A Model of Asset Pricing under Portfolio Delegation and Differential Information

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Abstract

Previous analyses of equilibrium asset prices often ignore the effects of delegated portfolio management and those of delegated portfolio management problems often ignore information and equilibrium asset prices. This paper develops a dynamic model that simultaneously considers optimal contracting and equilibrium asset prices under differential information. We consider a case in which investors can contract on various signals as well as a case in which investors constrain the contract form to be of a linear function of the terminal value of their portfolios. Optimal contracts and equilibrium asset prices are characterized in both cases. We examine the impact of portfolio delegation and risk sharing on portfolio managers' trading behavior, the autocorrelation in stock returns, and the persistence of fund performance. We find that due to the less optimal risk sharing contract, the risk premium on the stock as well as the autocorrelations in both stock and fund returns are substantially higher in case 1 than in case 2. In particular, we find that under certain conditions, the autocorrelations in fund returns are positive, suggesting the persistence of fund performance. The presence of differential information among funds reduces autocorrelations in stock and fund returns, but the costs associated with managing the portfolio enhance them.

1 Introduction

Mutual funds, pension funds, and other financial intermediaries have experienced rapid growth in the last two decades in terms of both the total assets under management by institutions and the number of funds available to investors. For example, Cuoco and Kaniel (2000) report that the value of the assets managed by mutual funds was $50 billion in 1977 and $4.5 trillion in 1997 and that the number of mutual funds grew from a few hundreds in the late 70's to more than 7000 in the late 90's. Pension funds experienced similar rapid expansion as well. Gompers and Metrick (1998) report that by December 1996, large institutions held discretionary control over more than half of the U.S. equity market.[1] Needless to say, these large institutions wield considerable market power in terms of their ability to affect market prices. As Allen and Santomero (1996) point out that "current asset pricing theories usually assume investors choose optimal portfolios directly. The fact that there is such extensive intermediation suggests that this approach may miss important features of actual markets. $\cdots$ In short, given the importance of intermediaries' trading in financial markets, asset pricing theories and intermediation theories need to be better integrated."[2] In our view, an integrated theory as envisioned by Allen and Santomero must simultaneously consider optimal contracting and equilibrium asset prices in an information-based framework.

Previous analyses of equilibrium asset prices under asymmetric or differential information often assume that investors trade for their own accounts. Consequently, these models may not be able to explain the trading behavior of portfolio managers. Previous analyses of delegated portfolio management problems often ignore differential information and equilibrium asset prices. Any attempts to construct an integrated model, however, would likely encounter two difficulties. On the one hand, different institutions may possess different information about the distribution of stock returns. Even in the absence of the delegated portfolio management problem, solving a model in which agents have differential information faces the so-called "infinite regress" problem. A rational expectations equilibrium often depends on the expectations of all agents. In a dynamic setup, the number of these expectations required to characterize the equilibrium increases with the path of agents' information and their trading behavior. In the continuous-time limit, the order of agents' expectations and the dimension of the problem become infinite, often making the problem intractable. On the other hand,

[1] Large institutions are defined as those with at least $100 million under management.

[2] See also Allen (2001).

even if one ignores the equilibrium aspects of asset prices, the contracting problem between portfolio managers and their clients or investors is extremely challenging. To characterize explicitly an optimal contract or a risk sharing rule between an investor and a manager, one must solve both the manager's and the investor's dynamic maximization problems simultaneously. The difficulty arises because these two problems are interconnected: the manager's portfolio selection depends upon the contract designed by the investor, and the investor's contract must take the manager's portfolio choice into account.

However, two recent developments have overcome the aforementioned obstacles in two partial situations, paving the way for a more integrated framework. First, He and Wang (1995) develop a multiperiod discrete-time rational expectations model of stock trading in which agents have differential information concerning the stock price.[3] In this model, agents receive different signals about the stock return and invest between a riskless bond and a risky stock. Under the assumption that there exists a continuum of agents, He and Wang overcome the infinite regress problem and show that all higher order expectations can be reduced to the first order ones. This model implicitly assumes that all agents trade for their own accounts and does not consider the delegated portfolio problem. In a recent paper, Ou-Yang (2000) studies a contracting problem faced by an individual investor who entrusts her funds to a professional portfolio manager. There is one riskless bond and multiple risky stocks available for trading. Ou-Yang analyzes the relationship between the investor and the manager in a continuous-time principal-agent framework and considers both the principal's and the agent's dynamic maximization problems simultaneously. Various optimal contracts with empirical implications are deduced in closed form from a large contract space. This model assumes that stock prices follow geometric Brownian motion processes, thus ignoring the equilibrium aspects of stock returns.

The objective of this paper is to develop a basic dynamic model of optimal contracting and asset pricing under differential information. In doing so, we aim to combine the models of He and Wang (1995) and Ou-Yang (2000). We analyze the relationship between managers and investors in a continuous-time principal-agent framework. Investors entrust their funds to and provide contracts for managers, forming institutional funds. Each fund observes a public signal as well as receives a distinctive private signal about the risky stock continuously over a finite period of time. Based

[3]See also Pfleiderer (1984), Singleton (1987), Brown and Jennings (1989), Grundy and McNichols (1989), Kim and Verrecchia (1991), and Foster and Viswanathan (1996).

on their information as well as their costs associated with managing portfolios, managers trade competitively in the market for both speculative and informational reasons. We consider two models that specify two types of contract spaces. In Model I the investor can contract on certain continuous processes of signals, and the potential conflict of interest between the investor and the manager within a fund is through the costs incurred by the manager for managing the portfolio. Model II assumes that investors observe only the terminal value of the portfolio, and constrains the contract form to be of a linear function of the terminal value. The equilibrium stock price and optimal contract are computed in both cases.

In Model I we demonstrate that the investor and manager achieve optimal risk sharing and that the basic results of He and Wang and of Ou-Yang obtained in their respective partial settings are quite robust in the context of our integrated model.[4] More specifically, the linear pricing function and the basic form for the manager's trading strategy of He and Wang still hold in the presence of portfolio delegation and optimal contracting, and the basic structure of the optimal contract of Ou-Yang is still valid with equilibrium asset pricing under differential information. The manager's trading strategy has two components, the manager's inferred value of the supply shock plus the difference between the estimated value of the expected dividend rate based upon the manager's information and the public information, respectively. The optimal contract is composed of a constant, a function of the state variables, the manager's costs associated with managing the portfolio, a fraction of the terminal value of the portfolio, plus a bonus or a penalty depending upon the excess return between the managed portfolio and a pre-specified benchmark.

Using the results from the model, we examine the impact of portfolio delegation including managers' operating costs on the managers' trading behavior, the autocorrelation in stock returns, and the persistence of fund performance. With portfolio delegation, the risk sharing effect reduces the sensitivity of the equilibrium price to the supply shock, thus allowing the manager to absorb risks more easily. As a result, the expected excess return of the stock goes down and the manager trades more aggressively than he would trade for his own account.

Due to less optimal sharing of the risk associated with supply shocks in Model II, both the risk premium on the risky stock and the autocorrelations in stock and fund returns are substantially higher than in Model I. The manager trades more cautiously in this case, and demands a higher risk

[4]Note that the He-Wang model is a special case of our model when the manager's compensation equals the entire wealth of the portfolio, and the Ou-Yang model is a special case when the stock price is exogenously given.

premium to clear the market in equilibrium. In both cases, the costs associated with managing the portfolio enhance these features, because with a higher cost, it is more difficult for the manager to absorb risks. Unlike in a typical principal-agent model, the sensitivity to performance in equilibrium does not necessarily increase with manager's cost function. Since given a higher cost of investing in the risky stock, the manager is less willing to absorb the supply shock. As a result, the price is more sensitive to the supply shock, or the price is more volatile with regard to the supply shock. Given a higher share of the portfolio, the manager has a higher incentive to invest in the risky stock, resulting in a lower sensitivity of the price to the supply shock. Therefore, there does not exist a monotone relationship between the cost function and pay-to-performance sensitivity in the linear contract.

Empirical studies indicate that there exists positive autocorrelation in short-horizon stock returns and negative autocorrelation in long-horizon stock returns, and that there is positive persistence (autocorrelation) in equity mutual fund performance from period to period in a short horizon.[5] Our numerical results for short horizons show that the autocorrelation in stock returns can be either negative or positive while that in fund returns is always positive. The presence of differential information among funds reduces the autocorrelation in stock returns as well as the persistence in fund performance.

The rest of the paper is organized as follows. The next subsection briefly reviews more related asset pricing models. Section 2 presents the basic setup that characterizes our economy. Explicit expressions for both the equilibrium stock price and the optimal contract are derived in Section 3. In Section 4, we describe three special cases for comparison purposes and the numerical procedure that solves our equilibrium. Section 5 examines how portfolio delegation affects various properties of a stock and a fund. Some concluding remarks regarding the model are offered in Section 6. All the proofs are given in the five appendices.

1.1 More Related Literature

In addition to the He-Wang model, other related asset pricing models include Brennan (1993), Campbell and Kyle (1993), Wang (1993), Foster and Viswanathan (1996), and Cuoco and Kaniel

[5] See, e.g., Conrad and Kaul (1988) and Lo and Mackinlay (1988) for positive autocorrelations and Fama and French, Lo and Mackinlay (1988), and Poterba and Summers (1988) for negative autocorrelations in stock returns. See, e.g., Grinblatt and Titman (1992), Hendricks, Patel, and Zeckhauser (1993), Mech (1993), Brown and Goetzmann (1995), and Sias and Starks (1999) for positive autocorrelations in fund returns.

(2000).[6] In perhaps the first attempt towards an integrated model, Brennan considers a one-period mean-variance asset pricing model which assumes that there are two types of investor. The first type is concerned with the mean and variance and the second type is a portfolio manager concerned with the return of his portfolio relative to that of a given benchmark portfolio. Brennan shows that the equilibrium expected stock return can be characterized by two factors, the market and the benchmark portfolios. Campbell and Kyle develop a model in which all informed agents receive the same signal about the stock price. They study the behavior of aggregate stock prices. They find that their model can explain the volatility and predictability of U.S. stock returns using either a low discount rate and a large constant risk discount on the stock price, or a higher discount rate and noise trading correlated with fundamentals. Wang develops a continuous-time asset pricing model under asymmetric information in which one group of investors is more informed about the expected growth rate of dividends than is the other group of investors. He finds that the existence of less informed investors increases the risk premium and that information asymmetry among investors can increase price volatility and negative autocorrelation in returns.

Generalizing the original Kyle (1985) model, Foster and Viswanathan (1996) analyze a multi-period model of strategic trading with differentially informed traders, liquidity traders, and a market maker. They show that the average of the signals is a sufficient statistic for the information known to all informed traders about the liquidation value of the asset. This means that the market maker and informed traders need be concerned only about inferring the average of the signals from the order flow, allowing them to overcome the infinite regress problem. All traders and the market maker are risk neutral. Investors trade for their own accounts in Campbell and Kyle, Wang, and Foster and Viswanathan. Cuoco and Kaniel (2000) analyze the asset pricing implications of the fund managers' compensation schemes in an equilibrium model. They find that the manager's compensation form can have significant effects on the equilibrium prices of stocks included in the benchmark portfolio, significant effects on their equilibrium Sharpe ratios, and marginal effects on their equilibrium volatilities. In this model, equilibrium asset prices are endogenously determined, but the contract form is exogenously given. All these models do not consider any costs associated

[6]See also Grossman (1976), Hellwig (1980), Diamond and Verrecchia (1981), Glosten and Milgrom (1985), Kyle (1985), Easley and O'Hara (1987), Holden and Subrahmanyam (1992), Foster and Viswanathan (1993), and Spiegel and Subrahmanyam (1994). For delegated portfolio management problem in the absence of equilibrium asset prices, see, e.g., Bhattacharya and Pfleiderer (1985), Dybvig and Spatt (1986), Starks (1987), Kihlstrom (1988), Stoughton (1993), Heinkel and Stoughton (1994), and Das and Sundaram (1998). Unlike the Ou-Yang paper, the objectives of these papers are not to solve for optimal contracts.

with managing the portfolio.

2 The Basic Setup

We consider a continuous-time equilibrium model of stock trading and optimal contracting in a simple setting. Investors entrust their funds to and provide compensation schemes for portfolio managers, forming various asset management firms. After signing contracts with investors, managers start trading on behalf of the investors based on their information concerning the stock. The point is that investors' contracts must induce managers to adopt portfolio policies that maximize both managers' and investors' expected utility functions over their respective terminal wealth. The time horizon is taken to be $[0, T]$. We shall adopt the notations of Wang (1993), He and Wang (1995), and Ou-Yang (2000). The model is further outlined as follows.

2.1 Financial Assets

There is a riskless bond and a risky stock available for trading. Each share of the risky asset pays a cumulative dividend D_t given by

$$D_t = \int_0^t (G du + \sigma_D dB_u) \quad \text{or} \quad dD_t = G dt + \sigma_D dB_t, \tag{1}$$

where B_t denotes a (2×1) vector of independent standard Brownian motion processes, where σ_D is a (1×2) vector of constants, and where G is a constant and can be interpreted as the expected rate of dividend payment. G is not directly observed and must be inferred over time by all the agents in the economy. At time 0, we assume that all agents possess the same information about G and that the prior distribution is $G \sim N(G_0, b_G^2)$. Also assume that the riskless asset yields a positive constant rate of return denoted by r.

Denote by Θ_t the total number of shares of the risky stock available in the market at time t. Θ_t follows an Ornstein-Uhlenbeck (O-U) process

$$d\Theta_t = -a_\Theta \Theta_t dt + \sigma_\Theta dB_t, \tag{2}$$

where a_Θ is a positive constant and σ_Θ denotes a (1×2) vector of constants. This process implies that the total supply of the risky stock is stochastic, which is equivalent to the notion of the existence of noise traders.[7]

[7]See, e.g., Kyle (1985). Following He and Wang, we assume that the total number of shares of the stock is 1 and that the noise traders have inelastic demand of $1 - \Theta$ shares of the stock, leaving the remaining Θ shares to the market.

2.2 Information Structures and Notations

Each manager or fund observes continuously the equilibrium price of the risky asset P_t as well as the dividend process D_t, which form the public information set $\mathcal{F}^c$. In addition, each fund observes a private signal about G, which is not shared by other funds,[8] denoted by Y_t^i:

$$dY_t^i = Gdt + \sigma_Y dB_t^i, \tag{3}$$

where σ_Y is a constant and where B_t^i is a one-dimensional Brownian motion process. Following He and Wang, we assume that the signals are uncorrelated across managers, i.e., $< dB_t^i, dB_t^j >= 0$ for $i \neq j$, and that the signals are uncorrelated with the dividend process or the supply shock, i.e., $< dB_t, dB_t^i >= 0 \ \forall i$.

For notational purpose, denote the information structures which are used throughout by

- $\mathcal{F}_t^c \equiv \{D_s, P_s, 0 \leq s \leq t\}$ denotes the common information available to everyone at time t;
- $\mathcal{F}_t^{Pi} \equiv \{Y_s^i, 0 \leq s \leq t\}$ denotes the private signal available to manager i at time t;
- $\mathcal{F}_t^i \equiv \{\mathcal{F}_0, D_s, P_s, Y_s^i, 0 \leq s \leq t\}$ denotes the total information available to manager i at time t, where $\mathcal{F}_0$ represents the initial information set as given by the prior distributions.

In addition, at each point in time, the inferred values of G and Θ with respect to different information sets are denoted by

$$G_t^c \equiv E\left[G|\mathcal{F}_t^c\right], \qquad \Theta_t^c \equiv E\left[\Theta_t|\mathcal{F}_t^c\right],$$
$$G_t^{Pi} \equiv E\left[G|\mathcal{F}_t^{Pi}\right], \qquad \Theta_t^{Pi} \equiv E\left[\Theta_t|\mathcal{F}_t^{Pi}\right],$$
$$G_t^i \equiv E\left[G|\mathcal{F}_t^i\right], \qquad \Theta_t^i \equiv E\left[\Theta_t|\mathcal{F}_t^i\right].$$

Finally, denote by f_t^c and f_t^i respectively the variances of G with respect to the common and individual information sets:

$$f_t^c = Var\left[G|\mathcal{F}_t^c\right], \qquad f_t^i = Var\left[G|\mathcal{F}_t^i\right].$$

[8] For example, a Fidelity fund and a Vanguard fund may have different information about the stock. Again, we assume that the investor and the manager that form a fund have the same information, but different funds may possess different information.

