AN ENGINEERING THEORY OF SOIL BEHAVIOUR IN UNLOADING AND RELOADING

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SOMMARIO. Si propone una legge costitutiva per descrivere il legame sforzi-deformazioni dei terreni sottoposti a processi di scarico e ricarico. Le equazioni costitutive sono formulate a tratti e definite su un dominio limitato dai luoghi di inversione di carico. Nella formulazione della legge costituitiva verranno introdotte delle variabili generalizzate di sforzo riferite all'ultimo punto di inversione di carico. Queste variabili sono legate alle deformazioni, riferite anche esse allo stato relativo all'ultimo punto di inversione di carico, da un tensore di cedevolezza variabile. Un'adeguata formulazione di questo tensore permette di modellare l'effetto di densificazione sotto carico deviatorico ciclico.

Questa legge costitutiva interpreta bene i risultati sperimentali su argille normalmente consolidate e sovraconsolidate. La teoria permette anche di descrivere la dipendenza del percorso degli sforzi efficaci in condizioni non drenate dal grado di sovraconsolidazione, la mobilità ciclica dell'argilla in condizioni non drenate e il percorso degli sforzi efficaci in un processo di scarico e ricarico in un edometro.

Per identificare il modello sono necessari solo tre parametri oltre a quelli necessari per identificare il comportamento del terreno vergine.

SUMMARY. A constitutive law is proposed for describing the stress-strain characteristic of soils in unloading-reloading. The constitutive equations are valid piecewisely between subsequent, appropriately formulated, stress reversal loci. The stress-strain relationships are formulated in terms of generalized stress and strain differences referred to the last stress reversal point and connected through a variable compliance tensor. The shear compaction effect is modelled by a suitable formulation of the compliance tensor.

Specialization to conventional triaxial condition is given. As well as fitting available experimental data in unloading--reloading of normally consolidated and overconsolidated clays, the proposed constitutive relation can model the dependence on OCR of the shape of the undrained effective stress paths, the phenomenon of cyclic mobility of clay in undrained compression and the unloading-reloading stress paths in the oedometer. The theory requires the identification of only three material constants in addition to those pertinent to the usual elastoplastic theory of soil with which it may be easily combined.

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1. INTRODUCTION.

The paper deals with the modelling of the loading, unloading and reloading behaviour of soils, rocks and other media which are sensitive to the mean stress. Special emphasis is laid on the description of hysteresis which occurs in unloading and reloading.

The geotechnical interest in the question comes from the fact that most natural deposits are overconsolidated. Therefore if loaded they undergo in fact a reloading process, at least for moderate loads. On the other hand normally consolidated soils may experience in earth works non-negligible unloading so that any further loading appears effectively to be again a reloading. It is thus of practical importance to accurately model the reloading behaviour of geologic media.

As a matter of fact, the majority of the available mathematical models adequately deal mainly with loading, understood as an elastic-plastic process, and unloading, recognized as a purely, often non-linear, elastic one. In contrast there is a scarcity of theoretical and experimental investigations as far as reloading is concerned. Indeed, the reloading behaviour is often considered identical to the unloading one. On the other hand, experimental evidence shows that this is justifiable only, but not always, at low stress ratios or for moderate unloading. If the material has experienced an advanced plastic strain, wide hysteresis loops occur in unloading-reloading cycle. In such cases, it often happens, especially for rocks, that the loop widths are larger than the plastic deformations. Fig. 1 illustrates this feature for a sample of Katowice coal tested in uniaxial compression. It is clear then, that the description of the material behaviour under variable loading should be complemented by taking into account the occurence of the strains relative to the loop.

The peculiarity of the behaviour of soils, rocks and other granular or frictional materials under shear stress consists in the dependence on the mean normal effective stress p (1).

First of all, it is known that the shear stress at failure is a function of p as well as the current yield locus, as understood

(1) Since in this paper only effective stresses will be considered, the usual dash denoting effective stresses will be omitted for the sake of simplicity of notation.

e.g. in the theory of Cam Clay, Schofield and Wroth (1968). Moreover, at variance with the behaviour of non-frictional materials, such as steel, the shear strain rates do depend on the mean effective stress rate. In addition, experimental evidence shows, e.g. Wroth (1965), that for a given stress path in a standard triaxial test, a unique relation can be established between the deviatoric strain and the ratio of the deviatoric stress and p rather than with the deviatoric stress itself.

Another well recognized feature of the p dependence of the behaviour of geologic media is the non-linear relation between p and volumetric strain in isotropic tests. It is commonly accepted that both in loading and unloading the stress strain relationship can be well approximated by a logarithmic function. The reloading behaviour is different from the unloading one but shows similar features.

For the aforementioned reasons a multidimensional (tensorial) constitutive law of the material behaviour, in the hysteretic range will be formulated, in terms of strains and of «generalized stresses», denoting with that a convenient normalization of stresses with respect to the mean pressure.

The approach that will be presented in this paper is an extension of the main ideas developed in previous papers by the authors (Hueckel and Nova, 1979[a] - 1979[b]). Two different modes of behaviour will be distinguished: the elastoplastic one and the hysteretic. The former is intended as in the theory of hardening plasticity in Soil Mechanics. The elastoplastic model that will be used here is an extension of Cam Clay (Nova (1977), Nova (1979), Nova and Wood (1979)). The latter is described piecewisely between subsequent appropriately formulated loci.

The validity of the theory proposed will be checked by means of several comparisons with available experimental data. It will be shown that besides the modelling of hysteretic behaviour, the theory encompasses also phenomena related to shear compaction and cyclic mobility.

In the following, the material under consideration will be regarded as isotropic, inviscid and deformed in the range of small strains. The term stress will always mean effective stress. Stress and strain will be taken positive in compression.

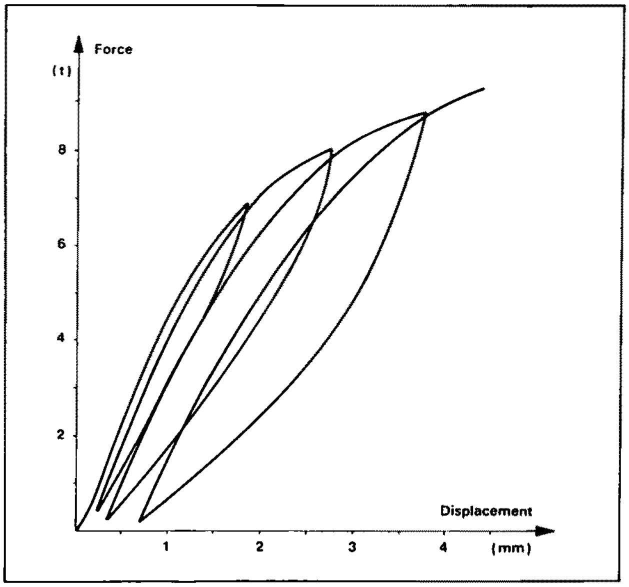

Fig. 1. Hysteresis loops of Katowice coal in uniaxial compression.

2. PREVIEW OF THE STRUCTURE OF THE THEORY.

The theory of hardening plasticity is strictly connected with the notion of a yield locus, understood as a history dependent limiting surface, in the stress space, which separates elastoplastic and elastic behaviour. During an elastoplastic loading, the locus undergoes a variation in dimensions, shape or position depending on the type of hardening or softening considered.

The elastoplastic model adopted here employes an isotropic hardening, which depends not only on the volumetric but also on the deviatoric plastic strain. This implies that failure and critical state may be not coincident both for normally consolidated and overconsolidated soils. The theory encompasses also the softening behaviour. The form of the yield locus is that shown in Fig. 2.

Within the yield locus the behaviour is not considered as elastic but instead hysteretic or «paraelastic» as defined in (Hueckel and Nova, (1979[a] - 1979[b])). In those papers a theory of hysteresis had been presented with special emphasis on the material memory. For the sake of clarity the constitutive relation between stress and strain had been taken as simple as possible. The assumed stress strain relation in unloading reloading rests on the experimental results and the model for unidimensional paths presented by Hardin and Drnevich (1972). In that spirit it was assumed that in unloading the strain difference between the current value of the strain and the value of the strain when unloading began, is linked to the corresponding stress difference by the relation

$$\Delta\epsilon = C\,\Delta\sigma \tag{2.1}$$

where the compliance modulus C depends on $\Delta\epsilon$ in the following way

$$C = C_0(1 + \omega\,\Delta\epsilon) \tag{2.2}$$

where C_0 denotes the initial value of the compliance and ω is a material parameter.

If the load is reversed, the equations (2.1) and (2.2) still

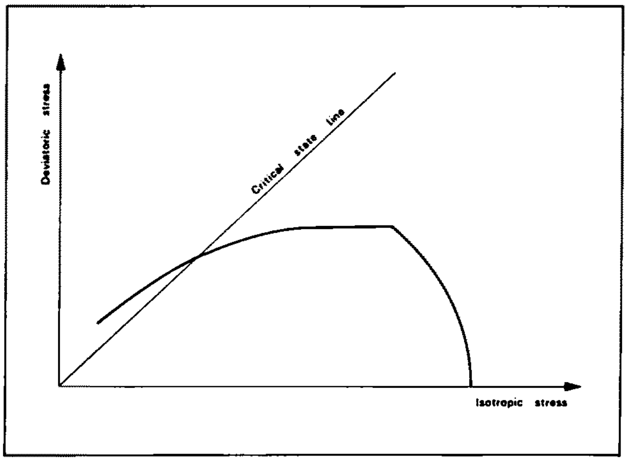

Fig. 2. Assumed form of the yield locus.

