhere. It should be noted however, that any unloading in the sense of the theory of plasticity should be treated as a stress reversal. Moreover, what is in agreement with the experimental evidence, plastic strains occur from the very beginning of loading process, i.e. no virgin "elastic" range exists.

The structure of the present theory (stiffness matrix plus set of inequalities) complies with that of the plasticity theory and makes it possible to implement it into existing finite element computer codes without much additional effort.

The proposed relation for the non-linear hysteretic behaviour is based on full reversibility of strain over closed stress circuits between two subsequent stress reversal points and in particular over a unidimensional circuit. A tensorially linear stress–strain law of the Cauchy material kind with variable moduli is the simplest kind of non-linear relation enjoying piecewise path independence. The fact that any stress reversal is followed by the initial stiffness restoration makes it possible to construct any internal loop by means of the same material functions. In consequence, only two supplementary material constants are needed in addition to linear elastoplasticity.

There are two parameters that govern the loading process as understood in this theory, i.e. the local law origin $\hat{\underline{\sigma}}$ and the local strain amplitude parameter. An evolution rule for the first parameter is established in such a way that for any increment of stress it determines the origin with respect to which stress and paraelastic strain differences are referred in the current constitutive law. The rule rests on the appropriately formulated criteria of continuation, stress reversal or reactivation of some previous local law. The peculiarity of $\hat{\underline{\sigma}}$ is that it is a non-continuous (step-wise) function of the natural time of the process. The other parameter is defined locally between stress reversal or possible reactivation points.

Obviously a lot of work has been done in the field of alternating loading, as briefly mentioned in the introduction, and it would be of interest to relate the present considerations to the previous theories, mostly to that of Mróz[6, 7], Mróz and Lind[21] and Dafalias and Popov[8, 9]. There are three elements which constitute these theories, i.e. stress–strain relation, ranges of relation validity and hardening rule. In the theory of Mróz and Lind[12] and the one presented here the stress–strain relation is non-linear, formulated piecewisely in finite quantities (thus full reversibility is assumed over the hysteretic monodimensional circuits) and are piecewisely path independent. In [6–9] the stress–strain relation has an intrinsic incremental form: therefore all circuits are continuously path dependent. The loci in the present theory undergo piecewisely continuous isotropic hardening and a kind of kinematic hardening in discrete form (generation at new stress reversal points) in which it resembles[12], while hardening in[7–9] has more classical kinematic/isotropic character.

The question of hardening rule is closely connected with the employed strain amplitude parameter. It appears that various cyclic effects may be modelled by its appropriate choice. In this presentation a direct measure of the loop advance was used but it is easily conceivable to apply, e.g. a distance-type parameter as in[8] to describe "cyclic creep" or "cyclic softening". It is however still lacking a broader experimental basis that could make possible a construction of a more general theory. In this theory the emphasis was put on the phenomenological meaning of the formal structures used in the constitutive law and on the other hand on the more rigorous mathematical description of the discrete and hierarchic memory of the material.

The theory formulated in such a spirit does not describe the occurrence of irreversible strains on closed cycles within the yield locus. Experimental evidence shows however that in p constant closed cycles, non-negligible irrecoverable volumetric strains occur even if the stress path is entirely within the yield locus. This fact is reflected also in cyclic undrained tests by a pore water pressure built-up leading to eventual liquefaction of sand samples. Therefore the application of the theory at the present stage is limited to those cases in which the number of cycles is moderate.

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APPENDIX

The positive semi-definiteness of the matrix $\underset{\sim}{A}\underset{\sim}{C}$ may be shown by considering the following quadratic form

$$^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{A}\underset{\sim}{C}\,^{\Delta L}\underset{\sim}{\sigma} = {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}_x\,^{\Delta L}\underset{\sim}{\sigma}\cdot{}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}\underset{\sim}{C}\,^{\Delta L}\underset{\sim}{\sigma}. \tag{A.1}$$

Taking account of the positive definiteness of $\underset{\sim}{C}$ and positive semi-definiteness of $\underset{\sim}{C}_x$ both quadratic forms of the right hand side of (A.1) appear to be at least non-negative.

To prove the positiveness of $\chi-\chi^*$ it is sufficient to show that so is the following expression:

$$(\chi)^2-(\chi^*)^2 = {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}\underset{\sim}{C}\,^{\Delta L}\underset{\sim}{\sigma} - {}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}\underset{\sim}{C}_x\,^{\Delta L}\underset{\sim}{\sigma}\,{}^{\Delta L}\tilde{\underset{\sim}{\sigma}}\underset{\sim}{C}_x\underset{\sim}{C}\,^{\Delta L}\underset{\sim}{\sigma} \tag{A.2}$$

$$= \tilde{\underset{\sim}{M}}\underset{\sim}{M} - \tilde{\underset{\sim}{M}}\underset{\sim}{G}\tilde{\underset{\sim}{G}}\underset{\sim}{M} \tag{A.3}$$

where:

$$\underset{\sim}{M} = \underset{\sim}{C}\,^{\Delta L}\underset{\sim}{\sigma} \tag{A.4}$$

$$\underset{\sim}{G} = \underset{\sim}{C}_x\,^{\Delta L}\underset{\sim}{\sigma} \tag{A.5}$$

$$\underset{\sim}{M}_0 = \underset{\sim}{C}_0\,^{\Delta L}\underset{\sim}{\sigma}. \tag{A.6}$$

By virtue of (3.7)

$$\underset{\sim}{G} = (\underset{\sim}{M}-\underset{\sim}{M}_0)\frac{1}{(\tilde{\underset{\sim}{M}}\underset{\sim}{M})} \tag{A.7}$$

and thus the r.h.s. of (A.3) becomes

$$\tilde{\underset{\sim}{M}}\underset{\sim}{M}\cdot\tilde{\underset{\sim}{M}}\underset{\sim}{M}_0(2\tilde{\underset{\sim}{M}}\underset{\sim}{M}-\tilde{\underset{\sim}{M}}_0\underset{\sim}{M}). \tag{A.8}$$

Under the assumption that matrices $\underset{\sim}{C}$ and $\underset{\sim}{C}_x$ have the same eigenvectors it may be shown that their product is positive semi-definite. In fact

$$\underset{\sim}{C}\underset{\sim}{C}_x = \tilde{\underset{\sim}{T}}\,\mathrm{diag}\,[\lambda_i]\cdot\mathrm{diag}\,[\lambda_{\chi i}]\tilde{\underset{\sim}{T}} \tag{A.9}$$

where λ_i, and $\lambda_{\chi i}$, are the eigenvalues of $\underset{\sim}{C}$ and $\underset{\sim}{C}_\chi$ respectively, certainly all non-negative. $\underset{\sim}{T}$ is the orthogonal matrix composed of the common eigenvectors of $\underset{\sim}{C}$ and $\underset{\sim}{C}_\chi$. Thus

$$\tilde{\underset{\sim}{M}}(\underset{\sim}{M}-\underset{\sim}{M}_0) = {}^{\Delta L}\underset{\sim}{\sigma}\underset{\sim}{C}\underset{\sim}{C}_x\,^{\Delta L}\underset{\sim}{\sigma}\chi^L \geqslant 0. \tag{A.10}$$

By the same way of reasoning it can be shown that the product $\tilde{\underset{\sim}{M}}\underset{\sim}{M}_0$ is positive. In consequence, since the term in the parenthesis of (A.8) is greater over $\tilde{\underset{\sim}{M}}\underset{\sim}{M}>0$ than the value of the l.h.s. of (A.10). Thus (A.8) is never negative and finally

$$\chi-\chi^*>0. \tag{A.11}$$