2.3 Economic Agents/Market Participants

There is a continuum of funds available in the economy. Each fund i is owned by one representative investor (principal) but is managed by a professional manager (agent) i. At time 0, the representative investor and the manager sign a compensation scheme S_T^i, where S_T^i is a short-hand notation for a contract that is a function of all relevant contractible variables to be specified below. The manager then decides how much to invest in the bond and the stock and continuously adjusts his portfolio positions.

The manager incurs a cumulative cost $\int_0^t c_u^i du$ associated with managing the fund between time 0 and t. Assume that the cost rate is given by

$$c_t^i = \gamma W_t^i + \frac{1}{2} k^i(t) {A_t^i}^2, \tag{4}$$

where γ is a constant and $k^i(t)$ is a deterministic function of time t, and where A_t^i denotes the number of shares invested in the stock at time t. The first term represents a cost function widely adopted in the mutual fund industry where funds typically charge a fraction of the total assets under management as operating expenses. The second term is for tractability of the solution and is a typical cost function used in the principal-agent literature with A_t being the effort level. In adopting this cost function, we are assuming that the more the manager invests in the risky stock, the higher costs he incurs. Equivalently, we are implicitly assuming that when the manager invests more shares in the risky stock, he must expend a higher level of effort in acquiring information on the stock. As in almost all the asset pricing models under asymmetric or differential information, the mechanism of information acquisition is beyond the scope of our current model.[9]

S_T^i is payable only at the terminal date T, Therefore, the investor's terminal wealth is $W_T^i - S_T^i$ and the manager's terminal wealth is $S_T^i - \int_0^T c_t^i dt$. For tractability of the model, we assume that both the manager and the investor consume at the terminal date only. If S_T^i affords the manager at least his reservation utility at time 0, he undertakes the job, with the understanding that he may not quit it between 0 and T.

Adopting from He and Wang and Ou-Yang, we assume that both the investors and the managers

[9] We may also interpret $\frac{1}{2}k^i(t){A_t^i}^2 dt$ as a fixed transactions cost for trading the risky stock between time t and $t+dt$, irrespective of the number of trades that takes place in the time interval. This fixed fee schedule changes continuously over time, depending upon the number of shares traded at time t. The higher the number of shares traded at t, the higher the fixed transactions cost for the time interval dt. $\frac{1}{2}\int_0^T k^i(t){A_t^i}^2 dtdt$ then represents the total "fixed" trading costs.

have negative exponential utility functions with risk aversion coefficients being R_p and R_a respectively.[10] The investor designs a contract to maximize the expected utility over her terminal wealth. Her problem is then given by

$$\sup_{\{A_t^i\},S_T^i} E\left[-\frac{1}{R_p}\exp\left[-R_p\left(W_T^i - S_T^i\right)\right]\right], \tag{5}$$

subject to various constraints to be specified in the next section. Given S_T^i, the manager makes investment decisions so as to maximize his own expected utility over his terminal wealth:

$$\sup_{\{A_t^i\}} E\left[-\frac{1}{R_a}\exp\left[-R_a\left(S_T^i - \int_0^T c_t^i dt\right)\right]\right]. \tag{6}$$

The point is that the optimal policies to both the investor's and the manager's problems must coincide in equilibrium. Otherwise, we say that the investor's contract does not implement her optimal policy.

3 Equilibrium

To derive a rational expectations equilibrium of our economy, we begin with each manager's and each investor's budget constraint and optimization problem using a conjectured price process P_t. Given the well-known property of CARA preferences under normal distributions of payoffs and signals, we shall adopt the linear pricing function of He and Wang in our setup.

In equilibrium, each manager's participation constraint and incentive compatibility constraint must be taken into account in solving the investor's problem, i.e.,

$$\sup_{\{A_t^i\},S_T^i} E\left[-\frac{1}{R_p}\exp\left[-R_p\left(W_T^i - S_T^i\right)\right]\right], \tag{7}$$

subject to

$$dW_t^i = rW_t^i dt + A_t^i(dP_t - rP_t dt + dD_t) \equiv rW_t^i + A_t^i dQ_t^i, \tag{8}$$

$$\{A_t^i\} = \text{argmax} \quad E\left[-\frac{1}{R_a}\exp\left[-R_a\left(S_T^i - \int_0^T c_t^i dt\right)\right]\right], \tag{9}$$

where $dQ_t^i = dP_t + dD_t - rP_t dt$ and

$$E\left[-\frac{1}{R_a}\exp\left(-R_a\left[S_T^i - \int_0^T c_t^i dt\right]\right)\right] \geq -\frac{1}{R_a}\exp\left(-R_a\mathcal{E}_0\right). \tag{10}$$

[10]In other words, as in He and Wang, the differentially informed managers have the same exponential utility function.

We shall adopt from Ou-Yang to arrive at the optimal solutions for $\{A_t^i\}$ and S_T^i by solving the above investor's dynamic maximization problem. In addition, the market must clear at every point in time, i.e.,

$$\int_i A_t^i di = \Theta_t.$$

Note that G cannot be observed and must be learned by all parties involved. The equilibrium pricing rule at each stage may depend on the historical information, which helps resolve uncertainties about G. This would lead to the problem of unbounded dimensionality of state variables. As in He and Wang, our assumption of a continuum of funds with differential (but same quality of) information helps reduce the unbounded dimensionality to a finite one that includes only the first order expectations.

3.1 Equilibrium Pricing Function

Following He and Wang, we conjecture that the equilibrium price function P_t is given by the following linear form:

$$P_t = \lambda_{0t} + \lambda_{1t} G + \lambda_{2t}\Theta_t + \lambda_{3t} G_t^c + \lambda_{4t} D_t, \tag{11}$$

where all the λ's are deterministic functions of time t alone. Define $\xi_t = \lambda_{1t}G + \lambda_{2t}\Theta_t$. Since observing P_t is equivalent to observing ξ_t, we can write the information sets $\mathcal{F}_t^c$ and $\mathcal{F}_t^i$ as

$$\mathcal{F}_t^c = \{D_s, \xi_s, 0 \leq s \leq t\}; \qquad \mathcal{F}_t^i = \{\mathcal{F}_0, D_s, \xi_s, Y_s^i, 0 \leq s \leq t\}.$$

Notice that G rather than G^i, the manager i's inferred value of G based upon his total information set $\mathcal{F}_t^i$, appears in the pricing function. Intuitively, the equilibrium price should depend on the average of all managers' expectations of G. In our setup as well as in that of He and Wang, there are an infinite number of managers and the noise terms of their signals are i.i.d.. By the Law of Large Numbers, the average of all private signals is G. The absence of G^i from the price function makes the solution tractable.

In the next three subsections, we shall validate the above linear pricing rule and solve for the optimal contract and optimal portfolio policy for each manager. For simplicity of exposition, let

$$\sigma_D = \begin{pmatrix} b_D & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_\Theta = \begin{pmatrix} 0 & b_\Theta \end{pmatrix},$$

where b_D and b_Θ are two constants.

3.2 Managers' Expectations

Before treating a manager's maximization problem for his optimal portfolio policy, we must solve the manager's filtering problem. In general, the manager's portfolio policy depends upon the manager's first order expectations as well as his higher order expectations. Adapting from He and Wang, this subsection demonstrates that all of the manager's higher order expectations can be reduced to a function of his first order expectations in our integrated model. Consequently, the manager's optimal portfolio policy depends only upon his first order expectations. The next proposition extends Lemma 3 of He and Wang to a continuous-time setting.

Proposition 1. *G_t^i can be expressed as a weighted average of G_t^c and $\overline{Y}_t^i = Y_t^i/t$:*

$$G_t^i = \pi_t G_t^c + (1-\pi_t)\overline{Y}_t^i, \tag{12}$$

where $\pi_t = \frac{\sigma_Y^2}{t f_t^c + \sigma_Y^2}$. Furthermore, the managers' higher order expectations can be expressed as linear functions of their first order expectations.

Proof: See Appendix A.

This proposition states that manager $i's$ expectation of G, conditional on his total information, is a weighted average of his first order expectations conditional on his common information and his private signal. This is the reason that G rather than G^i appears in the equilibrium price function. Higher order expectations include the market average of all investors' expectations of G and an investor's expectation of the market average of all investors' expectations of G.

We now identify the state variables needed to solve a manager's maximization problem using the dynamic programming approach. There are five state variables, W_t, G, Θ_t, G_t^c, and D_t. For simplicity, let Ψ_t be a column vector defined by ${\Psi_t}^T \equiv (1, G, \Theta_t, G_t^c, D_t)$ where T denote the transpose. The $d\Psi_t$ process may be written in a compact form as

$$d\Psi_t = a_\Psi \Psi_t dt + b_\Psi dB_\Psi. \tag{13}$$

By definition, $\Psi_t^{iT} = E\left[{\Psi_t}^T | \mathcal{F}_t^i\right] = (1, G_t^i, \Theta_t^i, G_t^c, D_t)$. Using Lemma **??** stated in Appendix A and the manager's information sets $\mathcal{F}_t^i$ and $\mathcal{F}_t^c$, we can arrive at expressions for G_t^i, Θ_t^i, and G_t^c. Similarly, $d\Psi_t^i$ can be written in a compact form as

$$d\Psi_t^i = a_\Psi^i \Psi_t^i dt + b_\Psi^i dB_{\Psi^i}. \tag{14}$$

Here, the expressions for coefficients a_Ψ, b_Ψ, a^i_Ψ, and b^i_Ψ are given in Appendix A.

To determine the wealth process, we first look at the excess return process. Define the excess return process dQ^i as

$$dQ^i = dD_t + dP_t - rP_t dt \equiv a^i_Q \Psi^i dt + b^i_Q dB_{\Psi^i}. \tag{15}$$

The wealth process can then be written as

$$dW_t = rW_t dt + A^i_t dQ^i = \left[rW_t + A^i_t a^i_Q \Psi^i \right] dt + A^i_t b^i_Q dB_{\Psi^i}, \tag{16}$$

where b^i_Q can be expressed in terms of b_{Ψ^i} as

$$b^i_Q = \Omega_{Q\Psi} b_{\Psi^i}. \tag{17}$$

The explicit forms for coefficients a^i_Q, b^i_Q, and $\Omega_{Q\Psi}$ are given in Appendix A.

Having worked out the processes for the state variables in Appendix A, we are now in a position to specify the contract space and then solve the manager's and the investor's dynamic programming problems. We next describe two models that adopt two different contract forms. In Model I investors may contract on various signals, and in Model II investors constrain the contract form to be of a linear function of the terminal values of their portfolios.

Model I: The Optimal Risk Sharing Contract

For tractability of the model and as a benchmark case, we assume that there is no information asymmetry about the risky stock between a manager and his investor. In other words, the investor or the investment firm that hires the manager observes the same information about the stock. This is understandably a strong assumption.[11] Justifications for the employment of a portfolio manager may include the manager's lower marginal costs in managing funds and the investors' lack of time for active investment.

In a portfolio with one stock and one bond, if investors observe the wealth process of the portfolio continuously, then they can infer precisely the manager's portfolio policy by quadratic variation of

[11] In the case of a mutual fund, there are two types of potential conflicts: the conflict between the outside investors and the mutual fund firm and that between the firm and the manager that actively manages the fund. Our model and all other exisiting models capture only one of the conflicts. The assumption that there is no asymmetric information might hold well if the principal is an investment firm and the agent is the manager hired by the firm. The firm and the manager naturally share all the information about the stock. Most likely, mutual fund firms hire financial analysts to provide their portfolio managers with information.

the wealth process.[12] In practice, investors do not know a fund's minute-by-minute value and only observe the value of the fund at the end of the day. Therefore, we assume that investors observe the terminal value of the portfolio W_T^i only.[13] Consequently, the contract S_T^i may depend upon W_T^i, $\mathcal{F}_t^i, t \in [0, T]$, and other variables that can be inferred jointly by both the manager and the investor. We next define a specific contract space.

Adapting from Ou-Yang, we assume that the contract space S_T^i is of the following form:

$$S_T^i = \mathcal{E}_T + \int_0^T \alpha_t^i(t, W_t^i, \Psi_t^i)dt + \int_0^T \beta_t^i(t, W_t^i, \Psi_t^i)dW_t^i + \int_0^T \Phi_t^i(t, W_t^i, \Psi_t^i)d\Psi_t^i, \tag{18}$$

where $\mathcal{E}_T$ is an arbitrary function of T, W_T, and Ψ_T^i and where $\Phi_t^i(t, W_t^i, \Psi_t^i)$ is a 5×1 vector.[14] Notice that this contract space includes certain path-dependent functions of W_t. Since the investor does not observe the W_t process continuously, we must exclude from our solution contracts that are functions of W_t.

Given the above contract form, we define the value function of manager i as

$$V^i(t, W_t^i, \Psi_t^i) = \sup_{\{A_u\}} \left[-\frac{1}{R_a} \exp\left\{ -R_a \left[\mathcal{E}_T + \int_t^T \left(\alpha_u^i(\cdot)du + \beta_u^i(\cdot)dW_u^i + \Phi_u^i(\cdot)d\Psi_u^i - c_u^i du \right) \right] \right\} \right]. \tag{19}$$

Appendix B presents the manager's Bellman equation. We next provide an expression for the equilibrium compensation scheme in terms of the manager's value function, α_t^i, β_t^i, and Φ_t^i using the manager's Bellman equation and his participation constraint.

Proposition 2. *The equilibrium compensation scheme is given by*

$$S_T^i = \mathcal{E}_0 + \int_0^T c_t^i dt + \frac{R_a}{2} \int_0^T \left[\bar{\beta}^i A_t^i b_Q^i + \bar{\Phi}^i b_{\Psi^i} \right] \left[\bar{\beta}^i A_t^i b_Q^i + \bar{\Phi}^i b_{\Psi^i} \right]^T dt + \left[\bar{\beta}^i A_t^i b_Q^i + \bar{\Phi}^i b_{\Psi^i} \right] dB_{\Psi^i}, \tag{20}$$

where $\bar{\beta}^i \equiv \beta^i - \frac{V_W}{R_a V^i}$ *and* $\bar{\Phi}^i \equiv \Phi^i - \frac{V_{\Psi^i}}{R_a V^i}$.

[12] As pointed out in Ou-Yang that even in the case where there are multiple stocks, if investors observe both the stock price and the wealth of the portfolio continuously, then they can infer precisely the manager's portfolio policy. Therefore, for any continuous-time delegated portfolio management problem to be more challenging, one must assume that investors do not observe the wealth and the stock prices simultaneously.

[13] For practical purposes, we may assume that forcing contracts in which investors can force managers to take any portfolio policies are not admissible, or that investors' contracts must induce managers to adopt their portfolio policies voluntarily.

[14] We assume that $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy the regularity conditions given in Ou-Yang. Since there is no asymmetric information about the state variables between the investor and the manager, the inferrence about them can be included in the contract. It can be seen that adding a term involving the stock price P_t such as $\int_0^T \beta_P(t, W_t^i, P_t, \Psi_t^i)dP_t$ is redundant, because $\beta_P(\cdot)dP_t$ can always be expressed as a function of $\Phi_t^i(\cdot)d\Psi_t^i$. Consequently, our contract form implicitly includes a passive index as a potential benchmark when $\int_0^T \Phi_t^i(\cdot)d\Psi_t^i = -nP_T$, where n is a constant and denotes the number of shares invested in the risky stock.

Proof: See Appendix B.

As in Holmstrom and Milgrom (1987), Schattler and Sung (1993), and Ou-Yang (2000), this optimal compensation scheme has an intuitive interpretation. At the terminal date, investors pay the manager his reservation wage, reimburse the manager's operating costs, compensate the manager in the third term for the risk that he bears in the fourth term, and the fourth term involves Brownian motion processes and represents the risk in the manager's compensation scheme. Note that if the manager is risk-neutral, then the third term vanishes, because a risk-neutral manager does not require compensation for bearing risk. It is easy to show that given this compensation scheme, the manager's participation constraint is satisfied, which greatly simplifies the investor's maximization problem.