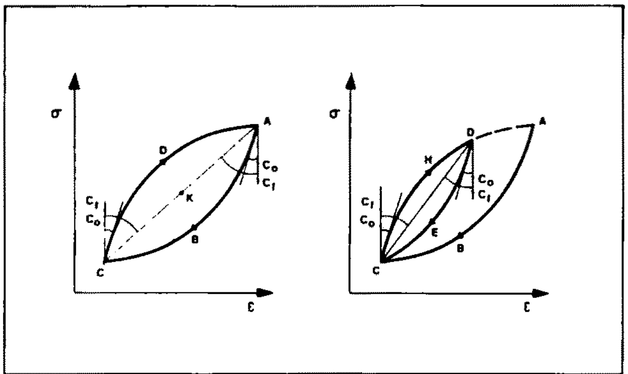

Fig. 3. Elementary hysteretic loops.

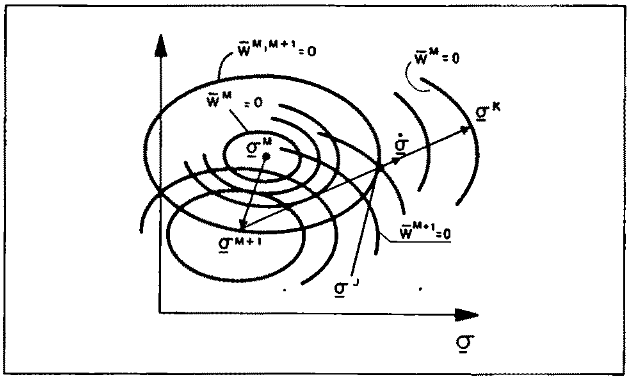

Fig. 4. Generation, growth and reactivation of stress reversal loci.

hold but the point to which one should refer stress and strain is now the load reversal point. In this way the initial compiance modulus is restored after reversal and the unloading--reloading curve is polar symmetric with respect to the center of the loop as shown in Fig. 3.

The extension to multiaxial conditions of the stress strain relation is as follows. If ${}^{\Delta L}\epsilon_{ij}$ and ${}^{\Delta L}\sigma_{hk}$ are the strain and stress differences tensors, respectively, referred to the Lth stress reversal point a tensorially linear relationship is postulated such that

$$ {}^{\Delta L}\epsilon_{ij} = C_{ijhk}\,{}^{\Delta L}\sigma_{hk} \tag{2.3} $$

where C_{ijhk} is an isotropic compliance tensor. The intrinsic loop non-linearity is reflected by the variation of the compliance C_{ijhk} with the norm of the strain difference, referred to as strain amplitude parameter

$$ \chi = ({}^{\Delta L}\epsilon_{ij}\,{}^{\Delta L}\epsilon_{ij})^{1/2} \tag{2.4} $$

in the following way

$$ C_{ijhk} = C^{0}_{ijrs}(\delta_{rshk} + \chi\Omega_{rshk}) \tag{2.5} $$

δ_{rshk} is defined as $\delta_{rshk} = 1/2\,(\delta_{rh}\delta_{sk} + \delta_{rk}\delta_{sh})$. It acts on C^{0}_{ijrs} that is by hypothesis symmetric, as a unit tensor. Ω_{rshk} is a tensor whose components may be experimentally determined, as shown in Sec. 6.

The tensor C^{0}_{ijrs} is the initial compliance tensor. The strain difference part which is related through C^{0}_{ijrs} to ${}^{\Delta L}\sigma_{rs}$ is referred to as elastic part, while the part related to ${}^{\Delta L}\sigma_{rs}$ through $C_{ijrs} - C^{0}_{ijrs}$ will be called microplastic.

In multiaxial conditions it is not a priori clear what the terms stress reversal, unloading and reloading, mean. The criterion for stress reversal adopted in the theory was based on the stress rate orientation. A locus in the stress space is conceived which determines the limit between stress rates which give rise to the continuation of the current law or to the stress reversal. For the former case the stress rate vectors are directed outwards the locus whilst in the latter they are oriented inwards. The continuation condition was connected with the condition that the strain amplitude does not decrease $\dot{\chi} \geqslant 0$. The stress rate that would violate the condition is inadmissible within the current law and is then thought to results in a stress reversal. The equation of the current locus is consequently, from (2.3), (2.4),

$$ \bar{W}^{L} = C_{ijhk}\,{}^{\Delta L}\sigma_{hk}\,C_{ijhk}\,{}^{\Delta L}\sigma_{hk} - \text{const} = 0. \tag{2.6} $$

The locus for which a stress reversal occurs is called dead locus and is stored in the memory of the material.

Investigations of simple uniaxial cycles allow to infer that the material enjoys a discrete memory organised in two levels in a hierarchic way. The first level, active, governs the current constitutive law, the second remembers all the dead loci in stack. If a stress path, say $M, M+1, K$ Fig. 4 touches and crosses one of these loci, say $\bar{W}^{M,M+1} = 0$ the memory is updated. All the «younger» dead loci and the current locus, say $\bar{W}^{M+1}$ are forgotten, whilst the current origin is shifted back to the point which was the origin for the crossed dead locus, say M. This locus is no more a dead one but it is reactivated together with the corresponding stress strain law. The constitutive law of the material can be then considered as piecewisely holonomic and ruled by a hierarchic discrete memory. The plastic yield locus acts as the oldest dead locus, that is the hierarchically most important. If the stress path crosses the plastic yield locus all the hysteresis loci are then canceled from the memory of the soil.

The simple linearized constitutive law employed in the aforementioned papers made it possible to formulate a mathematical theory of hysteresis and of the structure of the material memory. In this paper, it will be shown that main concepts of that theory, i.e. the concept of stress reversal, the polar symmetry of the stress strain relation in radial tests, the piecewise holonomic tensorially linear constitutive law and some effects of the structure of the memory give rise to good agreement between available experimental data and calculated results in many different tests on soil.

However, to do that the highly non linear behaviour of the soil in the considered range must be taken into account. First of all in a cycle of unloading and reloading of a soil under purely isotropic pressure, experimental evidence shows-see for instance Fig. 5 after Som (1968), that the hysteresis loop in the plane isotropic pressure, p, versus volumetric strain, υ, does not directly exhibit the property of polar symmetry. However, by introducing a new set of variables υ, $\ln p/p_0$ in which the above results are replotted (Fig. 6), it is shown that an acceptable idealisation of the loop rests on the assumption that the property of polar symmetry holds. The constant p_0 denotes a convenient reference pressure.

Moreover, as far as deviatoric stress and strain, q and ϵ

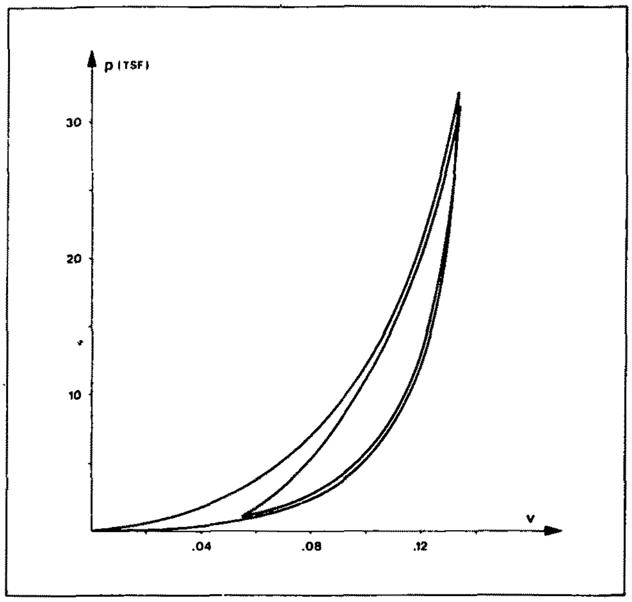

Fig. 5. Unloading-reloading in isotropic compression-data after Som (1968).

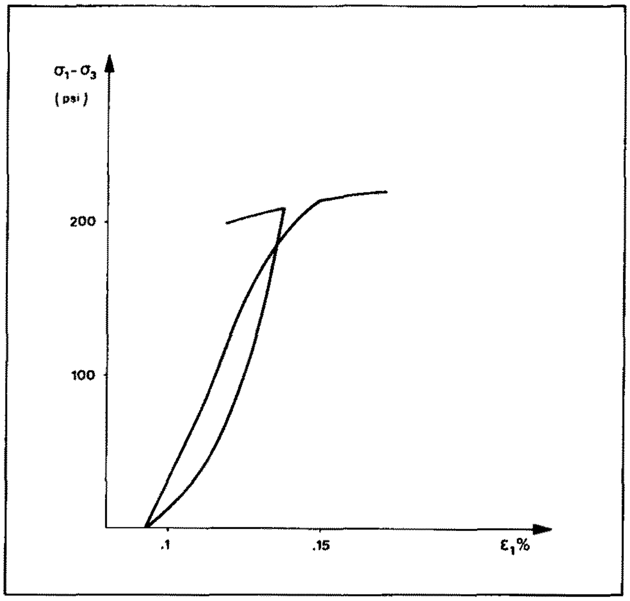

Fig. 7. Constant cell pressure drained triaxial test on Ottawa sand-data after Holubec (1966).

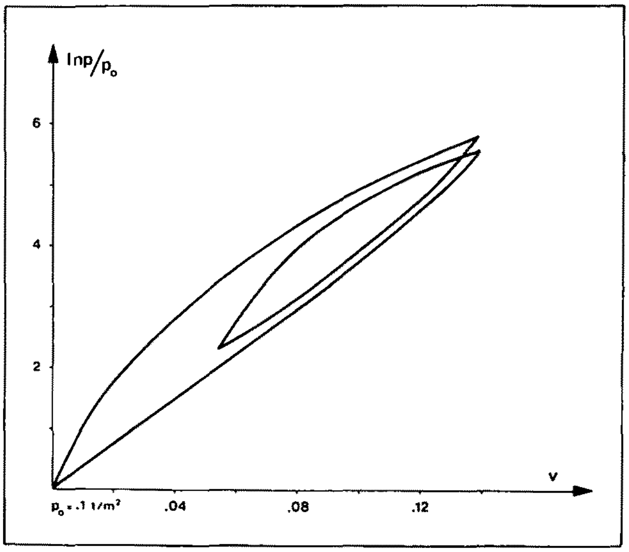

Fig. 6. Data of Fig. 5 plotted in a semi-log scale.

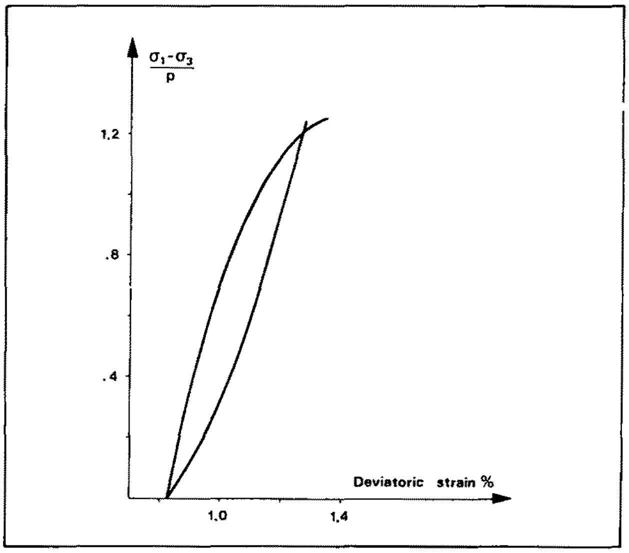

Fig. 8. Data of Fig. 7 plotted in the $\eta - \epsilon$ plane.

respectively, are concerned, this kind of symmetry is not obeyed in triaxial constant cell pressure tests, as shown in Fig. 7, data after Holubec (1966). For intance, the concavity of the essential part of the reloading branch is the same as that of the unloading one. However, by replotting these data in the plane of stress ratio, $\eta = q/p$ and vertical strain ϵ_1, the polar symmetry of the loop is restored (Fig. 8). Clearly, in p constant tests in both coordinate systems this property is respected, as confirmed by experimental data (Fig. 9) after Parry (1956).

Thus, in the following, the constitutive law, linear in tensorial variables, will be formally written as in (2.3) but the strain differences $^{\Delta L}\epsilon_{ij}$ will be linked through a compliance tensor C_{ijhk} not to the stress differences $^{\Delta L}\sigma_{hk}$ but to a tensor $^{\Delta L}t_{hk}$ of generalized stress that will be properly defined in the next section to take into account the aforementioned experimental findings.

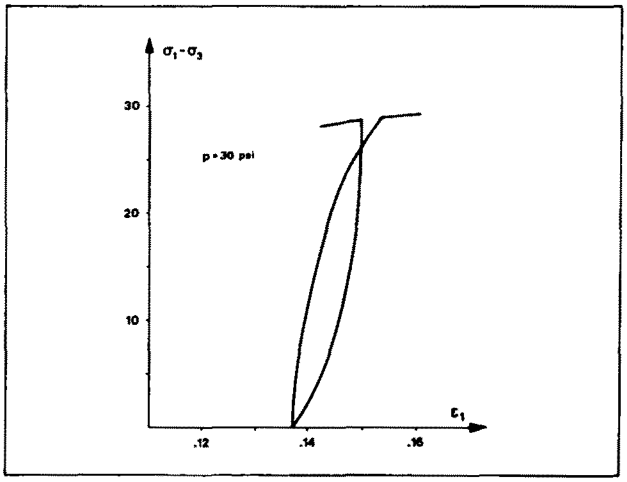

Fig. 9. p-constant drained triaxial test on clay, unloading-reloading loop-data after Parry (1956).

Also, a convenient definition of ${}^{\Delta L}t_{hk}$ will allow to describe the accumulation of volumetric strains in purely deviatoric unloading-reloading cycles and therefore to model the pressure built up in cyclic undrained tests.

The hysteretic and the elastoplastic models of behaviour are treated as independent. However, the amount of plastic strain experienced by the soil has a non-negligible influence on the parameters characterizing the hysteretic behaviour. To take account of this fact the tensor of initial compliances C^0_{ijrs} may be assumed to be a function of a plastic strain parameter. This dependence can be described by means of a concept of elastoplastic coupling (Hueckel (1975), Maier and Hueckel (1977)).

3. THE ENGINEERING STRESS-STRAIN RELATION IN THE HYSTERESIS RANGE.

In this Section then phenomenological observations discussed in Section 2 will be set into a mathematical multidimensional formulation. Consider a single branch of a hysteresis loop. For the sake of simplicity assume first that the current origin, to which the actual stresses and strains are referred in the sense specified above, is characterized by zero strain and stress state equal to the atmospheric pressure p_a.

In the isotropic test the observed strain-stress relation can be formally written as follows

$$\epsilon_{ii} = B \ln \frac{\sigma_{ii}}{3p_a} \tag{3.1}$$

where ϵ_{ij} and σ_{ij} are the strain and stress tensors. The Einstein summation convention is adopted throughout the paper. B is referred to as the bulk compliance modulus and is a function of the strain amplitude parameter χ.

For a generic stress path the observed relation between deviatoric strains e_{ij} and eviatoric stresses s_{ij}, defined as

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij}; \qquad s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3.2}$$

where δ_{ij} denotes the Kronecker symbol, can be written as follows by use of a proper normalization of the stress deviator with respect to the isotropic pressure:

$$e_{ij} = L\eta_{ij}; \qquad \eta_{ij} \equiv \frac{3s_{ij}}{\sigma_{kk}} \tag{3.3}$$

where L is the shear compliance modulus and is a function of χ. The tensor η_{ij} is referred to as the stress ratio tensor. The tensor η_{ij} and $\ln \sigma_{ii}/3p_a$ will be called generalized stress variables.

To take account of the shear compaction phenomenon for a generic stress path the volumetric strains will be assumed as given by

$$\epsilon_{ii} = B\left(\ln \frac{\sigma_{ii}}{3p_a} + \theta I_{2\eta}\right) \tag{3.4}$$

where $I_{2\eta}$ is the second invariant of the stress ratio tensor η_{ij}

$$I_{2\eta} \equiv (\eta_{kl}\eta_{kl})^{1/2} = \frac{3}{\sigma_{ii}}(s_{kl}s_{kl})^{1/2}. \tag{3.5}$$

The parameter θ is a material constant called the shear compaction parameter. Clearly, equation (3.1) is a special case of equation (3.4).

Formally, the equation (3.2), (3.4) can be put in a more compact form similar to equation (2.3) by introducing another generalized stress tensor t_{kl} defined as

$$t_{kl} = \eta_{kl} + \frac{1}{3}\left(\ln \frac{\sigma_{ii}}{3p_a} + \theta I_{2\eta}\right)\delta_{kl}. \tag{3.6}$$

Note that the tensor η_{kl} is the deviator of the tensor t_{kl}. Consequently, from (3.3) and (3.4) it results that

$$\epsilon_{ij} = C_{ijkl}(\chi)\, t_{kl}. \tag{3.7}$$

The isotropic compliance tensor C_{ijkl} may be expressed in terms of moduli B and L, as follows (see e.g. Fung, (1965), p. 415)).

$$C_{ijkl} = \frac{1}{3}(B - L)\,\delta_{ij}\delta_{kl} + \frac{1}{2}L(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{3.8}$$

The relation (7.7) is linear in the tensorial variables ϵ_{ij}, t_{kl} whilst its non-linearity consists in the dependence of the moduli B and L on the *scalar* variable χ.

Equation (3.7) in a more-explicit form reads

$$\epsilon_{ij} = \frac{1}{3}B(x)\left(\ln \frac{\sigma_{ii}}{3p_a} + \theta I_{2\eta}\right)\delta_{ij} + L(x)\,\eta_{ij}. \tag{3.9}$$

In order to determine the stress tensor σ_{ij} from the components of the tensor t_{kl} it is useful to find first the isotropic term

$$\sigma_{kk} = 3p_a \exp\,(t_{ii} - \theta I_{2\eta}). \tag{3.10}$$

Then, by inverting (3.6) one arrives at

$$\sigma_{kl} = p_a \exp\,(t_{ii} - \theta I_{2\eta})\left[t_{kl} + \left(1 - \frac{1}{3}t_{ii}\right)\delta_{kl}\right]. \tag{3.11}$$

It will be shown that the introduction of the aforementioned set of generalized stress variables allows to model peculiarities of soil behaviour. On the other hand, it preserves the known mathematical structure of the tensorial time-scale independent constitutive laws.