Notice that the expression for S_T^i represents only the equilibrium amount that the investor pays to the manager if the manager adopts the investor's optimal policy. It does not implement the investor's optimal policy.[15] In addition, since the manager's cost function is a function of W_t^i, which is not observed by the investor, this compensation scheme cannot be enforced. The investor's objective is to solve for an optimal contract that implements her optimal policy and that depends on $\{t, \Psi_t^i\}$ and W_T^i. Given the equilibrium compensation scheme in Proposition 2 that satisfies the manager's participation constraint, our strategy is to solve for A_t^i, β^i, and Φ_t^i so as to maximize the investor's expected utility. Given the optimal solutions A_t^{i*}, β^{i*}, and Φ_t^{i*}, we shall next construct an optimal contract in terms $\{t, \Psi_t^i\}$ and W_T^i, and then show that the resulting contract indeed implements the investor's optimal policy A_t^{i*} or that given the investor's contract, A_t^{i*} also solves the manager's dynamic maximization problem. The point is that the optimal contract must reduce to the equilibrium compensation scheme given in Proposition 2; otherwise, the contract may not be optimal. The next theorem summarizes a key result of the paper.

Theorem 1. *Given the conjectured price function (11), the optimal portfolio policy of manager i is given by*

$$A_t^{i*} = H\Psi^i; \quad H = \frac{a(t)\left[a_Q - 2\frac{R_a R_p}{R_a + R_p} b_Q^i b_{\Psi^i}^T V(t)\right]}{\Delta},$$

where

$$a(t) = \left(1 - \frac{\gamma}{r}\right) \mathrm{e}^{r(T-t)} + \frac{\gamma}{r} \quad \text{and} \quad \Delta = k + a^2(t) \frac{R_a R_p}{R_a + R_p} b_Q^i {b_Q^i}^T.$$

[15]See Section 3 of Ou-Yang for details.

The optimal contract for manager i is given by

$$
\begin{aligned}
S_T^i &= \mathcal{E}_0 - a(0)W_0^i + \frac{1}{2}\int_0^T k^i(t) A_t^{i*2} dt + \frac{R_a}{2}\int_0^T \left(\bar{\beta} A_t^{i*} b_Q^i + \Phi^i b_{\Psi^i}\right)\left(\bar{\beta} A_t^{i*} b_Q^i + \Phi^i b_{\Psi^i}\right)^T dt \\
&- \int_0^T [a_Q \Psi^i A_t^{i*} + a_{\Psi^i}\Psi^i] dt + \frac{R_p}{R_a + R_p}\left[W_T^i + 2\int_0^T \Psi^i V^T(t) d\Psi^i\right] \\
&+ \frac{R_a}{R_a + R_p}\left[W_T^i - \int_0^T a(t) A_t^{i*} \Omega_{Q\Psi} d\Psi^i\right] \\
&\equiv \mathcal{E}_0 + \frac{1}{2}\int_0^T k^i(t) A_t^{i*2} dt + U(\Psi^i) + \frac{R_p}{R_a + R_p} W_T^i \\
&+ \frac{R_a}{R_a + R_p}\left[W_T^i - a(0)W_0^i - \int_0^T a(t) A_t^{i*} \Omega_{Q\Psi} d\Psi^i\right] \qquad (21)
\end{aligned}
$$

where $\bar{\beta} = a(t)$ *and where*

$$
\Phi^i = \frac{2\Psi^i V^T(t) R_p - R_a a(t) A_t^{i*} \Omega_{Q\Psi}}{R_a + R_p},
$$

in which $V^T(t)$ *denotes the coefficient matrix in the investor's value function that is a solution to the system of ODEs given in Appendix C.*

For a special case in which $k^i(t) = 0$*, there exists an equivalent optimal contract given by*

$$
S_T^i = \mathcal{E}_0 + U'(\Psi^i) + \frac{R_p}{R_a + R_p} W_T^i + \frac{R_a}{R_a + R_p}\frac{\gamma}{r}\left[W_T^i - a(0)W_0^i - \int_0^T A_t^{i*} dQ^i\right], \qquad (22)
$$

where $U'(\Psi^i) = U(\Psi^i) + \int_0^T (\Omega_{Q\Psi^i} a_{\Psi^i} - a_Q)\Psi_t^i dt$*. Furthermore, if* $\gamma = 0$ *and* $k^i(t) = 0$*, then we arrive at a simple optimal contract given by*

$$
S_T^i = \text{constant} + \frac{R_p}{R_a + R_p} W_T^i. \qquad (23)
$$

Proof: See Appendix C.

Remark 1. These equilibrium contracts take essentially the same forms as those in Ou-Yang. In addition to functions of the state variables,[16] $U(\Psi^i)$ and $U'(\Psi^i)$, the agent's compensation is composed of a constant, the agent's cost associated with his trading, a fraction of the terminal value of the portfolio, plus a bonus or a penalty depending upon the excess return between the managed portfolio and a pre-specified benchmark. From the proof given in Appendix C, it can be seen that

[16] In Ou-Yang where the stock prices are assumed to follow geometric Brownian motion processes, $U(\Psi^i)$ and $U'(\Psi^i)$ are constants.

in equilibrium, the above optimal contract reduces to that in Proposition 2. When $k^i(t) = 0$, the problem has two equivalent solutions for optimal contracts. Not only are these two contracts equal in equilibrium, but also they implement the same portfolio policies.

Remark 2. Appendix C shows that our optimal contract allows investors and managers to achieve efficient risk sharing. In the absence of costs, the optimal contract takes a simple form, i.e., a linear function of the terminal value of the portfolio, independent of the intertemporal signals. In other words, observing the signals continuously does not add value to investors. In this case, since the manager does not have an incentive to underinvest in the risky stock, there is no need to use a benchmark to induce the manager to adopt certain trading strategies. Notice that both the investor and the manager consume only at the terminal date. Given the efficient sharing of the terminal value of the portfolio, the manager does not have an incentive to misinterpret the signals, therefore, contracting on intertemporal signals is not necessary.

Remark 3. Though Theorem 1 presents the expressions for the manager's demand function and the optimal contract, both of them have not been completely determined yet due to the presence of unknown parameters such as λ_{1t} and λ_{2t} in the expression for H through the a_Q and b_Q terms. They can be determined by imposing the market clearing conditions as follows.

3.3 Market Clearing

When the market clears in equilibrium, the total demand must equal to the total supply Θ_t at any time t. We thus have

$$\int_i A_t^{i*} = \int_i H\Psi^i = \Theta_t, \tag{24}$$

which determines the H vector in the manager's demand function, leading to a simple expression for A_t^{i*}.

Theorem 2. *In equilibrium, manager i's trading strategy can be expressed as*

$$A_t^{i*} = \Theta_t^i - \frac{\lambda_{1t}\pi_t}{\lambda_{2t}(1-\pi_t)} \left(G^i - G^c\right),$$

where $\lambda_{1t} > 0$ and $\lambda_{2t} < 0$, both of which can be computed by solving numerically the ODEs given in Appendix E.

Proof: The market clearing condition (24) leads to

$$H_1 + H_2 \int_i G_t^i + H_3 \int \Theta_t^i + H_4 G_t^c + H_5 D_t = \Theta_t$$

or

$$H_1 + H_2 G_t + H_3 \Theta_t + H_4 G_t^c + H_5 D_t = \Theta_t.$$

Using Eq. (**??**), $G_t = \pi_t G_t^c + (1-\pi_t)G$, and the fact that $\lambda_{1t} G + \lambda_{2t}\Theta_t = \lambda_{1t} G_t^c + \lambda_{2t}\Theta_t^c = \lambda_{1t} G_t^i + \lambda_{2t}\Theta_t^i$, we get

$$H_1 + H_2\left(\pi_t G^c + (1-\pi_t)G\right) + H_3 \frac{\lambda_{1t} G + \lambda_{2t}\Theta_t - \lambda_{1t}\left(\pi_t G^c + (1-\pi_t)G\right)}{\lambda_{2t}} + H_4 G_t^c + H_5 D_t = \Theta_t.$$

Equating the coefficients of all the state variables in the above equation, we obtain

$$\begin{aligned} &H_1 = 0; \quad H_3 = 1; \quad H_5 = 0; \\ &H_2 = -H_4 = -\frac{\lambda_{1t}\pi_t}{\lambda_{2t}(1-\pi_t)} = -\frac{\lambda_{1t}\sigma_Y^2}{\lambda_{2t} t f_t^c}. \end{aligned} \tag{25}$$

We thus have

$$A_t^{i*} = H_2 G_t^i + \Theta_t^i - H_2 G_t^c = \Theta_t^i - \frac{\lambda_{1t}\pi_t}{\lambda_{2t}(1-\pi_t)}(G_t^i - G_t^c).$$

Q.E.D.

As in He and Wang, the manager's demand function has two components. The first component is fund $i's$ estimation of the supply shock. The higher the supply, the higher the demand in equilibrium, reflecting the fund's position in accommodating the shocks. The second component is proportional to the difference between the fund's estimation of the stock's expected dividend rate and that based on the common information alone. It represents the fund's speculative position based on its private information. As expected, the manager's demand increases with his private estimation of the expected dividend rate.

In sum, this subsection demonstrates that the relations such as the manager's demand function of He and Wang and the optimal contract and its implications of Ou-Yang are quite robust and take the same form as in our integrated model.

Model II: The Optimal Linear Contract

In Model I we have assumed that investors and managers share the same signals. The only potential conflict is through managers' cost functions. In reality, however, investors usually observe neither the signals nor the continuous path of the wealth processes of their portfolios. All they observe might be the terminal values of their portfolios alone. Contracts do not usually depend upon signals regarding the underlying assets. As a matter of fact, most mutual fund companies charge their investors a fraction of the total assets under management.

For tractability, this model confines our contract space to be of the following linear form:

$$S_T^i = \alpha^i + \beta^i W_T^i,$$

where α and β are constants to be determined in equilibrium. In specifying this contract, we are assuming that the investor observes only the terminal value of her manager's portfolio. Given the above linear contract and conjecture that the equilibrium pricing rule takes the same linear form as in Model I, we first solve the manager's maximization problem, obtaining an optimal trading strategy in terms of β and the manager's information set. The market clearing condition determines the coefficients in the linear pricing function, and the manager's participation constraint gives a relation for α in terms of β. The investor's problem is then to choose β so as to maximize her ex-ante expected utility: $E_0\left[-\frac{1}{R_p}\mathrm{e}^{-R_p(W_T-S_T)}\right]$, with the wealth process being determined by the manager's optimal trading strategy. As in Model I, it is not possible to solve for the equilibrium price and the optimal demand function in closed form. We thus resort to the numerical algorithm as described in Appendix D. The Bellman equations for both the manager's and the investor's maximization problems are also presented in Appendix D.

4 Further Discussion and Simplification of the Equilibrium for Model I

To determine completely the optimal contract, the equilibrium price, and the equilibrium trading strategies, we must solve for the coefficients λ's numerically. This section further simplifies the equations that must be satisfied by these coefficients. To help understand model I presented in the last section, we shall also describe three benchmark cases. We shall first ignore differential information and consider two special cases in which G is a perfectly known constant and in which G cannot be observed and must be learned over time, respectively. For comparison with the previous literature, we shall also present the simplified case in which there is no delegated portfolio problem or managers trade for their own accounts.

4.1 Benchmark Case I: G Perfectly Known

In this case, since G is a perfectly known constant, there is no differential information among the managers. The vector of state variables except for W_t can then be written in a compact form as $\Psi = \begin{pmatrix} 1 \\ \Theta_t \\ D_t \end{pmatrix}$. The conjectured equilibrium pricing function takes a simple form:

$$P_t = \lambda_{0t} + \lambda_{1t}\Theta_t + \lambda_{2t}D_t.$$

Theorem 3. *The equilibrium price process can be expressed as*

$$P_t = \frac{G}{r}\left[1 - \mathrm{e}^{r(T-t)}\right] + \lambda_{1t}\Theta_t.$$

where λ_{1t} can be obtained by solving numerically the ODEs described in Appendix D.

Proof: See Appendix E.

Note that the current dividend level D_t drops out of the equilibrium price function.

4.2 Benchmark Case II : G unobservable, no differential information

In this case, we assume that no one receives the private signal Y_t^i and that the only signal that reveals information about G is the publicly observed dividend process D_t. Similarly, the vector of the state variables except for W_t can be formed as $\Psi = \begin{pmatrix} 1 \\ \Theta_t \\ G_t^c \\ D_t \end{pmatrix}$. The equilibrium price process then takes the following linear form of the state variables:

$$P_t = \lambda_{0t} + \lambda_{1t}\Theta_t + \lambda_{2t}G_t^c + \lambda_{3t}D_t,$$

which can be simplified as in the following theorem.

Theorem 4. *The equilibrium price process can be expressed as*

$$P_t = \frac{G_0 I_t + D_t}{(I_t + t)r}\left[1 - \mathrm{e}^{r(T-t)}\right] + \lambda_{1t}\Theta_t,$$

where $I_t = \frac{b_D^2}{b_G^2}$ and where λ_{1t} can be solved numerically as described in Appendix E.

Proof: See Appendix E.

As shown in Appendix E, G_t^c is a linear function of D_t in equilibrium, it thus drops out the equilibrium price function.

4.3 Differential Information Without Portfolio Delegation

To analyze the impact of portfolio delegation on the price formation and autocorrelation, we consider a continuous-time analog of the discrete-time model of He and Wang (1995) in the presence of a cost function. There is no contracting involved. A continuum of agents receive individual signals and trade for their own accounts. Unlike portfolio managers, these agents own the entire portfolio and trade to maximize their utility functions over the terminal value of the portfolio W_T^i minus the costs associated with managing the portfolio. The agent bears the total risk of the portfolio.

Therefore, each agent's problem becomes

$$\sup_{\{A_t^i\}} E\left[-\frac{1}{R_a}\exp\left[-R_a\left(W_T^i - \int_0^T c(t, W_t, A_t)dt\right)\right]\right],$$

subject to

$$dW_t^i = rW_t^i dt + A_t^i(dP_t - rP_t dt + dD_t) \equiv rW_t^i + A_t^i dQ_t^i.$$

The vector of the state variables is the same as in the general case. Using the equilibrium conditions as described in Section 2, we attain the following results:

Theorem 5. *Given the conjectured price function (11), the optimal portfolio policy of agent i is given by*

$$A^{*i} = H\Psi^i; \quad H = \frac{a(t)\left[a_Q - 2R_a b_Q^i b_{\Psi^i}^T V(t)\right]}{k + a^2(t) R_a b_Q^i {b_Q^i}^T},$$

where

$$a(t) = \left(1 - \frac{\gamma}{r}\right)\mathrm{e}^{r(T-t)} + \frac{\gamma}{r},$$

in which $V^T(t)$ is a coefficient matrix in the agent's value function that satisfies the system of ODEs given in Appendix E.

Proof: See Appendix E.

When the market clears in equilibrium, the total demand must equal to the total supply Θ. We thus have

$$\int_i A_t^{i*} = \int_i H\Psi^i = \Theta_t. \tag{26}$$

Like in the general case, the market clearing condition leads to an intuitive expression for A_t^{i*}:

$$A_t^{i*} = \Theta_t^i - \frac{\lambda_{1t}\pi_t}{\lambda_{2t}(1-\pi_t)}\left(G_t^i - G_t^c\right),$$

which takes the same form as in He and Wang or Theorem 2. Of course, the coefficients λ_{1t} and λ_{2t} differ from those in Theorem 2 where optimal contracting is involved.

4.4 The General Case

We now analyze the general case that we formulated in Section 3. The numerical procedure is described as follows.