Let us then adopt the above introduced variables in the constitutive relation of the hysteretic behaviour. These relations are formulated piecewisely between consecutive «stress reversal points» (S.R.P.) in the spirit of the paper by Hueckel and Nova (1979[b]). Suppose that the last S.R.P. occurred at $\sigma_{hk} = \sigma^L_{hk}$ to which correspond the strain $\epsilon_{hk} = \epsilon^L_{hk}$ The stress ratio difference tensor is then defined as

$${}^{\Delta L}\eta_{hk} \equiv \eta_{hk} - \eta^L_{hk}. \tag{3.12}$$

Note that ${}^{\Delta L}\eta_{hk} \neq 3\,{}^{\Delta L}s_{hk}/{}^{\Delta L}\sigma_{ii}$. The stress difference in terms of the logarithmic measure of the isotropic pressure may be expressed as follows

$$^{\Delta L}\ln \frac{\sigma_{kk}}{3p_a} \equiv \ln \frac{\sigma_{kk}}{3p_a} - \ln \frac{\sigma_{kk}^L}{3p_a} = \ln \frac{\sigma_{kk}}{\sigma_{kk}^L} \tag{3.13}$$

$$^{\Delta L}t_{kl} = {}^{\Delta L}\eta_{kl} + \frac{1}{3}\left(\ln \frac{\sigma_{ii}}{\sigma_{ii}^L} + \theta I_{2\,\Delta\eta}\right)\delta_{kl} \tag{3.14}$$

where the invariant of the stress ratio difference tensor $I_{2\,\Delta\eta}$ is given by

$$I_{2\,\Delta\eta} = ({}^{\Delta L}\eta_{kl}\,{}^{\Delta L}\eta_{kl})^{1/2}. \tag{5.15}$$

The inverse relation to (3.14) may be reached by finding

$$\sigma_{kk} = \sigma_{kk}^L \exp\,({}^{\Delta L}t_{kk} - \theta I_{2\,\Delta\eta}) \tag{3.16}$$

and thus

$$\sigma_{kl} = \frac{1}{3}\,\sigma_{kk}^L \exp\,({}^{\Delta L}t_{kk} - \theta I_{2\,\Delta\eta}) \cdot \tag{3.17}$$

$$\cdot \left[{}^{\Delta L}t_{kl} + \left(1 - \frac{1}{3}\,{}^{\Delta L}t_{ii}\right)\delta_{kl} - \eta_{kl}^L\right].$$

The current strain-generalized stress relation in terms of differences may be written by analogy to (2.3) in the form

$$^{\Delta L}\epsilon_{ij} = C_{ijkl}(\chi)\,{}^{\Delta L}t_{kl} \tag{3.18}$$

or directly in terms of bulk and shear compliance moduli

$$^{\Delta L}\epsilon_{ij} = \frac{1}{3}\,B\left(\ln \frac{\sigma_{kk}}{\sigma_{kl}^L} + \theta I_{2\,\Delta\eta}\right)\delta_{ij} + L\,{}^{\Delta L}\eta_{ij}. \tag{3.19}$$

The inverse relations may be easily obtained and read

$$\frac{\sigma_{ii}}{\sigma_{ii}^L} = \exp\left(\frac{^{\Delta L}\epsilon_{ii}}{B} - \frac{\theta}{L}\sqrt{^{\Delta L}e_{ij}\,{}^{\Delta L}e_{ij}}\right) \tag{3.20}$$

$$^{\Delta L}\eta_{ij} = \frac{1}{L}\,{}^{\Delta L}e_{ij}. \tag{3.21}$$

The above definition of $I_{2\,\Delta\eta}$ is adopted in order to describe the effect of shear compaction in cyclic unloading-reloading. In fact, $I_{2\,\Delta\eta}$ is always positive and in a constant mean stress test, $\sigma_{kk} = \sigma_{kk}^L$, the volume changes are proportional to $I_{2\,\Delta\eta}$ and thus give rise to compaction independently of the orientation of the stress path i.e.

$$^{\Delta L}\epsilon_{ii} = \theta B I_{2\,\Delta\eta} \tag{3.22}$$

similarly for a constant volume process, $^{\Delta L}\epsilon_{kk} = 0$

$$\ln \frac{\sigma_{ii}}{\sigma_{ii}^L} = -\,\theta I_{2\,\Delta\eta}. \tag{3.23}$$

The introduction of shear compaction into the constitutive law is paid with the fact that $^{\Delta L}t_{kl} \neq t_{kl} - t_{kl}^L$, so that the whole discussion of the memory effects can not be graphically represented in the t_{kl} space. On the other hand when considered in the space η_{kl}, $\ln \sigma_{ii}/3p_a$ the stress reversal loci are characterized by a corresponding suitably modified compliance tensor, which is not symmetric.

The reference point with respect to which the actual generalized stress-strain law is formulated undergo a switch when the stress reversal occurs in the sense specified in Section 2.

As already stated the stress reversal locus is expressed by the equation

$$\chi = \text{const} \tag{3.24}$$

where χ is defined by equation (2.4). The condition of continuation of the validity of the current constitutive law, $\dot{\chi} \geqslant 0$, may be expressed in terms of the generalized stress rates as in the previous paper by the authors (1979[b]) or directly by comparing the updated value of χ after a stress or strain increment with the value of χ prior to the increment. If the resulting difference appears to be positive the validity of the constitutive law still holds, otherwise a stress reversal occurs and the current stress point becomes the reference to the new portion of the law. Such approach allows to avoid the necessity of introduction of the rate equation.

4. GENERAL ELASTOPLASTIC FORMULATION.

In the preceding Sections the considerations have been confined to processes within the actual *yield* locus. The yield function $f = 0$ adopted in this paper will be assumed to depend on the generalized stress variables η_{kl} and $\ln (3\sigma_{ii}/p_a)$

$$f = f\left(I_{2\eta}, \ln \frac{3\sigma_{ii}}{p_a}, x\right) \tag{4.1}$$

where x is a hardening parameter.

For processes involving plastic yielding it will be assumed that the strain rate is the sum of an irreversible and a reversible part

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^r + \dot{\epsilon}_{ij}^i. \tag{4.2}$$

As already mentioned in Section 2, the yield locus can be treated as a particular stress reversal locus. Consequently, it is assumed that together with the development of the plastic yielding, a reactivation of the yield locus implies the reactivation of the law governing the reversible and microplastic deformations relevant to the origin of axes in the stress space.

The reversible part of the strain rate may be found from (2.5) to be

$$\dot{\epsilon}_{ij} = C_{ijkl}^0\,\dot{t}_{kl} \tag{4.3}$$

where C_{ijkl}^0 is the initial compliance tensor.

In this theory all the contributions to irreversible strain rate, i.e. microplastic, coupling induced and plastic yielding strain rates, are treated jointly. This implies the introduction of a flow rule, in general nonassociated, defining the joint irreversible strain rate as follows

$$\dot{\epsilon}_{ij}^i = \Lambda\,\frac{\partial g}{\partial t_{kl}} \tag{4.4}$$

where Λ is a plastic multiplier and $g = g(\eta_{kl}, \ln 3\sigma_{ii}/p_a)$ is the plastic potential.

For explicit forms of Eqs. (4.1) - (4.4), specialized to triaxial conditions see Section 6.

5. SPECIALIZATION TO «TRIAXIAL» CONDITIONS.

In the following Sections some typical laboratory loading path are discussed together with the available experimental data. The specialization of the constitutive relations of Section 3 to specific conditions of symmetry imposed in these tests is then required.

The conditions of symmetry assumed hereinafter are

$$\sigma_{22} = \sigma_{33}; \qquad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \tag{5.1}$$

$$\epsilon_{22} = \epsilon_{33}; \qquad \epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0. \tag{5.2}$$

The conditions are commonly accepted to be pertinent to tests such as standard triaxial and oedometric. The deviatoric stress and strain components are then respectively

$$\begin{aligned} s_{11} &= \frac{2}{3}(\sigma_{11} - \sigma_{33}); & e_{11} &= \frac{2}{3}(\epsilon_{11} - \epsilon_{33}) \\ s_{22} = s_{33} &= \frac{1}{3}(\sigma_{33} - \sigma_{11}); & e_{22} = e_{33} &= \frac{1}{3}(\epsilon_{33} - \epsilon_{11}) \\ s_{12} = s_{13} = s_{23} &= 0; & e_{12} = e_{13} = e_{23} &= 0. \end{aligned} \tag{5.3}$$

Defining as usual

$$p \equiv \frac{1}{3}\sigma_{kk} = \frac{1}{3}(\sigma_{11} + 2\sigma_{33}); \qquad q \equiv \sigma_{11} - \sigma_{33}; \qquad \eta \equiv \frac{q}{p} \tag{5.4}$$

$$v \equiv \epsilon_{kk} \equiv \epsilon_{11} + 2\epsilon_{33}; \qquad \epsilon \equiv \frac{2}{3}(\epsilon_{11} - \epsilon_{33}). \tag{5.5}$$

Eq. (3.19) can be reduced to a couple of scalar equations

$$^{\Delta L}\epsilon = \frac{2}{3} L\, ^{\Delta L}\eta \tag{5.6}$$

$$^{\Delta L}v = B\left(\ln\frac{p}{p_L} + \sqrt{\frac{2}{3}}\,\theta\, |^{\Delta L}\eta|\right) \tag{5.7}$$

or alternatively in the matrix form

$$\begin{Bmatrix} ^{\Delta L}v \\ ^{\Delta L}\epsilon \end{Bmatrix} = \begin{bmatrix} B & \sqrt{\frac{2}{3}}B\theta \operatorname{sgn}\eta \\ 0 & \frac{2}{3}L \end{bmatrix} \begin{Bmatrix} \ln\frac{p}{p_L} \\ ^{\Delta L}\eta \end{Bmatrix}. \tag{5.8}$$

As already mentioned the compliance matrix in the above coordinates is not symmetric. The compliance moduli B and L are functions of the strain amplitude parameter χ which becomes in triaxial conditions equal to

$$\chi \equiv (^{\Delta L}\epsilon_{11}^2 + 2\,^{\Delta L}\epsilon_{33}^2)^{1/2} = \left(\frac{1}{3}\,^{\Delta L}v^2 + \frac{3}{2}\,^{\Delta L}\epsilon^2\right)^{1/2} = \tag{5.9}$$

$$= \left\{\frac{1}{3}B^2\left(\ln\frac{p}{p_L} + \sqrt{\frac{2}{3}}\,\theta\,|^{\Delta L}\eta|\right)^2 + \frac{2}{3}L^2\eta^2\right\}^{1/2}.$$

The compliance moduli become

$$B = B_0(1 + \omega_v \chi) \tag{5.10}$$

$$L = L_0(1 + \omega_\epsilon \chi) \tag{5.11}$$

where ω_v and ω_ϵ are two material constants which can be experimentally determined, as shown in the following section

6. COMPARISON WITH TEST DATA.

By using the equations derived in the previous section (Eq. (5.8)) it is possible to compare theoretical predictions with available experimental results. In this paper the verification of the constitutive law only will be performed. The verification of the structure of the material memory in complex loading paths requires ad hoc tests implying several stress reversals and will be possibly pursued elsewhere.

Consider first an isotropic test. The behaviour in unloading-reloading only will be analyzed. From Eqs. (5.8), (5.9) it results that

$$^{\Delta L}v = B_0\left(1 + \frac{\omega_v}{\sqrt{3}}|^{\Delta L}v|\right)\ln\frac{p}{p_L} \qquad ^{\Delta L}\epsilon = 0. \tag{6.1}$$

Fig. 10 shows a comparison between calculated and experimental curve (for overconsolidated London clay) data after Som (1968). The parameter $B_0 = .00833$ has been determined from the initial tangent of the first reloading curve in the semilogarithmic plot. The parameter $\omega_v = 23.33$ has been found by imposing that the calculated curve goes through the point at which stress reversal occurs. The overall accuracy of the prediction seems reasonable.

Consider now a p constant test. Eqs. (5.8) give in this case

$$^{\Delta L}v = \alpha(1 + \omega_v \chi)\,|^{\Delta L}\eta| \tag{6.2}$$

$$^{\Delta L}\epsilon = \gamma(1 + \omega_\epsilon \chi)\,^{\Delta L}\eta \tag{6.3}$$

where for convenience $\sqrt{2/3}\,\theta B_0$ has been denoted with α

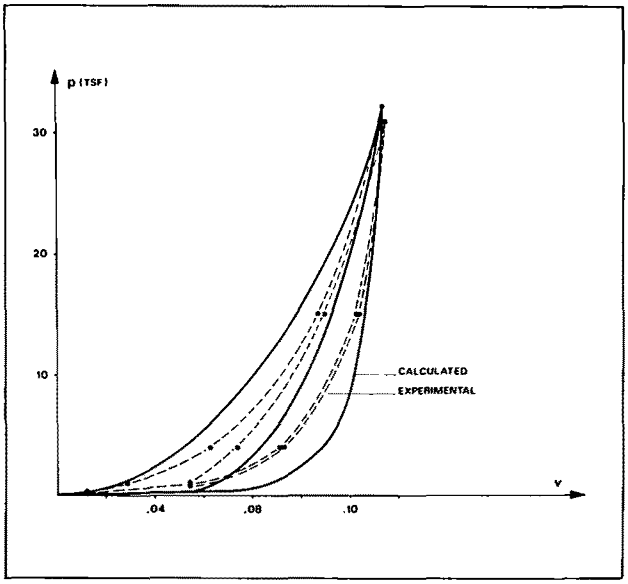

Fig. 10. Comparison between experimental and calculated results in isotropic unloading reloading-data of Fig. 5.

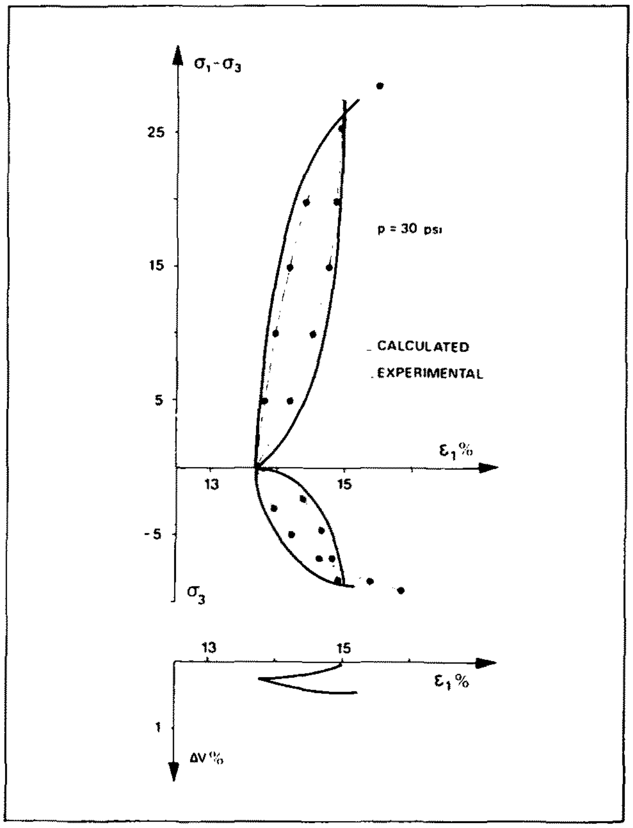

Fig. 11. Comparison between experimental and calculated results in *p*-constant test-data of Fig. 9.

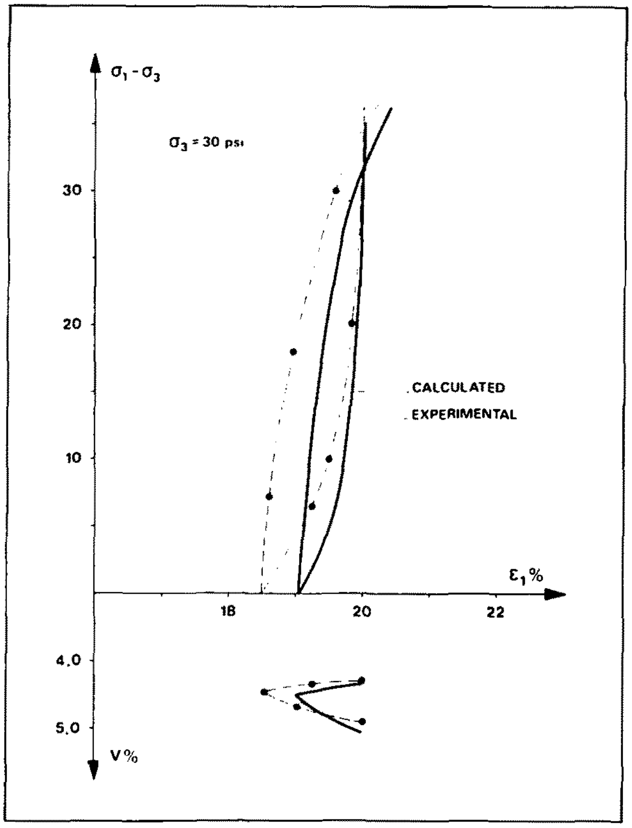

Fig. 12. Comparison between experimental and calculated results in constant cell pressure drained test-data after Parry (1956).

and $2/3L_0$ with γ. Eliminating χ from Eqs. (6.2), (6.3) and solving for $^{\Delta L}\upsilon$, ones gets

$$^{\Delta L}\upsilon = \left\{\frac{\omega_\upsilon}{\omega_\epsilon}\left(\frac{^{\Delta L}\epsilon}{\gamma\, ^{\Delta L}\eta} - 1\right) + 1\right\}\alpha\,|^{\Delta L}\eta|. \tag{6.4}$$

The deviatoric strain difference may be found by substituting Eq. (6.4) in Eq. (5.9) and then into Eq. (6.3) and squaring so that

$$\left\{\frac{^{\Delta L}\epsilon}{\gamma\, ^{\Delta L}\eta} - 1\right\}^2 = \frac{\omega_\epsilon^2}{3}\left\{\frac{\omega_\upsilon}{\omega_\epsilon}\left(\frac{^{\Delta L}\epsilon}{\gamma\, ^{\Delta L}\eta} - 1\right) + 1\right\}^2 \cdot \tag{6.5}$$

$$\cdot\, \alpha^2\, ^{\Delta L}\eta^2 + \frac{3}{2}\,\omega_\epsilon^2\, ^{\Delta L}\epsilon^2.$$

Solving Eq. (6.5) and excluding the solution for which

$$\frac{^{\Delta L}\epsilon}{\gamma\, ^{\Delta L}\eta} < 1 \tag{6.6}$$

that would violate the condition that χ is a non-negative quantity, the deviatoric strain difference reads

$$^{\Delta L}\epsilon = \gamma\, ^{\Delta L}\eta\left[1 + \frac{1}{3}\,\omega_\upsilon(\omega_\epsilon - \omega_\upsilon)\,\alpha^2\, ^{\Delta L}\eta^2 + \omega_\epsilon\,|^{\Delta L}\eta| \cdot \right. \tag{6.7}$$

$$\left. \cdot \sqrt{\frac{1}{3}\alpha^2 + \frac{3}{2}\gamma^2\left(1 - \frac{1}{3}(\omega_\epsilon - \omega_\upsilon)\,\alpha^2\, ^{\Delta L}\eta^2\right)}\,\right] \cdot$$

$$\cdot \left[1 - \left(\frac{\omega^2 \upsilon}{3}\,\alpha^2 + \frac{3}{2}\gamma^2\omega_\epsilon^2\right)\, ^{\Delta L}\eta^2\right]^{-1}.$$

Fig. 11 shows a comparison between calculated and experimental curve again for a specimen of Haslemere Clay-data after Parry (1980). The values of B_0 and ω_υ, which were not directly available, have been taken from the isotropic test performed by Som.