Define $Z = \lambda_{1t}\dot{\lambda}_{2t} - \lambda_{2t}\dot{\lambda}_{1t} - a_\Theta\lambda_{1t}\lambda_{2t}$ and $B = b_Q b_{\Psi^i} V(t)$. The market clearing conditions yield

$$\frac{a(t)}{\Delta}\left[\dot{\lambda}_{1t} - r\lambda_{1t} + 1 + \lambda_{4t} - 2\frac{R_a R_p}{R_a + R_p}B[2]\right] = -\frac{\lambda_{1t}\pi_t}{\lambda_{2t}(1-\pi_t)} = -\frac{\lambda_{1t}\sigma_Y^2}{\lambda_{2t} t f_t^c},$$

$$\frac{a(t)}{\Delta}\left[\dot{\lambda}_{2t} - (a_\Theta + r)\lambda_{1t} - 2\frac{R_a R_p}{R_a + R_p}B[3]\right] = 1,$$

where $B[2]$ and $B[3]$ denote the second and third elements in the B vector. This leads to the expression for Z:

$$Z = \left(1 + \lambda_{4t} + \frac{\lambda_{1t}\sigma_Y^2}{\lambda_{2t} t f_t^c}\right)\lambda_{2t} + 2\frac{R_a R_p}{R_a + R_p}\left(\lambda_{1t}B[3] - \lambda_{2t}B[2]\right) + \frac{\Delta\lambda_{1t}}{a(t)}, \tag{27}$$

where

$$\begin{aligned} B[2] &= (1+\lambda_{4t})f_t^i\left[1 - \frac{Z}{\lambda_{2t}}\right]\left[V(2,2) - \frac{\lambda_{1t}}{\lambda_{2t}}V(3,2)\right] + \lambda_{2t}b_\Theta^2 V(3,2) \\ &\quad + \lambda_{3t}f_t^{c2}\left[\frac{1}{b_D^2} + \frac{Z^2}{\lambda_{2t}^4 b_\Theta^2}\right]V(4,2) + b_D^2(1+\lambda_4 t)V(5,2); \end{aligned}$$

$$\begin{aligned} B[3] &= (1+\lambda_{4t})f_t^i\left[1 - \frac{Z}{\lambda_{2t}}\right]\left[V(3,2) - \frac{\lambda_{1t}}{\lambda_{2t}}V(3,3)\right] + \lambda_{2t}b_\Theta^2 V(3,3) \\ &\quad + \lambda_{3t}f_t^{c2}\left[\frac{1}{b_D^2} + \frac{Z^2}{\lambda_{2t}^4 b_\Theta^2}\right]V(4,3) + b_D^2(1+\lambda_4 t)V(5,3). \end{aligned}$$

Substituting the expressions for $B[2]$ and $B[3]$ into Eq. (27), we obtain a quadratic expression for Z given by

$$Z = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1},$$

where

$$C_1 = -\lambda_{2t}(1+\lambda_{4t}) - \left[1+\frac{\lambda_{1t}\sigma_Y^2}{a(t)tf_t^c}\right]\left[k+a^2(t)\frac{R_aR_p}{R_a+R_p}\left[b_D^2(1+\lambda_{4t})^2+\lambda_{2t}^2b_\Theta^2+\frac{\lambda_{3t}^2f_t^{c2}}{b_D^2}\right]\right]$$
$$+ \quad 2\frac{R_aR_p}{R_a+R_p}\left[\frac{\xi_1}{\lambda_{2t}}+\lambda_{2t}b_\Theta^2\left[\lambda_{2t}V(3,2)-\lambda_{1t}V(3,3)\right]+\frac{\xi_2}{b_D^2}\right],$$

$$C_2 = 1-2\frac{R_aR_p}{R_a+R_p}\frac{\xi_1}{\lambda_{2t}^2}, \quad C_3 = \frac{R_aR_p}{R_a+R_p}\frac{2}{\lambda_{2t}^4b_\Theta^2}\left[\xi_2-\lambda_{1t}\lambda_{3t}^2f_t^{c2}\right],$$

$$\xi_1 = f_t^i\left[\lambda_{2t}^2V(2,2)+\lambda_{1t}^2V(3,3)-2\lambda_{1t}\lambda_{2t}V(3,2)\right], \quad \xi_2 = \lambda_{3t}f_t^{c2}\left[\lambda_{2t}V(4,2)-\lambda_{1t}V(4,3)\right].$$

Given this explicit solution for Z, we can now convert our system of implicit ODEs (from Bellman equation (**??**)) to an explicit boundary value problem. The boundary conditions are given by $f_0^c = b_G^2$ and $V(T) = \lambda_{1T} = \lambda_{2T} = \lambda_{3T} = \lambda_{4T} = 0$. We conjecture an f_T^c and solve the system as an initial value problem.[17] We keep reiterating until the solution for f_0^c converges to b_G^2.

5 Equilibrium Price, Trading Strategy, and Autocorrelation

This section solves the equilibrium numerically and examines the impact of portfolio delegation on the equilibrium stock price and risk premium , the manager's trading strategy, and the serial correlation in stock and fund returns. We next present results for Models I and II separately.

Model I

5.1 Some Insights from Benchmark Case I

To gain insight into this model, we first examine Benchmark Case I in which the expression for the expected excess return can be obtained. Notice that this benchmark case differs from Campbell and Kyle (1993) or the symmetric information case of Wang (1993) on three aspects. First, the investors in this case delegate the responsibility to an agent and do not trade on their own. Second, our agents have a finite trading horizon. Third, the agents in our model have cost functions associated with both the total amount of funds under management and the magnitude of the position taken in the risky stock. One should therefore expect our results to differ from those of Campbell and Kyle and of Wang.

[17] This is feasible because all conditions other than that of f_t^c are at the terminal point. One can change the variable from t to $(T-t)$ to solve it like an initial value problem. Any standard software on numerical procedures such as MATLAB can perform the calculation.

For simplicity, we assume that $r = 0$. The excess return dQ is then given by

$$dQ = \Gamma_t \Theta dt + \left[\begin{array}{cc} b_D & \lambda_1 b_\Theta \end{array} \right] dB_t,$$

where

$$\Gamma_t = \left(2 \frac{R_a R_p}{R_a + R_p} \lambda_{1t} b_\Theta^2 V(2,2) + \frac{\Delta}{a(t)} \right).$$

Here $\Gamma_t \Theta_t$ is the expected excess return on the stock. According to Theorem 3, we have

$$P_t = G(T-t) + \lambda_{1t}\Theta_t; \quad a(t) = 1 - (T-t)\gamma.$$

The expected excess return is thus affected by two factors- the cost of action (which includes γ and $k^i(t)$) and the risk aversion coefficients of the investor and manager. Assuming that a typical investor is less risk-averse than the manager, the effective risk aversion coefficient $\frac{R_a R_p}{R_a + R_p}$ is lower with portfolio delegation or risk sharing than without it. The portfolio delegation thus helps reduce the expected excess return or risk premium. However, even if $R_p = 0$, i.e., investors are risk neutral, there is still a non-zero expected excess return due to the manager's cost function. The excess return is i.i.d. across periods only if the investors are risk neutral and $k^i(t) = 0$. Under this extreme scenario, the excess return of the fund $A^i dQ^i$ is given by $\Theta_t \left[\begin{array}{cc} b_D & 0 \end{array} \right] dB_t$, which is i.i.d. In other words, the funds will show no persistence of performance only under conditions that investors are risk neutral and that there are no costs associated with managing the portfolio. Therefore, it should not be surprising to see that portfolios exhibit persistence in performance under more general conditions in our model.

We next solve our general case as well as Benchmark Cases II and III for equilibrium prices, trading strategies, and autocorrelations. The numerical results are presented in the attached tables and figures.

5.2 The Impact of Portfolio Delegation on Equilibrium Prices and Trading Strategies

Figure 1 presents numerical results on the sensitivity or coefficient of the equilibrium price to the supply shock Θ with and without portfolio delegation. Note that portfolio delegation in our model allows investors to share risk with managers and in return managers manage funds at lower costs. With risk sharing, managers can absorb the supply shock more easily. As a result, the sensitivity of the equilibrium price to the supply shock decreases. In addition, as the risk aversion of the investor decreases, the price becomes less sensitive to supply shocks.

Figure 2 demonstrates the effect of portfolio delegation on the coefficient of G^c in the pricing function, i.e., λ_3. It can be seen that this coefficient is similar with or without portfolio delegation. Note that G is a constant in equilibrium and that G^c is chiefly determined by the dividend process D_t, which is not in the price process. Therefore, the risk sharing due to portfolio delegation results in little change in sensitivity of the price with respect to this variable. For example, in the benchmark cases, $G = G^c$ and the coefficient on G can be seen to be independent of the risk aversion coefficients of both the investor and the manager.

Figure 3 compares the sensitivities of the equilibrium price to G and G^c, respectively. As time proceeds, more information is revealed about G and therefore, the coefficient of G increases while that of G^c decreases. Consequently, the ratio $\frac{\lambda_1}{\lambda_3}$ increases. Similar to the results given in Figure 2, this ratio is largely unaffected by portfolio delegation.

Figure 4 illustrates the impact of the parameter k in our cost function on the sensitivity of the equilibrium price to the supply shock. It is seen that this sensitivity increases with k both with portfolio delegation and without it. This is because as k increases, the cost of absorbing any extra supply shock or trading in the risky stock increases, and the manager (or the investor in the case of no portfolio delegation) becomes more cautious with his demand. He thus demands a higher return whenever there is a positive supply shock. In equilibrium, since the market has to clear, the price becomes more sensitive to the supply shock.

Figure 5 studies agents' trading strategies with respect to their private information. It can be seen that with portfolio delegation, the managers trade more aggressively with respect to their private information than they would trade for their own accounts. With risk sharing, managers can afford to be more aggressive with their private information. This aggressiveness goes down as the manager's trading becomes more costly, i.e., as $k^i(t)$ increases.

5.3 Autocorrelation

Empirical studies find positive autocorrelations in stock returns over short horizons of days to months and negative autocorrelations over long horizons of 2-10 years.[18] Empirical studies on mutual fund

[18]See, e.g., Conrad and Kaul (1988) and Lo and MacKinlay (1988) on short horizons, and Fama and French (1987), Lo and Mackinlay (1988), and Poterba and Summers (1988) for long horizons.

performance find positive persistence in fund performance, i.e., past winners typically repeat their superior performance for some time.[19] Theoretically, Wang (1993) finds negative serial correlation in stock returns. Wang points out that the strong mean reversion in the stock supply can generate this result under both symmetric and asymmetric information. Under asymmetric information, the negative autocorrelation can be enhanced by the information effect of dividends and prices on the behavior of uninformed investors. The persistence in mutual fund performance has not been examined theoretically due to the absence of a dynamic asset pricing model with portfolio delegation. In this and the next subsections, we employ our model to study the autocorrelations in stock as well as delegated portfolio returns. Since a contract period is typically short in nature, we focus on the short-term horizons.

Following Wang, we consider the excess return to one share of stock, which is defined as $dQ = dP + dD - rPdt$. In order to examine the autocorrelation in stock returns, we consider the following:

$$\beta_1 \equiv Corr\left(Q_{t+\tau} - Q_t, Q_t - Q_{t-\tau}\right) = Corr\left(\int_{t-\tau}^{t} dQ(u), \int_{t}^{t+\tau} dQ(u)\right),$$

where Corr(.) denotes the autocorrelation between $\int_{t-\tau}^{t} dQ(u)$ and $\int_{t}^{t+\tau} dQ(u)$. Similarly, we can study the persistence of the fund performance by calculating the autocorrelation of the excess wealth $A^i dQ^i$ between the time intervals, $t-\tau$ and t, and t and $t+\tau$:

$$\beta_2 = Corr\left(\int_{t-\tau}^{t} A_u^i dQ^i, \int_{t}^{t+\tau} A_u^i dQ^i\right).$$

Furthermore, we study the persistence of the fund performance after adjusting for the payment to the manager:

$$\beta_3 = Corr\left(\int_{t-\tau}^{t} A_u^i dQ^i - \int_{t-\tau}^{t} dS(u), \int_{t}^{t+\tau} A_u^i dQ^i - \int_{t}^{t+\tau} dS(u)\right).$$

Here we interpret $dS(u)$ as the manager's compensation for his work between t and $t+dt$, which is given by

$$dS(u) = \frac{\mathcal{E}_0}{T}du + c_u^i du + \frac{R_a}{2}\left[\bar{\beta}^i A_u^i b_Q^i + \bar{\Phi}^i b_{\Psi^i}\right]\left[\bar{\beta}^i A_u^i b_Q^i + \bar{\Phi}^i b_{\Psi^i}\right]^T du + \left[\bar{\beta}^i A_u^i b_Q^i + \bar{\Phi}^i b_{\Psi^i}\right] dB_{\Psi^i}.$$

Notice that unlike in Model II where optimal linear contracts are considered, for small values of R_p such as $R_p = 0.1$, the autocorrelations generated by this model are not significantly different from

[19] See, e.g., Grinblatt and Titman (1992), Hendricks, Patel, and Zeckhauser (1993), Mech (1993), Brown and Goetzmann (1995), and Sias and Starks (1999).

zero. Tables 2 and 3 present a series of results on autocorrelations in stock and portfolio returns for $R_p = 1$ and $k = 1$. They show that the autocorrelations in portfolio returns are always positive, implying a persistence in fund performance. Note that agents in our model make portfolio selections based upon certain inferred variables such as the inferred supply shock and the inferred expected dividend growth rate based upon their private as well as public information. This learning process is a continuous one. If an agent has an informational advantage that leads to a positive excess return or premium, he would not lose that advantage over a short period of time, which would result in performance persistence. The autocorrelations increase with the parameter k in the manager's cost function. The reason is that with a lower k, the managers trade more aggressively with respect to their private information. They may profit more in the current period, but they are more likely to have lesser information advantage in the next period, leading to lower autocorrelations in returns. On the other hand, a higher k or a higher cost function makes the manager less aggressive and thus more likely to retain their information advantage, leading to a higher autocorrelation than with a lower cost function.

The positive autocorrelations in both the stock and the portfolio returns in our economy are perhaps due to the short horizons. Our economy may not reach a steady state since it has a finite trading horizon. As time moves forward, the probability of a supply shock reverting to its mean position decreases. Therefore, if a high premium is built up due to a large supply shock, then it is going to persist in the short run, leading to a positive autocorrelation. Furthermore, k in the cost function adds to this correlation. A higher k or a higher cost function means that the equilibrium pricing function is more sensitive to the supply shock or that it is more demanding to absorb the extra shock. Since the supply shock will persist in the short run, if we experience a higher return in the current period, we shall expect a higher return in the near future. A higher k enhances this feature or results in a higher positive autocorrelation.

Specifically, Table 2 compares the numerical estimates of β_1 and β_2 for different values of k in the absence of information asymmetry. We find positive autocorrelations in both stock and fund returns. For example, if there is a positive supply shock, then the expected return on the stock goes up in equilibrium. As a result, the return on the fund goes up since each fund holds an equal share of the excess supply in this symmetric case. In the short run, this supply shock will persist even though it is mean reverting over the long run. Therefore, the funds will continue to hold more at a higher return, thereby maintaining a good performance while the stock returns persist as well.

Table 3 presents the case where there is differential information among managers. It shows that while the autocorrelation in portfolio returns is always positive, the autocorrelation in stock returns can be either positive with a high cost function or negative with a low cost function. The intuition is as follows. Suppose that we observe the price to be low (i.e., lower than expected). The excess return is low for the current period. Since observing the price is equivalent to observing $(\lambda_1 G + \lambda_2 \Theta)$, where λ_2 is negative and its magnitude increases with k, and where λ_1 is positive and changes little with k. Conditional on a lower price, the manager would revise his inference about G downward, or about Θ upward, or a combination of both. When the magnitude of λ_2 is low as in the case of a low k, the manager's inference about Θ would be high. Given a high Θ and its mean reverting nature, the manager would expect Θ more likely to go down quickly, leading to a potentially negative autocorrelation. When λ_2 is high as in the case of a high k, the manager would revise his inference about Θ downward to a lesser degree. As a result, the manager would not expect Θ to revert to its mean quickly, leading to a potentially positive autocorrelation in the short period of time. For values in between, the autocorrelation can go from positive to negative, depending upon the length of the time period.