The value of $L_0 = .00397$ has been determined from the initial slope of the unloading portion of the loop in the $\eta - \epsilon$ plane. The values of $\omega_\epsilon = 274.$ and $\theta = .245$ have been obtained by imposing that the experimental and the calculated curves meet at the extremities of the cycle in the $\eta - \epsilon$ plane.

The comparison appears satisfactory. Note that because of the introduction of the off-diagonal term of the matrix in Eqs. (5.8) it is possible to model the irrecoverable axial strain after the completion of the stress cycle.

This fact is of fundamental importance in modelling cyclic loading where cumulation of strain is observed.

The verification of the pertinence of the above determined material constants can be performed in a conventional triaxial compression test at constant cell pressure. After some laborious algebra it can be found in a similar way as for Eq. (6.7) that

$$^{\Delta L}\upsilon = \left\{\frac{\omega_\upsilon}{\omega_\epsilon}\left(\frac{^{\Delta L}\epsilon}{\gamma\, ^{\Delta L}\eta} - 1\right) + 1\right\} \cdot \tag{6.8}$$

$$\cdot \left(\alpha\,|^{\Delta L}\eta| + B_0 \ln\frac{p}{p_L}\right)$$

$$\frac{^{\Delta L}\epsilon}{\gamma\, ^{\Delta L}\eta} = \left\{ 1 + \frac{1}{3}\,\omega_\upsilon(\omega_\epsilon - \omega_\upsilon)\left(\alpha\,|^{\Delta L}\eta| + B_0 \ln\frac{p}{p_L}\right)^2 + \right. \tag{6.9}$$

$$+\omega_\epsilon\left\{\frac{1}{3}\left(\alpha\left|{}^{\Delta L}\eta\right|+B_0\ln\frac{p}{p_L}\right)^2+\frac{3}{2}\gamma^2\,{}^{\Delta L}\eta^2\cdot\right.$$

$$\left.\cdot\left[1-\frac{1}{3}(\omega_\epsilon-\omega_v)\left(\alpha\left|{}^{\Delta L}\eta\right|+B_0\ln\frac{p}{p_L}\right)^2\right]^{1/2}\right\}\cdot$$

$$\cdot\left[1-\frac{\omega_v^2}{3}\left(\alpha^{\Delta L}|\eta|+B_0\ln\frac{p}{p_L}\right)^2-\frac{3}{2}\gamma^2\omega_\epsilon^2\,{}^{\Delta L}\eta^2\right]^{-1}.$$

Fig. 12 shows a comparison between calculated and experimental curves-data from Parry (1956). The tested clay is the same as in the p-constant test. The calculated curves enjoy the main features of the experimental ones. Namely, the volumetric compaction in reloading is essentially larger than that in unloading and the axial strain cumulation is qualitatively in good agreement with the actual one. Despite that some of the material constants have been only approximately evaluated, the overall behaviour appears to be acceptably described.

The proposed theory models also the behaviour of overconsolidated clays in conventional monotonic triaxial loading. As an example, the behaviour of an overconsolidated clay in undrained compression for various overconsolidation ratios (OCR) will be described.

Consider a set of saturated specimens of a normally consolidated clay under an effective isotropic pressure p_c that will be referred to as the preconsolidation pressure. Suppose now to unload each specimen to a different isotropic pressure $\hat{p}_i$. The samples are thus overconsolidated with $\mathrm{OCR}_i = p_c/\hat{p}_i$. Perform now on these samples a series of undrained compression tests. The kinematic condition on the material element is thus that the volume does not change from the beginning of the undrained portion of the test. Therefore the volumetric strain difference will be constrained to be

$$^{\Delta L}v \equiv {}^{\Delta L}v_i^*. \tag{6.10}$$

The point L to which the differences are referred is the preconsolidation pressure point. In fact it is assumed that no stress reversal occurs within the yield locus neither during the isotropic unloading nor during the undrained path. Then

$$^{\Delta L}v_i^* = -\frac{B_0\ln \mathrm{OCR}_i}{1-\dfrac{\omega_v}{\sqrt{3}}\ln \mathrm{OCR}_i}. \tag{6.11}$$

By virtue of the constitutive Eqs. (5.8)

$$(1+\omega_v\chi)\left(B_0\ln\frac{p}{p_c}+\alpha\left|{}^{\Delta L}\eta\right|\right)={}^{\Delta L}v_i^* \tag{6.12}$$

$$^{\Delta L}\epsilon=(1+\omega_\epsilon\chi)\gamma\,{}^{\Delta L}\eta \tag{6.13}$$

with

$$\chi=\left(\frac{1}{3}\,{}^{\Delta L}v_i^{*2}+\frac{3}{2}\,{}^{\Delta L}\epsilon^2\right)^{1/2}. \tag{6.14}$$

Substituting (6.14) in (6.13), and solving for $^{\Delta L}\epsilon$ one arrives at

$$^{\Delta L}\epsilon=\frac{1+\sqrt{1-(1-\gamma L_0\omega_\epsilon^2\,{}^{\Delta L}\eta^2)\left(1-\dfrac{\omega_\epsilon^2}{3}\,{}^{\Delta L}v_i^{*2}\right)}}{1-\gamma L_0\omega_\epsilon^2\,{}^{\Delta L}\eta^2}. \tag{6.15}$$

It is then possible to know χ for any value of $^{\Delta L}\eta$ and then the equation of the stress path can be easily determined from Eq. (6.12) so that

$$p=p_c\exp\left\{\frac{^{\Delta L}v_i^*}{B_0(1+\omega_v\chi)}-\frac{\alpha\left|{}^{\Delta L}\eta\right|}{B_0}\right\}. \tag{6.16}$$

If $^{\Delta L}\eta$ is large enough the stress path reaches the yield locus pertinent to the preconsolidation pressure. The further loading process involves then plastic yielding and the elastoplastic constitutive law must be employed. Following the elastoplastic theory quoted in Section 1 the yield locus (4.1), specialized to triaxial conditions takes the form

$$f=\frac{4\mu}{M^2}\eta^2+1-\left(\frac{p_c}{p}\right)^2=0 \qquad \eta\leqslant\frac{M}{2} \tag{6.17}$$

$$f=\eta-M+m\ln\frac{p}{p_u}=0 \qquad \eta\geqslant\frac{M}{2} \tag{6.18}$$

where M is the stress ratio at the critical state, p_u is the isotropic pressure at which the yield locus meets the critical state line in the $p-q$ plane, μ and m are experimental parameters linked to the form of the yield locus.

The flow rule is assumed to be associated for $\eta\leqslant M/2$. If $\eta\geqslant M/2$ the flow rule is non-associated and the plastic potential g reads (see (4.4))

$$g=\eta-\frac{M}{1-\mu}\left\{1-\mu\left(\frac{p}{p_g}\right)^{\frac{1-\mu}{\mu}}\right\}=0 \tag{6.19}$$

where p_g is the isotropic pressure at which the plastic potential meets the critical state line in the $p-q$ plane.

The incremental elastoplastic constitutive law is then given by

$$\dot{v}=B_0\frac{\dot{p}}{p}+\Lambda d \tag{6.20}$$

$$\dot{\epsilon}=\frac{2}{3}L_0\dot{\eta}+\Lambda \tag{6.21}$$

where d is the plastic dilatancy coefficient that may be found to be

$$d=\frac{M^2}{4\mu\eta} \qquad \eta\leqslant\frac{M}{2} \tag{6.22}$$

$$d=\frac{1}{\mu}(M-\eta) \qquad \eta\geqslant\frac{M}{2} \tag{6.23}$$

and Λ is the plastic multiplier of Eq. (4.4) that may be found through the Prager's consistency rule to be

$$\Lambda=(\lambda-B_0)\,\frac{(d+\eta)\dfrac{\dot{p}}{p}+\dot{\eta}}{d(d+\eta)} \qquad \eta\leqslant\frac{M}{2} \tag{6.24}$$

$$\Lambda=(\lambda-B_0)\,\frac{m\dfrac{\dot{p}}{p}+\dot{\eta}}{md} \qquad \eta\geqslant\frac{M}{2} \tag{6.25}$$

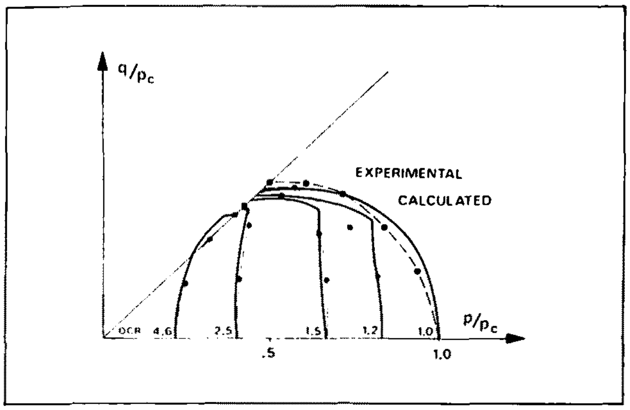

Fig. 13. Comparison between experimental and calculated effective stress paths in undrained compression tests on clay samples at various OCR-data after Wroth and Loudon (1967).

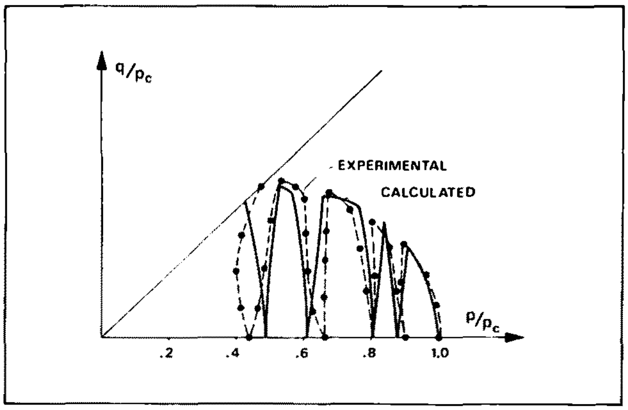

Fig. 14. Comparison between experimental and calculated effective stress paths in undrained cyclic loading on a normally consolidated kaolin-data after Wroth and Loudon (1967).

where λ is the slope of the $v - \ln p$ curve in an isotropic loading on a virgin material.

From Eq. (6.20), by imposing the condition of no change of volume, it is possible to derive the elastoplastic stress path. From Eq. (6.22), (6.23) and (6.24), (6.25) one has

$$\ln \frac{p}{p_c} = -\frac{1}{2}\left(1 - \frac{B_0}{\lambda}\right) \ln \left(1 + \frac{\eta}{d}\right) \qquad \eta \leqslant \frac{M}{2} \tag{6.26}$$

$$\ln \frac{p}{\bar{p}_i} = -\frac{1 - \dfrac{B_0}{\lambda}}{m}(\eta - \bar{\eta}_i) \qquad \eta \geqslant \frac{M}{2} \tag{6.27}$$

where $\bar{p}_i$ and $\bar{\eta}_i$ are the coordinates of onset of yielding.

Fig. 13 shows a comparison between calculated and experimental stress paths for a normally consolidated specimen of Cambridge kaolin and four overconsolidated samples at different OCRS-data after Wroth and Loudon (1967).

The values of the material constants have been evaluated in part on the basis of experimental results obtained on the same kind of clay and in part directly from the stress path followed by the considered normally consolidated sample. They are: $M = .96$, $\lambda = .113$, $B_0 = .022$, $\mu = .67$, $m = 7$, $\omega_v = 23$, $\omega_\epsilon = 150$, $L_0 = .00397$, $\alpha = .0022$.

The agreement between calculated and experimental results is satisfactory. In particular the variation of the shape of the stress path with OCR is matched. A discrepancy may be observed at the vicinity of the yield locus for lower values of OCR. The numerical calculations confirm the assumed hypotehsis that no stress reversal occurs in the undrained portion of the test, $\dot{\chi}$ always positive. Note that at the very end of the stress path for OCR = 4.6 softening occurs but the equation of the elastoplastic stress path is formally the same.

Consider now an undrained test on a normally consolidated clay under cyclic loading with variable amplitude. Assume the programme of loading to be that imposed by Wroth and Loudon (1967) on the aforementioned Kaolin. The first loading yields elastoplastic strains and stress path is given by Eq. (6.26). The first unloading path is in the hysteresis range and the origin to which refer stress and strain differences is the stress reversal point on the yield locus. By imposing that no volumetric strain occurs it is possible to find the stress path equation from Eqs. (5.8)

$$p = p_L \exp \left\{\frac{-\alpha}{B_0} |{}^{\Delta L}\eta|\right\}. \tag{6.28}$$

The corresponding strains are given by

$${}^{\Delta L}\epsilon = \frac{\gamma\, {}^{\Delta L}\eta}{1 - \omega_\epsilon \sqrt{\dfrac{3}{2}}\, \gamma\, {}^{\Delta L}\eta}. \tag{6.29}$$

When $\eta = 0$ the load is reversed so that the relations between further stress and strain differences should be referred to this new S.R.P. until the yield locus is reached anew. The further loading causes plastic yielding so that the hysteretic memory of the material is cleared. Therefore both in unloading and reloading the current constitutive law is always referred to the last S.R.P. Again for low values of p the reloading causes softening behaviour at yielding.

Fig. 14 shows the comparison between calculated and experimental stress paths determined in the aforementioned tests. The constitutive parameters applied in the calculation are as in the preceding test. It is seen that the travelling of the effective stress point towards the critical state is matched without substantial departures. Clearly, due to the imposed condition of zero volumetric strain, a pore water pressure build-up occurs. The number of cycles necessary to cause failure of the sample is correctly predicted. The off-diagonal term in the compliance matrix (5.8) and thus the coupling of the deviatoric and isotropic parts of the constitutive law play an essential role in the simulation of the above effects.

The proposed theory can be also used to predict the stress path followed by a specimen in unloading-reloading in confined compression. In this case the kinematic condition is

$${}^{\Delta L}\epsilon_{33} = 0 \tag{6.30}$$

so that

$${}^{\Delta L}v = {}^{\Delta L}\epsilon_{11} = \frac{3}{2}\, {}^{\Delta L}\epsilon, \qquad \chi = \frac{3}{2} |{}^{\Delta L}\epsilon|. \tag{6.31}$$

Therefore by substituting (6.31) in (5.8)

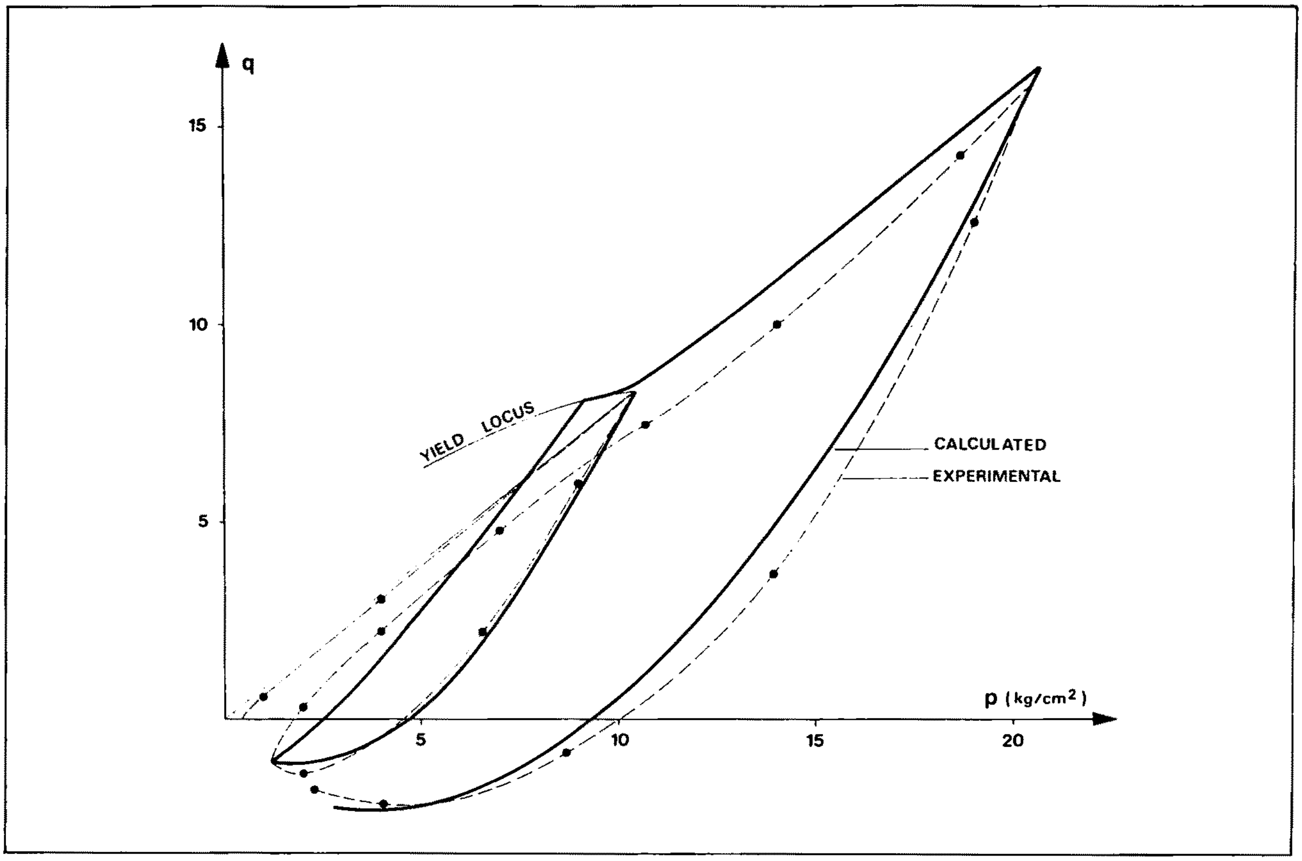

Fig. 15. Comparison between experimental and calculated stress paths in oedometric unloading-reloading-data courtesy of Studio Geotecnico Italiano.