Note that the positive autocorrelation in fund returns decreases slightly with differential information. Consider a simple case in which the equilibrium price is lower than expected because the current dividend turns out to be lower than expected while G and Θ have remained the same. In the case of symmetric information, the drop in dividend and price yields a lower return for the current period, it will not affect the expected future cash flow of the stock because the effect is fully incorporated into the current price. When there is differential information, given the lower dividend and price, managers will revise the estimated value of G downward and that of Θ upward, respectively. Given a higher estimated Θ, the manager would expect that the probability for Θ to go higher in the next period will be lower and as a result, the probability for the price to drop will be lower, leading to a potentially lower positive autocorrelation.

Model II

Since the presence of differential information does not change the autocorrelations in fund returns significantly but complicates the numerical calculations considerably, we present only results for the symmetric case in this model. Table 1 summarizes results for α and β for various values of k and γ in the cost function. As expected, we find that for zero costs, the pay-to-performance sensitivity β is given by $\beta = \frac{R_p}{R_a+R_p}$. However, as k or γ increases, β increases. This result differs from that obtained in a typical principal-agent model where the pay-to-sensitivity decreases with the cost of effort.[20] Our seemingly different result is due to the equilibrium aspect of the model. As the cost of investing in the risky asset increases, the manager becomes more cautious in his orders. Since in equilibrium, the market must clear, the stock price gets more discounted, i.e., $|\lambda_2|$ becomes larger or the price becomes more sensitive to supply shocks. By making the value of β higher, the manager has more stake in the portfolio under his management. He becomes more aggressive in trading the risky asset, and as a result, $|\lambda_2|$ goes down. A lower $|\lambda_2|$ reduces the volatility of the price process, which benefits the investor.

Figure 6 compares the risk premium (the expected excess return), $\dot{\lambda}_2 - (a_Q + r)\lambda_2$, under the cases of optimal risk sharing contract and the optimal linear contract. Due to less optimal risk sharing, the risk premium is much higher in the presence of a linear optimal contract in which the investor and manager cannot share the risk factors such as the supply shocks optimally. Consequently, in the presence of a cost function, the manager is effectively more risk averse and demands a higher risk premium.

Tables 4 and 5 present the autocorrelations in both stock and fund returns under very small risk aversion coefficient for the investor and small cost functions for the manager. It can be seen that for the short horizons considered in the calculations, the linear contract always generates positive autocorrelations in stock and fund returns, which increase with the parameter k in the cost function as in the optimal risk sharing case. The magnitude of the autocorrelation in this case is higher than that in the optimal risk sharing case. The reason is that in this less optimal risk sharing contract, the price is more sensitive to the supply shocks. For example, a positive supply shock leads to a higher return in this case than in the optimal risk sharing case, thus leading to a higher autocorrelation in the short run before the supply shock reverts to its mean.

[20]For example, in the well-known Holmstrom-Milgrom (1987) model, β is given by $\beta = \frac{1+R_p k\sigma^2}{1+(R_a+R_p)k\sigma^2}$.

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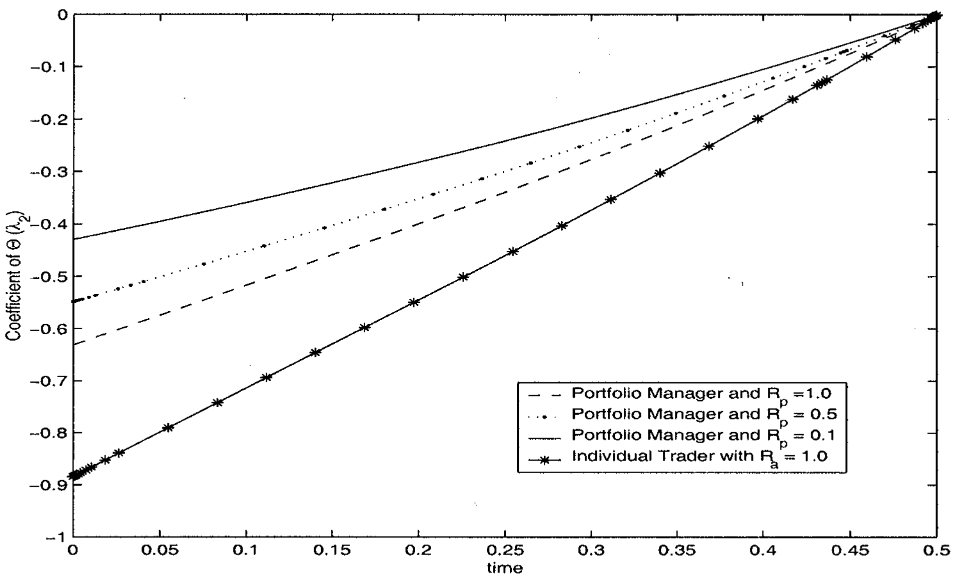

Figure 1: The sensitivity of price to Θ with portfolio delegation (as represented by portfolio manager) and without portfolio delegation (as represented by individual trader). The parameter values are $R_a = 1$, $b_G = 1$, $b_d = 1$, $b_\Theta = 1$, $b_Y = 1$, $a_\Theta = 1$, $r = .05$, $\gamma = 0.02$, $k(t) = 1$ and $T = 0.5$.

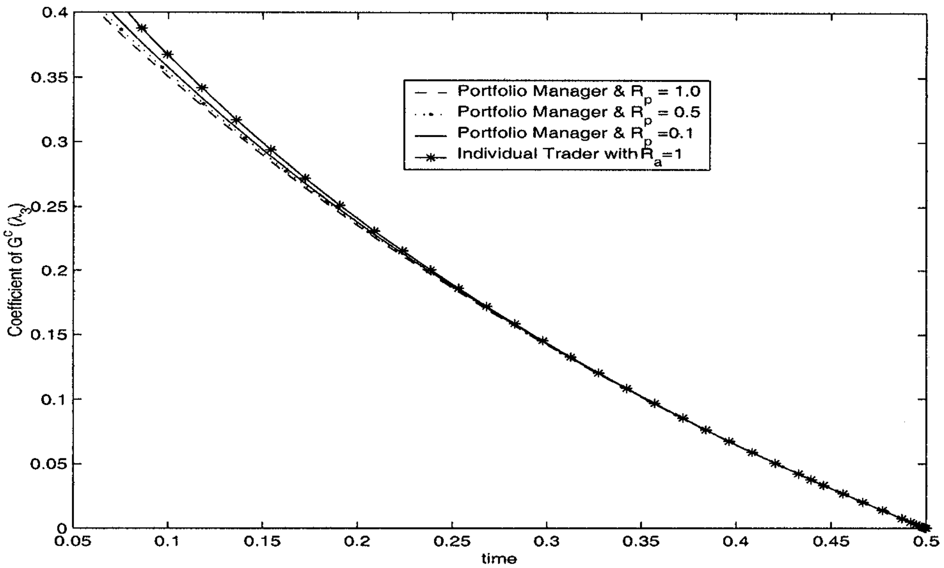

Figure 2: The sensitivity of price to G^c with portfolio delegation (as represented by portfolio manager) and without portfolio delegation (as represented by individual trader). The parameter values are $R_a = 1$, $b_G = 1$, $b_d = 1$, $b_\Theta = 1$, $b_Y = 1$, $a_\Theta = 1$, $r = .05$, $\gamma = 0.02$, k(t) = 1 and $T = 0.5$.

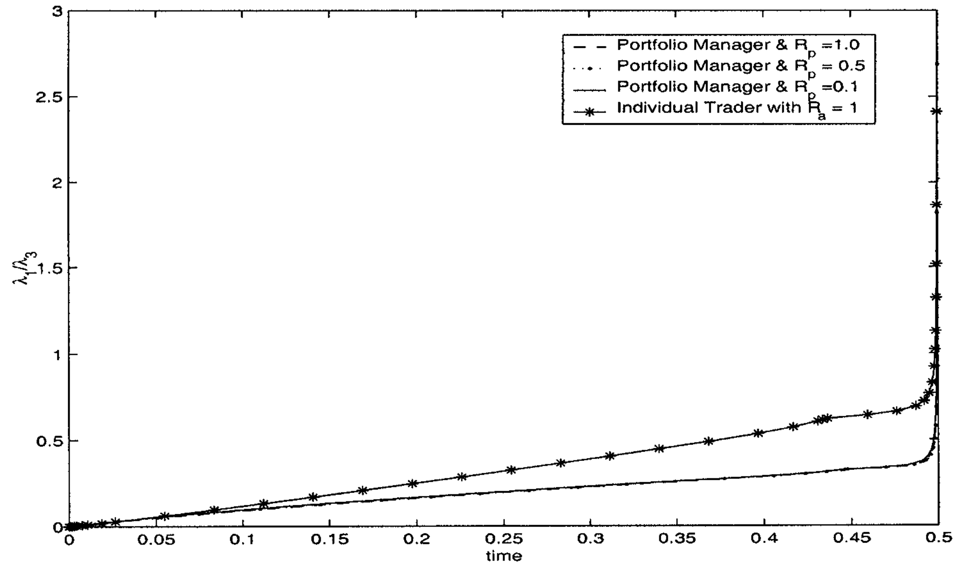

Figure 3: A comparison of sensitivity of price to G with the sensitivity to G^c (i.e. $\frac{\lambda_1}{\lambda_3}$) with portfolio delegation (as represented by portfolio manager) and without portfolio delegation (as represented by individual trader). The parameter values are $R_a = 1$, $b_G = 1$, $b_d = 1$, $b_\Theta = 1$, $b_Y = 1$, $a_\Theta = 1$, $r = .05$, $\gamma = 0.02$, k(t) = 1 and $T = 0.5$.

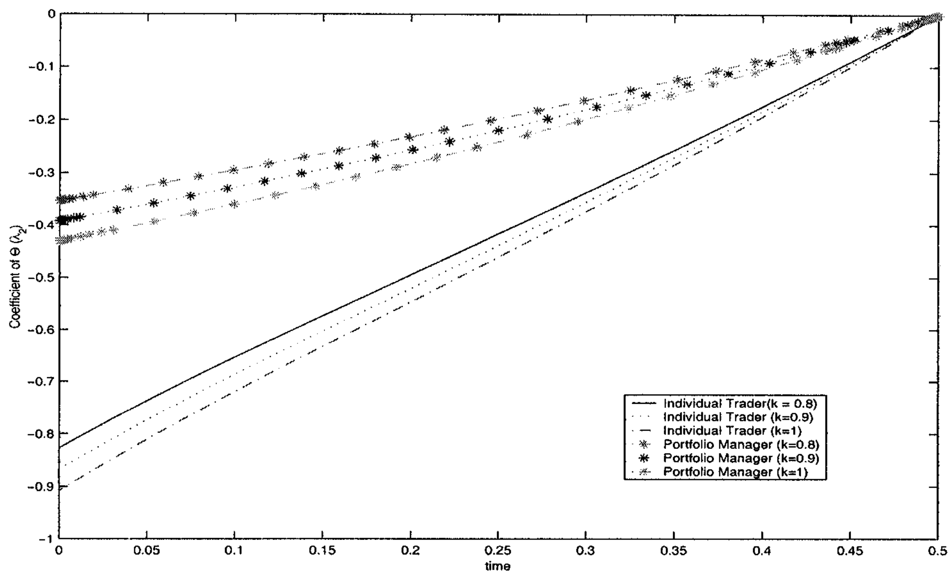

Figure 4: The sensitivity of price to Θ with portfolio delegation (as represented by portfolio manager) and without portfolio delegation (as represented by individual trader) for different values of k(t). The parameter values are $R_a = 1$, $R_p = 1$, $b_G = 1$, $b_d = 1$, $b_\Theta = 1$, $b_Y = 1$, $a_\Theta = 1$, $r = .05$, $\gamma = 0.02$ and $T = 0.5$.

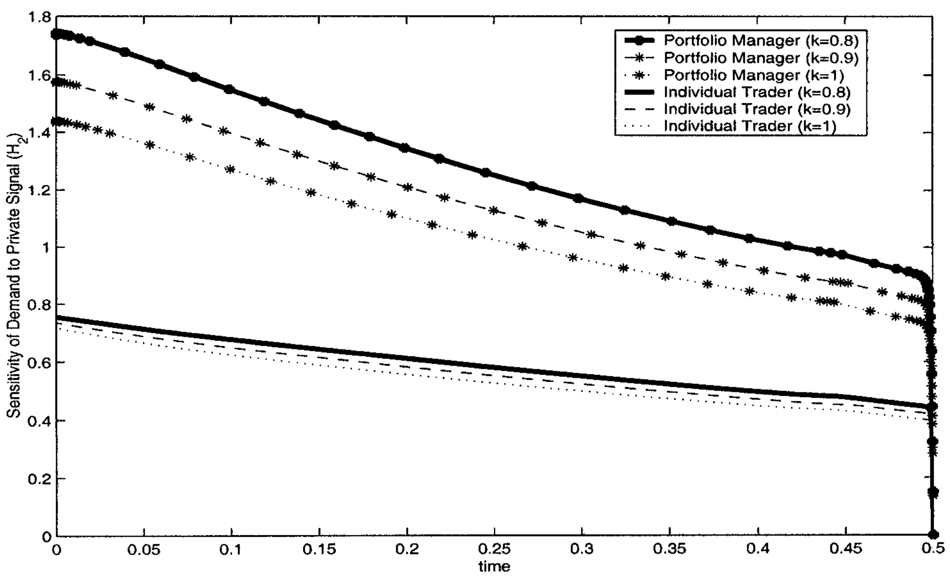

Figure 5: The response of demand of risky stock to $(G_t^i - G_t^c)$ with portfolio delegation (as represented by portfolio manager) and without portfolio delegation (as represented by individual trader). The parameter values are $R_a = 1$, $R_p = 1$, $b_G = 1$, $b_d = 1$, $b_\Theta = 1$, $b_Y = 1$, $a_\Theta = 1$, $r = .05$, $\gamma = 0.02$ and $T = 0.5$.

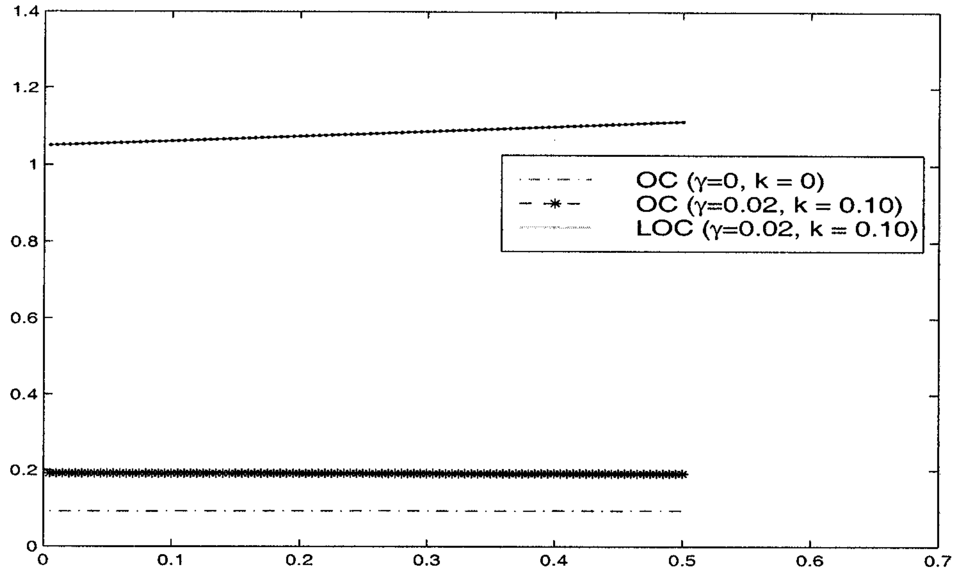

Figure 6: The risk premiums for the risky stock. OC and LOC represent the optimal risk sharing and optimal linear contracts, respectively. The parameter values are $R_a = 1$, $R_p = 0.1$, $b_G = 1$, $b_d = 1$, $b_\Theta = 1$, $b_Y = 1$, $a_\Theta = 1$, $r = .05$, and $T = 0.5$.