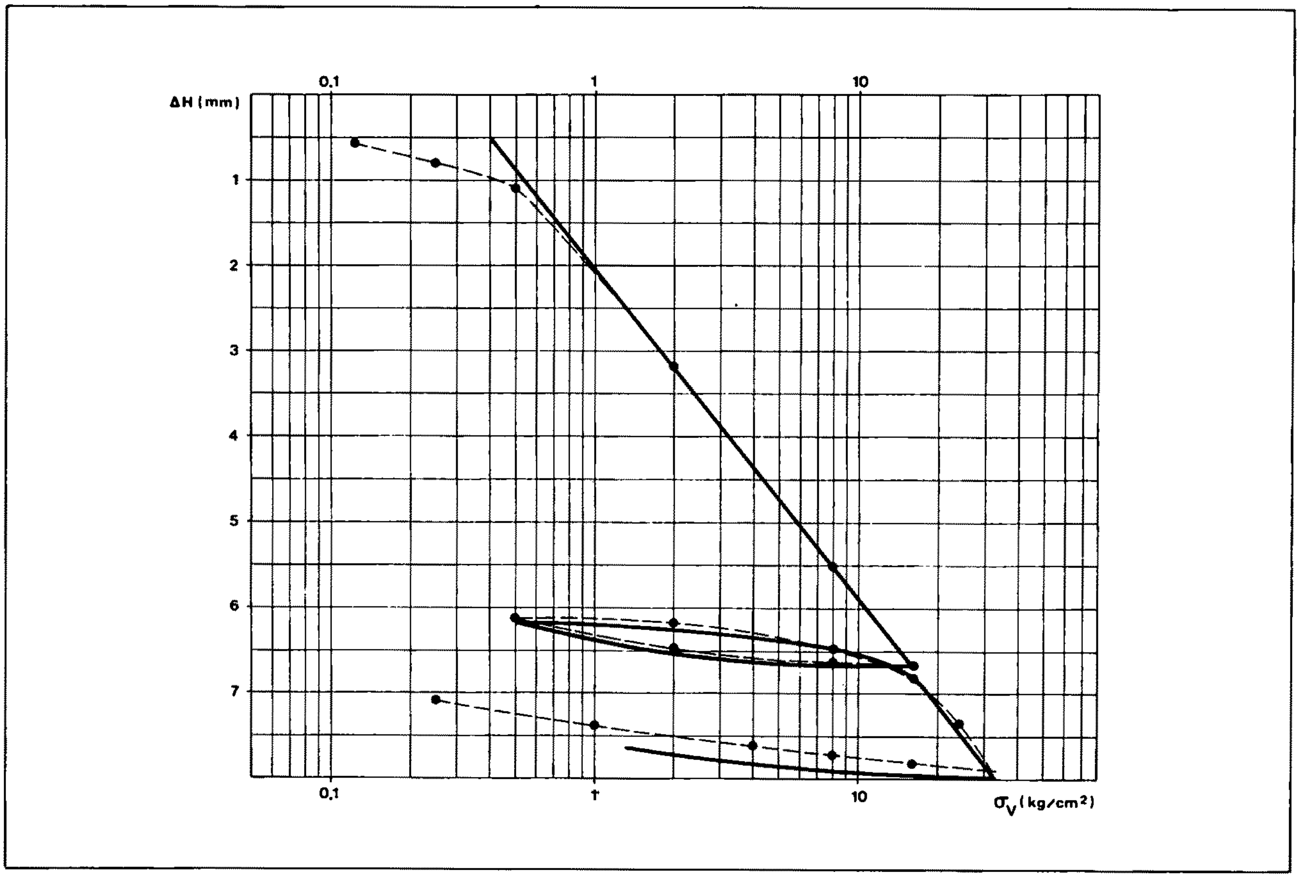

Fig. 16. Comparison between experimental and calculated results in oedometric unloading-reloading-data courtesy of Studio Geotecnico Italiano.

$$\left(1 + \frac{3}{2}\omega_v |^{\Delta L}\epsilon|\right)\left(B_0 \ln \frac{p}{p_L} + \alpha|^{\Delta L}\eta|\right) = \tag{6.32}$$

$$= L_0\left(1 + \frac{3}{2}|^{\Delta L}\epsilon|\,\omega_\epsilon\right)^{\Delta L}\eta$$

from which

$$\ln \frac{p}{p_L} = \frac{L_0\left(1 + \frac{3}{2}\omega_\epsilon |^{\Delta L}\epsilon|\right)^{\Delta L}\eta}{B_0\left(1 + \frac{3}{2}\omega_v |^{\Delta L}\epsilon|\right)} - \frac{\alpha}{B_0}|^{\Delta L}\eta|. \tag{6.33}$$

Since, by virtue of (6.31), (5.8)

$$|^{\Delta L}\epsilon| = \frac{\gamma|^{\Delta L}\eta|}{1 - \frac{3}{2}\gamma\omega_\epsilon |^{\Delta L}\eta|} \tag{6.34}$$

after some algebra it is possible to derive the equation of the stress path

$$p = p_L \exp\left\{\frac{1}{B_0}\right\}\frac{L_0{}^{\Delta L}\eta}{1 + (\omega_v - \omega_\epsilon)L_0|^{\Delta L}\eta|} - \alpha|^{\Delta L}\eta|\Big\}. \tag{6.35}$$

Fig. 15, 16 show a comparison between calculated and experimental curves for a silty clay at Gioia Tauro-courtesy of Studio Geotecnico Italiano. The constitutive parameters for the hysteretic range have been found by fitting the first unloading curve both for the stress path and the vertical stress-vertical strain curve. The derived values of the constants are $B_0 = .0028$, $L_0 = .0021$, $\omega_\epsilon = 230$, $\omega_v = 125$, $\alpha = .00028$.

The overall agreement looks acceptable although some discrepancies occur, mainly for the loading stress path. In fact, the convexity of the calculated path is different from the actual one. Moreover the path reaches the yield locus at a slighty higher value of η. To calculate the further part of the reloading path it has been assumed that the shape of the yield locus is given by Eq. (6.18) and that the material constants are $M = 1.2$, $m = .6$. Integrating Eq. (6.20), (6.21) under the condition of zero lateral strain the elastoplastic stress path has been determined to be:

$$m \ln \frac{p}{p_0} = (\eta_0 - \eta) + \left(1 + \frac{L_0}{B_0}m\right) \cdot \tag{6.36}$$

$$\cdot\left\{\frac{B_0}{\lambda}(\eta - \eta_0) + \left(\frac{\lambda}{B_0} - 1\right)\frac{B_0^2}{\lambda^2}\frac{3}{2}\mu \ln \cdot\right.$$

$$\left.\cdot\left[\frac{\frac{\lambda}{B_0}(\eta - \eta_0) - \frac{3}{2}\mu\left(\frac{\lambda}{B_0} - 1\right)}{\frac{\lambda}{B_0}(M - \eta) - \frac{3}{2}\mu\left(\frac{\lambda}{B_0} - 1\right)}\right]\right\}$$

where $\lambda = .066$ corresponds to the slope of the loading curve in Fig. 16 and $\mu = .54$ has been derived by assuming that the asymptote of the elastoplastic stress path coincides with the K_0 line for the virgin soil. It may be noted, however, that the calculated elastoplastic stress path approaches the asymptote from above, whilst the experimental one does it from below, although both curves exhibit a similar inflexion in the vicinity of the yielding onset.

7. CONCLUSIONS.

The presented mathematical model of soil behaviour under alternating loading combines theories pertinent to two kinds of material responses in the elastoplastic and in the «hysteretic» range. The latter, which is the subject of this paper, is dealt with in terms of a piecewise description between subsequent appropriately defined stress reversals. In order to tackle with the highly nonlinear behaviour of soils a set of generalized variables has been introduced, i.e. the tensor of stress ratio η_{kl} and the logarithmic measure of isotropic pressure. In terms of these variables the constitutive law for a single portion of the stress path is linear in the tensorial quantities, whilst the intrinsic nonlinearity of the hysteresis is described by a variation of the symmetric compliance tensor with a *scalar* strain amplitude parameter. Between two subsequent stress reversals the behaviour is thus treated as path independent in the space of generalized variables. On closed radial stress cycles the deviatoric strain are recoverable. The volumetric strain however is not restored after the cycle completion and may be accumulated under repeated loading. This is modelled by inserting into the constitutive law the term of shear compaction, which is insensitive to the load direection. It is believed that for the scope of engineering applications the symulation of ratcheting, cyclic mobility, etc., by means of the volumetric strain component only is accurate enough and still relatively simple. The comparison of the model predictions with the experimental results in p-constant; σ_{33}-constant, oedometric, undrained tests on overconsolidated clays, cyclic undrained on normally consolidated clay gives an encouraging support.

An alternative approach to cyclic loading may be made via anisotropic work-hardening plasticity formulated in terms of rates as in the papers by Mròz (1967), Dafalias, Popov (1975), Prévost (1977), Mròz, Norris, Zienkiewicz (1978). Such theories take into account the path dependence of soil response even in the hysteretic range what inevitably leads to (sometimes laborious) incremental step-by-step procedures.

The present approach is based on piecewisely path-independent relation between strain and generalized stress differences, and thus its path dependence is reduced to single points of the stress path i.e. the stress reversal points. This allows for the direct evaluation of material response by relatively simple formulas. More complex problems, e.g. plane stress problems, may be treated by adapting the generalized stress-strain relations given in tensorial form to a particular finite element computer code. The concept of the sensitivity of the material response only to discrete points of its history has been developed with reference to cyclic behaviour by Mròz and Lind (1972), Mròz (1972) and Hueckel and Nova (1979[a]). In general, in such approach the use of a kind of a relatively simple potential law implies usually a remarkable difference of the rate response in reversal and continuation ranges for very close stress rates, what can be hardly avoided without recours to a rate formulation.

In this paper attention has been confined to the stress--strain behaviour and its verification whilst the concept of stress reversal locus and the structure of the material memory have been discussed in the preceding paper by the authors (1979[b]). Their physical meaning is still an open question and, specifically, the form of the stress reversal locus, requires an extensive experimental confirmation.

This paper is directed towards engineering and laboratory applications. With this scope the very particular non-linear functions $\ln p/p_a$, $\eta_{kl} = s_{kl}/p$ have been used, which are however generalizations of scalar variables traditionally used in the Soil Mechanics School of Cambridge. Clearly, other more general non-linear functions may be used within the framework of this theory. The chosen formulation requires only three material constants in addition to those necessary for an elastoplastic theory not taking the hysteresis effects into account. The constants are easily determinable from basic laboratory tests. By setting those «hysteretic constants» equal to zero, the theory reduces to an elastoplastic theory what offers the advantage of using a unified numerical algorithm in a suitably reduced version without following the whole loading history.

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