γ	k	α	β
0.00	0.00	1.5985	0.0909
0.00	0.01	1.5969	0.0910
0.00	0.02	1.5954	0.0920
0.00	0.03	1.5885	0.096
0.00	0.04	1.5870	0.096
0.00	0.05	1.5854	0.096
0.00	0.06	1.5786	0.1010
0.00	0.07	1.5771	0.1010
0.00	0.08	1.5755	0.1010
0.00	0.09	1.5687	0.1060
0.00	0.10	1.6872	0.1060
0.00	0.20	1.5363	0.1210
0.01	0.10	1.5670	0.1110
0.02	0.10	1.5775	0.1110
0.03	0.10	1.5774	0.1110
0.04	0.10	1.5850	0.1110
0.05	0.10	1.5743	0.1210

Table 1: Optimal linear contracts, $S_T = \alpha + \beta W_T$, under symmetric information. The parameter values are $R_a = 1$, $R_p = 0.1$, $b_G = 1$, $a_\Theta = 1$, $b_D = 1$, $b_\Theta = 1$, $b_Y = 1$, $r = .05$, $\gamma = 0.01$ and $T = 0.5$.

	β_1			β_2			β_3		
	k			k			k		
τ	0.10	0.50	0.80	0.10	0.50	0.80	0.10	0.50	0.80
0.01	0.00	0.01	0.01	0.00	0.02	0.02	0.01	0.02	0.01
0.02	0.00	0.01	0.02	0.00	0.01	0.03	0.01	0.01	0.02
0.03	0.01	0.01	0.03	0.01	0.02	0.04	0.01	0.02	0.03
0.04	0.01	0.02	0.04	0.02	0.03	0.06	0.03	0.04	0.06
0.05	0.01	0.02	0.04	0.02	0.04	0.08	0.03	0.05	0.07
0.06	0.01	0.02	0.05	0.01	0.04	0.08	0.03	0.05	0.08
0.07	0.01	0.03	0.05	0.02	0.05	0.10	0.04	0.05	0.09
0.08	0.02	0.04	0.06	0.02	0.05	0.10	0.03	0.06	0.09
0.09	0.02	0.04	0.07	0.02	0.06	0.11	0.04	0.07	0.10
0.10	0.02	0.04	0.07	0.02	0.07	0.12	0.04	0.08	0.11
0.11	0.02	0.05	0.08	0.03	0.08	0.12	0.05	0.09	0.11
0.12	0.02	0.05	0.08	0.03	0.09	0.13	0.05	0.10	0.12
0.13	0.02	0.06	0.09	0.03	0.09	0.14	0.06	0.10	0.13
0.14	0.02	0.06	0.10	0.04	0.10	0.15	0.06	0.11	0.13
0.15	0.02	0.06	0.10	0.04	0.10	0.16	0.06	0.11	0.14
0.16	0.02	0.06	0.10	0.04	0.11	0.16	0.06	0.12	0.14
0.17	0.02	0.07	0.11	0.04	0.12	0.17	0.07	0.12	0.15
0.18	0.02	0.07	0.11	0.04	0.12	0.17	0.07	0.12	0.15
0.19	0.03	0.07	0.12	0.04	0.12	0.17	0.07	0.13	0.15
0.20	0.03	0.08	0.12	0.04	0.12	0.18	0.07	0.13	0.15

Table 2: A comparison of autocorrelation of excess returns on the risky stock and on fund i for different time-horizons τ under homogenous information and portfolio delegation. $\beta_1 = Corr\left(\int_{t-\tau}^{t} dQ(u), \int_{t}^{t+\tau} dQ(u)\right)$, $\beta_2 = Corr\left(\int_{t-\tau}^{t} A_u dQ(u), \int_{t}^{t+\tau} A_u dQ(u)\right)$ and $\beta_3 = Corr\left(\int_{t-\tau}^{t} (A_u dQ(u) - dS(u)), \int_{t}^{t+\tau} (A_u dQ(u) - dS(u))\right)$. A random value of G is selected and 100 simulated paths are generated. We thus get a vector of 100 excess returns for different horizons (τ). The autocorrelations for this sample are computed. This procedure is repeated 600 times. The autocorrelations reported here are the average of autocorrelations. The parameter values are $R_a = 1$, $R_p = 1$, $b_G = 1$, $a_\Theta = 1$, $b_D = 1$, $b_\Theta = 1$, $b_Y = 1$, $r = .05$, $\gamma = 0.01$ and $T = 0.5$.

	β_1			β_2			β_3		
	k			k			k		
τ	0.10	0.50	0.80	0.10	0.50	0.80	0.10	0.50	0.80
0.01	-0.00	0.00	0.00	-0.01	0.00	0.02	-0.01	0.00	0.02
0.02	-0.00	0.01	0.00	0.00	0.02	0.02	0.00	0.02	0.03
0.03	-0.00	-0.00	0.01	0.00	0.02	0.04	0.01	0.02	0.04
0.04	-0.01	-0.01	0.01	0.00	0.03	0.05	0.01	0.03	0.04
0.05	-0.01	-0.00	0.00	0.01	0.04	0.05	0.02	0.04	0.05
0.06	-0.02	-0.00	0.01	0.02	0.04	0.06	0.03	0.05	0.06
0.07	-0.03	-0.01	0.01	0.02	0.04	0.07	0.04	0.05	0.06
0.08	-0.03	-0.01	0.01	0.02	0.05	0.08	0.04	0.05	0.07
0.09	-0.04	-0.01	0.01	0.03	0.05	0.09	0.05	0.06	0.08
0.10	-0.04	-0.01	0.01	0.03	0.06	0.10	0.05	0.07	0.09
0.11	-0.04	-0.01	0.01	0.04	0.08	0.10	0.06	0.08	0.09
0.12	-0.04	-0.02	0.02	0.04	0.08	0.12	0.06	0.09	0.11
0.13	-0.05	-0.01	0.02	0.04	0.08	0.13	0.06	0.09	0.12
0.14	-0.05	-0.02	0.02	0.04	0.08	0.14	0.07	0.09	0.12
0.15	-0.06	-0.02	0.02	0.04	0.08	0.14	0.07	0.08	0.12
0.16	-0.06	-0.03	0.02	0.04	0.08	0.15	0.06	0.09	0.13
0.17	-0.07	-0.03	0.03	0.04	0.08	0.16	0.07	0.09	0.14
0.18	-0.07	-0.02	0.02	0.04	0.09	0.16	0.07	0.09	0.14
0.19	-0.07	-0.03	0.03	0.04	0.09	0.17	0.07	0.09	0.14
0.20	-0.08	-0.03	0.03	0.04	0.10	0.17	0.07	0.10	0.15

Table 3: A comparison of autocorrelation of excess returns on the risky stock and on fund i for different time-horizons τ under differential information and portfolio delegation. $\beta_1 = Corr\left(\int_{t-\tau}^{t} dQ(u), \int_{t}^{t+\tau} dQ(u)\right)$, $\beta_2 = Corr\left(\int_{t-\tau}^{t} A_u dQ(u), \int_{t}^{t+\tau} A_u dQ(u)\right)$ and $\beta_3 = Corr\left(\int_{t-\tau}^{t} \left(A_u dQ(u) - dS(u)\right), \int_{t}^{t+\tau} \left(A_u dQ(u) - dS(u)\right)\right)$. A random value of G is selected and 100 simulated paths are generated. We thus get a vector of 100 excess returns for different horizons (τ). The autocorrelations for this sample are computed. This procedure is repeated 600 times. The autocorrelations reported here are the average of autocorrelations. The parameter values are $R_a = 1$, $R_p = 1$, $b_G = 1$, $a_\Theta = 1$, $b_D = 1$, $b_\Theta = 1$, $b_Y = 1$, $r = .05$, $\gamma = 0.01$ and $T = 0.5$.

	β_1		β_2		β_3	
	k		k		k	
τ	0.05	0.10	0.05	0.10	0.05	0.10
0.01	0.00	0.01	0.00	0.02	0.00	0.01
0.02	0.00	0.01	0.01	0.02	0.00	0.02
0.03	0.01	0.02	0.02	0.03	0.01	0.03
0.04	0.01	0.02	0.02	0.04	0.01	0.05
0.05	0.01	0.02	0.03	0.05	0.01	0.06
0.06	0.01	0.03	0.03	0.06	0.02	0.07
0.07	0.02	0.03	0.03	0.07	0.02	0.08
0.08	0.02	0.04	0.03	0.07	0.02	0.09
0.09	0.02	0.04	0.03	0.08	0.03	0.09
0.10	0.02	0.05	0.04	0.10	0.03	0.10
0.11	0.02	0.06	0.04	0.11	0.04	0.12
0.12	0.02	0.06	0.04	0.12	0.04	0.13
0.13	0.02	0.07	0.04	0.13	0.05	0.13
0.14	0.02	0.07	0.05	0.14	0.05	0.14
0.15	0.03	0.08	0.05	0.14	0.05	0.15
0.16	0.03	0.08	0.05	0.15	0.06	0.16
0.17	0.03	0.09	0.06	0.17	0.07	0.17
0.18	0.03	0.10	0.06	0.18	0.07	0.18
0.19	0.03	0.10	0.06	0.18	0.07	0.19
0.20	0.04	0.11	0.07	0.19	0.08	0.20

Table 4: A comparison of autocorrelation of excess returns on the risky stock and on fund i for different time-horizons τ in the presence of an optimal linear contract. $\beta_1 = Corr\left(\int_{t-\tau}^{t} dQ(u), \int_{t}^{t+\tau} dQ(u)\right)$, $\beta_2 = Corr\left(\int_{t-\tau}^{t} A_u dQ(u), \int_{t}^{t+\tau} A_u dQ(u)\right)$ and $\beta_3 = Corr\left(\int_{t-\tau}^{t} (A_u dQ(u) - dS(u)), \int_{t}^{t+\tau} (A_u dQ(u) - dS(u))\right)$. A random value of G is selected and 100 simulated paths are generated. We thus get a vector of 100 excess returns for different horizons (τ). The autocorrelations for this sample are computed. This procedure is repeated 600 times. The autocorrelations reported here are the average of autocorrelations. The parameter values are $R_a = 1$, $b_G = 1$, $a_\Theta = 1$, $b_D = 1$, $b_\Theta = 1$, $b_Y = 1$, $r = .05$, $\gamma = 0.01$ and $T = 0.5$.

	β_1		β_2		β_3	
	k		k		k	
τ	0.05	0.10	0.05	0.10	0.05	0.10
0.01	0.00	0.01	0.01	0.01	0.01	0.01
0.02	0.00	0.01	0.01	0.03	0.01	0.03
0.03	0.01	0.03	0.01	0.04	0.01	0.04
0.04	0.01	0.03	0.02	0.05	0.02	0.05
0.05	0.01	0.03	0.02	0.07	0.02	0.07
0.06	0.01	0.04	0.02	0.07	0.02	0.07
0.07	0.02	0.05	0.03	0.09	0.03	0.09
0.08	0.02	0.06	0.04	0.10	0.03	0.10
0.09	0.02	0.06	0.04	0.11	0.04	0.11
0.10	0.02	0.06	0.04	0.12	0.03	0.12
0.11	0.02	0.07	0.04	0.13	0.04	0.12
0.12	0.02	0.07	0.04	0.13	0.04	0.13
0.13	0.02	0.08	0.04	0.14	0.04	0.14
0.14	0.02	0.09	0.05	0.15	0.05	0.15
0.15	0.03	0.09	0.05	0.17	0.05	0.16
0.16	0.03	0.10	0.06	0.17	0.05	0.17
0.17	0.03	0.10	0.06	0.18	0.06	0.18
0.18	0.03	0.11	0.06	0.19	0.06	0.19
0.19	0.03	0.11	0.07	0.20	0.06	0.20
0.20	0.03	0.12	0.07	0.21	0.07	0.20

Table 5: A comparison of autocorrelation of excess returns on the risky stock and on fund i for different time-horizons τ in the presence of an optimal linear contract. $\beta_1 = Corr\left(\int_{t-\tau}^{t} dQ(u), \int_{t}^{t+\tau} dQ(u)\right)$, $\beta_2 = Corr\left(\int_{t-\tau}^{t} A_u dQ(u), \int_{t}^{t+\tau} A_u dQ(u)\right)$ and $\beta_3 = Corr\left(\int_{t-\tau}^{t} \left(A_u dQ(u) - dS(u)\right), \int_{t}^{t+\tau} \left(A_u dQ(u) - dS(u)\right)\right)$. A random value of G is selected and 100 simulated paths are generated. We thus get a vector of 100 excess returns for different horizons (τ). The autocorrelations for this sample are computed. This procedure is repeated 600 times. The autocorrelations reported here are the average of autocorrelations. The parameter values are $R_a = 1$, $b_G = 1$, $a_\Theta = 1$, $b_D = 1$, $b_\Theta = 1$, $b_Y = 1$, $r = .05$, $\gamma = 0.02$ and $T = 0.5$.

A Proof of Proposition 1

The filtering problem that we treat in this paper is a special case of the following lemma whose proof can be found in textbooks such as Liptser and Sheryayev (1978). A more general version of the lemma is presented in Wang (1993).

Lemma 1 *Suppose that:*

$$dZ_t = [a_{z0} + a_{zz}Z_t]\, dt + b_z dB_t, \tag{A1}$$

$$dS_t = [a_{s0} + a_{sz}Z_t]\, dt + b_s dB_t, \tag{A2}$$

where Z_t is an n-vector of unobserved state variables, S_t is an m-vector of signals and B_t is a k-vector standard Brownian motion process and a_{z0}, a_{zz}, a_{s0}, a_{sz}, b_z and b_z are respectively $(n \times 1)$, $(m \times 1)$, $(n \times n)$, $(m \times m)$, $(n \times k)$ and $(m \times k)$ matrices of constants. Let $q_{zz} = b_z b_z^T$, $q_{ss} = b_s b_s^T$ and $q_{zs} = b_z b_s^T$. Suppose the prior is $z(t_0) \sim N(Z_0, o_0)$. Define the information set $\mathcal{F}_t^s = \{S_u, 0 \leq u \leq t\}$, the continuous observation of S_u between 0 and t. Conditioned on $\mathcal{F}_t^s$, the posterior mean of Z_t, i.e. the filter $\hat{Z}_t = E\left[Z|\mathcal{F}_t^s\right]$, is given by the stochastic differential equation:

$$d\hat{Z}_t = \left[a_{z0} + a_{zz}\hat{Z}_t\right] dt + \left[o_t a_{sz}^T + q_{zs}\right] q_{ss}^{-1/2} d\tilde{B}_t, \tag{A3}$$

where $o_t = E\left[(Z_t - \hat{Z}_t)(Z_t - \hat{Z}_t)^T|\mathcal{F}_t^s\right]$ is the positive semi-definite conditional variance-covariance matrix of Z_t given by the solution to the Riccati equation:

$$\dot{o}_t = a_{zz}o_t + o_t a_{zz}^T + q_{zz} - \left[o_t a_{sz}^T + q_{zs}\right] q_{ss}^{-1} \left[a_{sz}o_t + q_{zs}^T\right]. \tag{A4}$$

The innovation process, $\tilde{B}_t$, defined by

$$d\tilde{B}_t = q_{ss}^{-1/2}(dS_t - \left[a_{z0} + a_{zz}\hat{Z}_t\right] dt), \tag{A5}$$

is a Wiener process with respect to $\mathcal{F}_t^s$.

We define the vector of unobserved variables as Z_t:

$$Z_t \equiv \begin{pmatrix} G \\ \Theta_t \end{pmatrix}.$$

Denote the filtered values of Z_t under different information sets as

$$Z_t^c \equiv \begin{pmatrix} G_t^c \\ \Theta_t^c \end{pmatrix}; \qquad Z_t^{Pi} \equiv \begin{pmatrix} G_t^{Pi} \\ \Theta_t^{Pi} \end{pmatrix}; \qquad Z_t^i = \begin{pmatrix} G_t^i \\ \Theta_t^i \end{pmatrix}.$$

Note that Z_t follows a stochastic process given by

$$dZ_t = a_{zz}Z_t dt + b_z dB_t, \tag{A6}$$

where matrices a_z and b_z are given by

$$a_{zz} = \begin{pmatrix} 0 & 0 \\ 0 & -a_\Theta \end{pmatrix}; \qquad b_z = \begin{pmatrix} 0 & 0 \\ 0 & b_\Theta \end{pmatrix}.$$

The vector of commonly observed signals is denoted by S_t^c:

$$S_t^c \equiv \begin{pmatrix} D_t \\ \xi_t \end{pmatrix}.$$

S_t^c follows a stochastic process given by

$$dS_t^c = a_{sz}^c Z_t dt + b_s^c d\tilde{B}_t^c,$$

where

$$a_{sz}^c = \begin{pmatrix} 1 & 0 \\ \dot{\lambda}_{1t} & \dot{\lambda}_{2t} - a_\Theta \lambda_{2t} \end{pmatrix} \quad \text{and} \quad b_s^c = \begin{pmatrix} b_D & 0 \\ 0 & \lambda_{2t} b_\Theta \end{pmatrix}.$$

Applying Lemma 1, we obtain

$$dZ_t^c = a_{zz} Z_t^c dt + b_{z^c} d\tilde{B}_t^c, \tag{A7}$$

where

$$b_{z^c} = \left(o_t^c a_{sz}^T + q_{zs} \right) \Sigma_s^{c-\frac{1}{2}}, \tag{A8}$$

$$dB_t^c = \begin{bmatrix} \frac{1}{b_D}\left(dD - G_t^c dt\right) \\ \frac{1}{\lambda_{2t} b_\Theta}\left(d\xi_t - \left(\dot{\lambda}_{1t} G_t^c + \left(\dot{\lambda}_{2t} - a_\Theta \lambda_{2t} \right) \Theta^c \right) \right) dt \end{bmatrix}. \tag{A9}$$

Since $\lambda_{1t}G + \lambda_{2t}\Theta_t = \xi_t$, which is observable, it is easy to see that $\lambda_{1t}G_t^c + \lambda_{2t}\Theta_t^c = \xi_t$. It follows that

$$E\left[(\Theta_t - \Theta_t^c)^2 | \mathcal{F}_t^c \right] = E\left[\left(\frac{\lambda_{1t}}{\lambda_{2t}} \right)^2 (G - G_t^c)^2 | \mathcal{F}_t^c \right] = \left(\frac{\lambda_{1t}}{\lambda_{2t}} \right)^2 f_t^c, \tag{A10}$$

$$E\left[(\Theta_t - \Theta_t^c)(G - G_t^c) | \mathcal{F}_t^c \right] = E\left[-\frac{\lambda_{1t}}{\lambda_{2t}} (G - G_t^c)^2 | \mathcal{F}_t^c \right] = -\frac{\lambda_{1t}}{\lambda_{2t}} f_t^c. \tag{A11}$$

Therefore,

$$o_t^c = Var(Z_t | \mathcal{F}_t^c) = \begin{bmatrix} f_t^c & -\frac{\lambda_{1t}}{\lambda_{2t}} f_t^c. \\ -\frac{\lambda_{1t}}{\lambda_{2t}} f_t^c & \frac{\lambda_{1t}^2}{\lambda_{2t}^2} f_t^c \end{bmatrix}. \tag{A12}$$

Applying (A4) to (A12) yields

$$\dot{f}_t^c = -f_t^{c2}\left[\frac{1}{b_D^2} + \frac{\left(\dot{\lambda_{1t}}\lambda_{2t} - \dot{\lambda_{2t}}\lambda_{1t} + a_\Theta\lambda_{1t}\lambda_{2t}\right)^2}{\lambda_{2t}^4 b_\Theta^2}\right]. \tag{A13}$$

Let S_t^i be the vector of all observed signals by manager i:

$$S_t^i \equiv \begin{pmatrix} D_t \\ \xi_t \\ Y_t^i \end{pmatrix}.$$

S_t^i follows a stochastic process given by

$$dS_t^i = a_{sz}^i Z_t dt + b_s^i d\tilde{B}_t^i, \tag{A14}$$

where

$$a_{sz}^i = \begin{pmatrix} 1 & 0 \\ \dot{\lambda_{1t}} & \dot{\lambda_{2t}} - a_\Theta\lambda_{2t} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b_s^i = \begin{pmatrix} b_D & 0 & 0 \\ 0 & \lambda_{2t}b_\Theta & 0 \\ 0 & 0 & \sigma_Y \end{pmatrix}. \tag{A15}$$

Similar steps yield

$$dZ_t^i = a_{zz}Z_t^i dt + b_{z^i}d\tilde{B}_t^i, \tag{A16}$$

where

$$b_{z^i} = \left(o_t^i {a_{sz}^i}^T + q_{zs}^i\right)\Sigma_s^{i\,-\frac{1}{2}}, \tag{A17}$$

$$d\tilde{B}_t^i = \begin{bmatrix} \frac{1}{b_D}\left(dD - G_t^i dt\right) \\ \frac{1}{\lambda_{2t}b_\Theta}\left(d\xi_t - \left(\dot{\lambda}_{1t}G_t^i + \left(\dot{\lambda}_{2t} - a_\Theta\lambda_{2t}\right)\Theta^i\right)\right) \\ \frac{1}{\sigma_Y}\left(dY^i - G_t^i dt\right) \end{bmatrix}. \tag{A18}$$

and

$$o_t^i = \begin{bmatrix} f_t^i & -\frac{\lambda_{1t}}{\lambda_{2t}}f_t^i \\ -\frac{\lambda_{1t}}{\lambda_{2t}}f_t^i & \frac{\lambda_{1t}^2}{\lambda_{2t}^2}f_t^i \end{bmatrix}, \tag{A19}$$

$$\dot{f}_t^i = -{f_t^i}^2\left[\frac{1}{b_D^2} + \frac{\left(\dot{\lambda_{1t}}\lambda_{2t} - \dot{\lambda_{2t}}\lambda_{1t} + a_\Theta\lambda_{1t}\lambda_{2t}\right)^2}{\lambda_{2t}^4 b_\Theta^2} + \frac{1}{\sigma_Y^2}\right]. \tag{A20}$$

Equations (A13) and (A20) imply

$$\frac{\dot{f}_t^c}{f_t^{c2}} = \frac{\dot{f}_t^i}{{f_t^i}^2} + \frac{1}{\sigma_Y^2}.$$

Solving this with the initial condition $f_0^i = f_0^c = \sigma_G^2$ leads to

$$\frac{1}{f_t^i} = \frac{1}{f_t^c} + \frac{t}{\sigma_Y^2}. \tag{A21}$$

We are now in a position to prove Proposition 1.

A.1 Proof of Proposition 1

Applying Lemma 1, we have

$$dG_t^c = \frac{f_t^c}{b_D^2}\left[dD_t - G_t^c dt\right] - \frac{f_t^c}{\lambda_{2t}^3 b_\Theta^2}\left(\dot{\lambda_{2t}}\lambda_{1t} - \dot{\lambda_{1t}}\lambda_{2t} - a_\Theta\lambda_{1t}\lambda_{2t}\right)\left[d\xi_t - \dot{\lambda}_{1t}\hat{G}_t^c - (\dot{\lambda_{2t}} - a_\Theta\lambda_{2t})\Theta^c dt\right], \tag{A22}$$

$$\begin{aligned} dG_t^i &= \frac{f_t^i}{b_D^2}\left[dD_t - \hat{G}_t^i dt\right] - \frac{f_t^i}{\lambda_{2t}^3 b_\Theta^2}\left(\dot{\lambda_{1t}}\lambda_{2t} - \dot{\lambda_{2t}}\lambda_{1t} + a_\Theta\lambda_{1t}\lambda_{2t}\right)\left[d\xi_t - \dot{\lambda}_{1t}G_t^i - (\dot{\lambda_{2t}} - a_\Theta\lambda_{2t})\Theta^i dt\right] \\ &\quad + \frac{f_t^i}{\sigma_Y^2}\left[dY^i - G_t^i dt\right]. \end{aligned} \tag{A23}$$

Using the fact that $\lambda_{1t}G_t^c + \lambda_{2t}\Theta_t^c = \lambda_{1t}G_t^i + \lambda_{2t}\Theta_t^i = \xi_t$, we get

$$\Theta^c = \frac{\lambda_{1t}(\xi_t - \hat{G}_t^c)}{\lambda_{2t}}, \tag{A24}$$

$$\Theta^i = \frac{\lambda_{1t}(\xi_t - \hat{G}_t^i)}{\lambda_{2t}}. \tag{A25}$$

Substituting the above Θ^c and Θ^i into Eqs. (A22) and (A23), and using Eqs. (A13) and (A20), respectively, we obtain

$$dG_t^c - G_t^c\frac{\dot{f}_t^c}{f_t^c}dt = f_t^c\Omega_t, \tag{A26}$$

$$dG_t^i - G_t^i\frac{\dot{f}_t^i}{f_t^i}dt = f_t^i\left(\Omega_t + \frac{dY^i}{\sigma_Y^2}\right), \tag{A27}$$

where

$$\Omega_t = \frac{dD_t}{b_D^2} - \frac{\lambda_{1t}(\dot{\lambda_{2t}}\lambda_{1t} - \dot{\lambda_{1t}}\lambda_{2t} - a_\Theta\lambda_{1t}\lambda_{2t})}{\lambda_{2t}^3 b_\Theta^2}\left[d\left(\frac{\xi_t}{\lambda_{1t}}\right) - \frac{(\dot{\lambda_{2t}}\lambda_{1t} - \dot{\lambda_{1t}}\lambda_{2t} - a_\Theta\lambda_{1t}\lambda_{2t})}{\lambda_{1t}^2\lambda_{2t}}\xi_t dt\right].$$

Solving Eqs. (A26) and (A27) yields

$$d\left[\frac{G_t^i}{f_t^i} - \frac{G_t^c}{f_t^c}\right] = \frac{dY}{\sigma_Y^2}.$$

Using the initial conditions, $G_0^c = G_0^i$ and $f_0^c = f_0^i = \sigma_G^2$, we get

$$\frac{G_t^i}{f_t^i} = \frac{G_t^c}{f_t^c} + \frac{1}{\sigma_Y^2}\int_0^t dY_s^i.$$

Using Eq. (A21), we arrive at

$$G_t^i = \pi_t G_t^c + \frac{1-\pi_t}{t}\int_0^t dY_s^i, \tag{A28}$$

where $\pi_t = \frac{\sigma_Y^2}{\sigma_Y^2 + t f_t^c}$.

We next show that the higher order expectations are linear combinations of the first order expectations. Using the Law of Large Numbers, we have

$$\int_i dY^i = G dt. \tag{A29}$$

Let $\hat{G}_t \equiv \int_i G_t^i$ be the market expectation of G, and $\hat{\hat{G}}_t^i \equiv E_t^i[\hat{G}_t]$ be manager i's expectation of G. Then, $\hat{\hat{G}}_t \equiv \int_i \hat{\hat{G}}_t^i$ is the market average of the managers' expectation of $\hat{G}_t$. Using Eqs. (A28) and (A29), we get

$$\begin{aligned} \hat{G}_t &= \pi_t G_t^c + (1-\pi_t)G, \\ \hat{\hat{G}}_t^i &= \pi_t G_t^c + (1-\pi_t)\hat{G}_t^i, \\ \hat{\hat{G}}_t &= \pi_t(2-\pi_t)G_t^c + (1-\pi_t)^2 G. \end{aligned} \tag{A30}$$

It means that all higher order expectations of G can be expressed as a linear combination of the market expectation and the private expectation. In fact, it can be shown that the n^{th} order private expectation of G is given by $[1-(1-\pi_t)^n]\, G_t^c + (1-\pi_t)^n \hat{G}_t^i$ while the n^{th} order market expectation is given by $[1-(1-\pi_t)^n]\, G_t^c + (1-\pi_t)^n G$. As expected, as the order increases, the market expectation converges to G_t^c. **Q.E.D.**

A.2 Some Expressions Used in the Main Text

$$dB_\Psi = \begin{pmatrix} dB_D \\ dB_\Theta \\ \frac{1}{b_D}\left[dD_t - \hat{G}_t^c dt\right] \\ \frac{1}{\lambda_{2t} b_\Theta}\left[d\xi_t - \lambda_{1t}\hat{G}_t^c - (\dot{\lambda}_{2t} - a_\Theta \lambda_{2t})\hat{\Theta}^c dt\right] \end{pmatrix}; \qquad d\hat{B}_{\hat{\Psi}^i} = \begin{pmatrix} \frac{1}{b_D}\left[dD_t - \hat{G}_t^i dt\right] \\ \frac{1}{\lambda_{2t} b_\Theta}\left[d\xi_t - \lambda_{1t}\hat{G}_t^i - (\dot{\lambda}_{2t} - a_\Theta \lambda_{2t})\hat{\Theta}^i dt\right] \\ \frac{1}{\sigma_Y}\left[dY^i - \hat{G}_t^i dt\right] \\ \frac{1}{b_D}\left[dD - \hat{G}_t^c dt\right] \\ \frac{1}{\lambda_{2t} b_\Theta}\left[d\xi_t - \lambda_{1t}\hat{G}_t^c - (\dot{\lambda}_{2t} - a_\Theta \lambda_{2t})\hat{\Theta}^c dt\right] \end{pmatrix}.$$

$$a_\Psi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_\Theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}; \qquad b_\Psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_\Theta & 0 & 0 \\ 0 & 0 & \frac{f_t^c}{b_D} & \frac{f_t^c}{\lambda_{2t}^2 b_\Theta} \\ b_D & 0 & 0 & 0 \end{pmatrix}.$$

$$b^i_\Psi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{f^i_t}{b_D} & \frac{f^i_t}{\lambda^2_{2t} b_\Theta}\left(\lambda_{2t}\dot{\lambda}_{1t} - \lambda_{1t}\dot{\lambda}_{2t} + a_\Theta \lambda_{1t}\lambda_{2t}\right) & \frac{f^i_t}{\sigma_Y} & 0 & 0 \\ -\frac{\lambda_{1t}}{\lambda_{2t}}\frac{f^i_t}{b_D} & -\frac{\lambda_{1t}}{\lambda_{2t}}\frac{f^i_t}{\lambda^2_{2t} b_\Theta}\left(\lambda_{2t}\dot{\lambda}_{1t} - \lambda_{1t}\dot{\lambda}_{2t} + a_\Theta \lambda_{1t}\lambda_{2t}\right) + b_\Theta & -\frac{\lambda_{1t}}{\lambda_{2t}}\frac{f^i_t}{\sigma_Y} & 0 & 0 \\ 0 & 0 & 0 & \frac{f^c_t}{b_D} & \frac{f^c_t}{\lambda^2_{2t} b_\Theta} \\ b_D & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$a^i_Q = \left[\dot{\lambda}_t + (a_{\hat{\Psi}^i} - r)\lambda^T_t + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}\right]; \qquad b^i_Q = \lambda_t b_{\Psi^i} + \begin{pmatrix} b_D & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\Omega_{Q\Psi} = \lambda_t + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here $\lambda_t \equiv (\lambda_{0t}\ \lambda_{1t}\ \lambda_{2t}\ \lambda_{3t}\ \lambda_{4t})$.

B Proof of Proposition 2

This proposition is a special case of Corollary B of Ou-Yang (2000). For completeness, we present the proof here. The manager's Bellman equation is given by[1]

$$\begin{aligned} &\sup_{A^i}\Big\{-V^{Pi}\Big(R_a\left[\alpha^i + \beta^i\left(rW^i + a_Q A^i \Psi^i\right) + \Phi^i a_\Psi \Psi^i - c\right] - \frac{1}{2}R^2_a\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]^T\Big) \\ &+V^i_t + V^i_W\left[\left(rW^i + a_Q A^i\Psi^i\right) - R_a\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]{b^i_Q}^T A^i\right] + \frac{1}{2}V^i_{WW}{A^i}^2 b^i_Q {b^i_Q}^T \\ &+(V^i_{\Psi^i})^T\left[a_\Psi\Psi^i - R_a b_{\Psi^i}\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]^T\right] + \frac{1}{2}tr\left(V^i_{\Psi^i\Psi^i} b_{\Psi^i} b^T_{\Psi^i}\right) + b^i_Q b_{\Psi^i} V^i_{W\Psi^i} A^i\Big\} = 0. \end{aligned}$$

By Ito's lemma, $dV^i(t, W, \Psi^i)$ is given by

$$\begin{aligned} dV^i &= [V^i_t + V^i_W\left(rW + a_Q A^i\Psi^i\right) + \frac{1}{2}V^i_{WW}{A^i}^2 b^i_Q {b^i_Q}^T + V^i_{\Psi^i} a_\Psi \Psi^i + \frac{1}{2}tr\left(V^i_{\Psi^i\Psi^i} b_{\Psi^i} b^T_{\Psi^i}\right) \\ &\quad + b^i_Q b_{\Psi^i} V^i_{W\Psi^i} A^i]dt + \left[V^i_W A^i b^i_Q + (V^i_{\Psi^i})^T b_{\Psi^i}\right] dB_{\Psi^i}. \end{aligned}$$

Combining the above two equations yields

$$\begin{aligned} dV^i &= V^i\left(R_a\left[\alpha^i + \beta^i\left(rW + a_Q A^i\Psi^i\right) + \Phi^i a_\Psi \Psi^i - c\right] - \frac{1}{2}R^2_a\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]\left[\beta A^i b^i_Q + \Phi^i b_{\Psi^i}\right]^T\right) \\ &+ V^i_W\left[R_a\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]{b^i_Q}^T A^i\right] + (V^i_{\Psi^i})^T\left[R_a b_{\Psi^i}\left[\beta^i A^i b^i_Q + \Phi^i b_{\Psi^i}\right]^T\right] \\ &+ \left[V^i_W A^i b^i_Q + (V^i_{\Psi^i})^T b_{\Psi^i}\right] dB_{\Psi^i}. \end{aligned}$$

Define an $\mathcal{E}_t$ process as $R_a\mathcal{E}_t = -\log[-R_a V^i(t, W, \Psi)]$. Using the expression for dV^i and with some manipulations, we obtain

$$R_a d\mathcal{E}_t = -\frac{dV^i}{V^i} + \frac{1}{2}\left(\frac{dV^i}{V^i}\right)^2$$

[1] We assume that the regularity condition of Ou-Yang is satisfied or that the Bellman equation is both a necessary and a sufficient condition.

$$
\begin{aligned}
= & \;-R_a\left[\alpha^i+\beta^i\left(rW^i+a_QA^i\Psi^i\right)+\Phi^i a_\Psi\Psi^i-c-\frac{1}{2}R_a^2\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]^T\right]dt \\
& +R_a\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]dB_{\Psi^i}-R_a\left[\beta^iA^ib_Q^i+\Phi^ib_{\Psi^i}\right]dB_{\Psi^i}.
\end{aligned}
$$

where $\bar{\beta}^i\equiv\beta^i-\frac{V_W^i}{R_aV^i}$ and $\bar{\Phi}^i\equiv\Phi^i-\frac{V_{\Psi^i}^i}{R_aV^i}$. Therefore, we have

$$
\begin{aligned}
& \alpha^idt+\beta^idW_t^i+\Phi^id\Psi_t^i \\
= & \; d\mathcal{E}_t+cdt+\frac{R_a}{2}\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]^Tdt+\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]dB_{\Psi^i}.
\end{aligned}
$$

Integrating the above expression between 0 and T yields

$$
S_T^i=\mathcal{E}_0+\int_0^T cdt+\frac{R_a}{2}\int_0^T\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]^Tdt+\int_0^T\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]dB_{\Psi^i}.
$$

Imposing the manager's participation constraint yields that $\mathcal{E}_0$ equals the manager's reservation wage at time 0. **Q.E.D.**

C Proof of Theorem 1

Define the investor's value function as

$$
\begin{aligned}
J_t^i \;=\; & E_t^i\left[-\frac{1}{R_p}\exp\left[-R_p\left[W_T-\left[\int_t^T c_udu+\frac{R_a}{2}\int_t^T\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]^Tdu\right.\right.\right.\right.\right. \\
& \left.\left.\left.\left.\left.+\int_t^T\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]dB_{\Psi^i}\right]\right]\right]\right].
\end{aligned}
$$

The principal's Bellman equation is then given by

$$
\begin{aligned}
& \sup_{A^i,S_T^i}\left\{J\left[R_pc+\frac{R_p}{2}(R_a+R_p)\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]^T\right]+J_t\right. \\
+ & \; J_W\left[\left(rW^i+a_QA^i\Psi^i\right)+R_p\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]{b_Q^i}^TA^i\right]+\frac{1}{2}J_{WW}{A^i}^2b_Q^i{b_Q^i}^T \\
+ & \; J_{\Psi^i}^T\left[a_\Psi\Psi^i+R_pb_{\Psi^i}\left[\bar{\beta}^iA^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]^T\right]+\frac{1}{2}tr\left(J_{\Psi^i\Psi^i}b_{\Psi^i}b_{\Psi^i}^T\right)+\left.b_Q^ib_{\Psi^i}J_{W\Psi^i}A^i\right\}=0.
\end{aligned}
$$

The first order conditions are-

w.r.t. $\bar{\beta}^i$:

$$
J(R_a+R_p)\left[\bar{\beta}A^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]{b_Q^i}^TA^i+J_W{A^i}^2b_Q^i{b_Q^i}^T+J_{\Psi^i}^TA^ib_{\Psi^i}{b_Q^i}^T=0. \tag{A31}
$$

w.r.t. $\bar{\Phi}^i$:

$$
J(R_a+R_p)\left[\bar{\beta}A^ib_Q^i+\bar{\Phi}^ib_{\Psi^i}\right]b_{\Psi^i}+J_WA^ib_Q^ib_{\Psi^i}+J_{\Psi^i}^Tb_{\Psi^i}b_{\Psi^i}=0. \tag{A32}
$$

w.r.t. A^i:

$$J\left[R_p c_{A^i} + (R_a+R_p)R_p\left[\bar{\beta}^i A^i b_Q^i + \bar{\Phi}^i b_{\Psi^i}\right]\bar{\beta}^i {b_Q^i}^T\right] + J_W\left[a_Q\Psi^i + 2R_p\bar{\beta}^i A^i b_Q^i {b_Q^i}^T + R_P\bar{\Phi}^i b_{\Psi^i}{b_Q^i}^T\right]$$
$$+\quad J_{WW}A^i b_Q^i {b_Q^i}^T + R_p\bar{\beta}^i b_Q^i b_{\Psi^i}J_{\Psi^i} + b_Q^i b_{\Psi^i}J_{W\Psi^i} = 0. \tag{A33}$$

Conditions (A31) and (A32) are satisfied if the following relationship holds:

$$\bar{\beta}A^i b_Q^i + \bar{\Phi}^i b_{\Psi^i} = -\frac{J_W A^i b_Q^i + J_{\Psi^i}^T b_{\Psi^i}}{J(R_a+R_p)} \tag{A34}$$

Conjecture that the investor's value function is given by

$$J = -\frac{1}{R_p}\exp\left[-R_p\left(a(t)W^i + {\Psi^i}^T V(t)\Psi^i\right)\right].$$

with the boundary conditions that $a(T)=1$, $V(T)=0$. This gives rise to

$$\bar{\beta}^i A^i b_Q^i + \bar{\Phi}^i b_{\Psi^i} = \frac{a(t)A^i b_Q^i + 2{\Psi^i}^T V^T(t) b_{\Psi^i}}{R_a+R_p}R_p. \tag{A35}$$

We now discuss the solutions for $\bar{\beta}^i$ and $\bar{\Phi}^i$. Similarly to Ou-Yang, to obtain an optimal contract that is path-independent and that implements the investor's optimal policy, we arrive at a set of solutions:

$$\bar{\beta}^i \;=\; a(t), \tag{A36}$$
$$\bar{\Phi}^i b_{\Psi^i} \;=\; \frac{2\Psi^i V^T(t) b_{\Psi^i} R_p - R_a a(t) A^i b_Q^i}{R_a+R_p}. \tag{A37}$$

Using Eq. (**??**), we have

$$\bar{\Phi}^i \;=\; \frac{2\Psi^i V^T(t) R_p - R_a a(t) A^i \Omega_{Q\Psi}}{R_a+R_p}. \tag{A38}$$

To eliminate the W_t^i terms from the investor's Bellman equation, we have

$$a(t) = \left(1-\frac{\gamma}{r}\right)\mathrm{e}^{r(T-t)} + \frac{\gamma}{r}.$$

Substituting the above equations into Eq. (A33), we arrive at the optimal policy A^{i*}:

$$A^{i*} = H\Psi^i,$$

where H is given by

$$H = \frac{a(t)\left[a_Q - 2\frac{R_aR_p}{R_a+R_p}b_Q^i b_{\Psi^i}^T V(t)\right]}{k + a^2(t)\frac{R_aR_p}{R_a+R_p}b_Q^i {b_Q^i}^T} \equiv \frac{a(t)\left[a_Q - 2\frac{R_aR_p}{R_a+R_p}b_Q^i b_{\Psi^i}^T V(t)\right]}{\Delta}. \tag{A39}$$

This leads to

$$\bar{\beta}^i A^i b_Q^i + \bar{\Phi}^i b_{\Psi^i} = \frac{R_p}{R_a + R_p} \frac{\Psi^{iT}\left[a^2(t) a_Q^T b_Q^i + 2kV^T(t) b_{\Psi^i}\right]}{\Delta}. \tag{A40}$$

The optimal fee structure can then be expressed as

$$\begin{aligned}
S_T^i &= \mathcal{E}_0 + \int_0^T \left[\frac{1}{2} k^i(t) A^{i*2} + \gamma W_t\right] dt \\
&+ \frac{R_a}{2} \int_0^T \left[\frac{a A^{*i} b_Q^i + 2\Psi^{iT} V^T(t) b_{\Psi^i}}{R_a + R_p} R_p\right] \left[\frac{(t) A^{i*} b_Q^i + 2\Psi^{iT} V^T(t) b_{\Psi^i}}{R_a + R_p} R_p\right]^T dt \\
&+ \int_0^T a(t) \left[dW^i - [rW^i + a_Q \Psi^i A^{i*}] dt\right] + \int_0^T \bar{\Phi}^i \left[d\Psi^i - a_{\Psi^i} \Psi^i dt\right],
\end{aligned}$$

which simplifies to

$$\begin{aligned}
S_T^i &= \mathcal{E}_0 - W_0 a(0) + W_T^i + \frac{1}{2} \int_0^T k^i(t) A^{*i2} dt + \frac{R_a}{2} \left(\frac{R_p}{R_a + R_p}\right)^2 \int_0^T [a(t) A^{*i} b_Q^i + 2\Psi^{iT} V^T(t) b_{\Psi^i}] \\
&\quad [a(t) A^{*i} b_Q^i + 2\Psi^{iT} V^T(t) b_{\Psi^i}]^T dt - \int_0^T a(t) a_Q \Psi^i A^{*i} dt + \int_0^T \bar{\Phi}^i \left[d\Psi^i - a_{\Psi^i} \Psi^i dt\right] \\
&= \mathcal{E}_0 - a(0) W_0^i + \frac{1}{2} \int_0^T k^i(t) A_t^{i*2} dt + \frac{R_a}{2} \int_0^T \left(\bar{\beta} A_t^{i*} b_Q^i + \Phi^i b_{\Psi^i}\right) \left(\bar{\beta} A_t^{i*} b_Q^i + \Phi^i b_{\Psi^i}\right)^T dt \\
&\quad - \int_0^T [a_Q \Psi^i A_t^{i*} + a_{\Psi^i} \Psi^i] dt + \frac{R_p}{R_a + R_p} [W_T^i + 2 \int_0^T \Psi^i V^T(t) d\Psi^i] \\
&\quad + \frac{R_a}{R_a + R_p} [W_T^i - \int_0^T a(t) A_t^{i*} \Omega_{Q\Psi} d\Psi^i] \\
&\quad \equiv \mathcal{E}_0 + \frac{1}{2} \int_0^T k^i(t) A_t^{i*2} dt + U(\Psi^i) + \frac{R_p}{R_a + R_p} W_T^i \\
&\quad + \frac{R_a}{R_a + R_p} [W_T^i - a(0) W_0^i \int_0^T a(t) A_t^{i*} \Omega_{Q\Psi} d\Psi^i]
\end{aligned} \tag{A41}$$

The investor's Bellman equation can now be written as

$$\begin{aligned}
&\dot{V} + \frac{a(t)}{2} \left[a_Q^T H + H^T a_Q\right] + \left[V^T a_\Psi + a_\Psi^T V\right] + \\
&\frac{R_p^2}{2(R_a + R_p)\Delta^2} \left[a^2(t) a_Q^T b_Q + 2kV^T b_{\Psi^i}\right] \left[a^2(t) a_Q^T b_Q + 2kV^T b_{\Psi^i}\right]^T \\
- \;& \frac{1}{2} H^T H \left[k + a^2(t) R_p b_Q b_Q^T\right] - a(t) R_p \left[V^T b_{\Psi^i} b_Q^T H + H^T b_Q b_{\Psi^i}^T V\right] \\
&-2R_p b_{\Psi^i} b_{\Psi^i}^T V V^T + m_{11}^N tr \left[V^T b_{\Psi^i} b_{\Psi^i}^T\right] = 0,
\end{aligned} \tag{A42}$$

where m_{11}^N denotes an $N \times N$ matrix with the element (1,1) = 1 and all other elements being zero, and with N being the dimensionality of the vector Ψ^i. All the terms are deterministic function of time t. $V(t)$ must satisfy the above equation with the boundary condition given by $V(T) = 0$.

We now verify that our optimal contract implements the investor's optimal portfolio policy A_t^*. To do so, we need to derive the manager's Bellman equation using this contract. Conjecture that given the investor's optimal contract, the manager's value function is given by

$$V^i(t, W, \Psi) = -\frac{1}{R_a} \exp\left[-R_a\left[f_1(t)W_t + f_2(t)\right]\right], \tag{A43}$$

with the boundary conditions that $f_1(T) = 1$ and $f_2(T) = 0$. The manager's Bellman equation can then be written as

$$\begin{aligned}
&\sup_{A^i}\Bigg\{-V^i\left[\frac{R_a^2}{2}\left(\frac{R_p}{R_a+R_p}\right)^2 [a(t)A^{i*}b_Q^i + 2{\Psi^i}^T V^T(t) b_{\Psi^i}][a(t)A^{i*}b_Q^i + 2{\Psi^i}^T V^T(t) b_{\Psi^i}]^T - a(t)a_Q\Psi^i A^{i*}\right] \\
&-V^i\left[\frac{1}{2}k^i(t){A^{*i}}^2 - \frac{1}{2}k^i(t){A^i}^2 - \gamma W^i - \frac{1}{2}R_a\bar{\Phi}^i b_{\Psi^i} b_{\Psi^i}^T \bar{\Phi}^{iT}\right] + V_t^i \\
&+V_W^i\left[rW^i + a_Q A^i\Psi^i - R_a\bar{\Phi}_t^i A^i b_{\Psi^i} {b_Q^i}^T\right] + \frac{1}{2}V_{WW}^i b_Q^i {b_Q^i}^T {A^i}^2\Bigg\} = 0.
\end{aligned}$$

Using Eq. (A35), the above Bellman equation becomes

$$\begin{aligned}
&\sup_{A^i}\Bigg\{-V^i\left[R_a\left(\bar{\beta}A^{*i}b_Q^i + \Phi^i b_{\Psi^i}\right)\left(\bar{\beta}A^{*i}b_Q^i + \Phi^i b_{\Psi^i}\right)^T\right] \\
&-V^i\left[\frac{k^i(t)}{2}{A^{*i}}^2 - \frac{k^i(t)}{2}{A^i}^2 - \gamma W^i - \frac{1}{2}R_a\bar{\Phi}^i b_{\Psi^i} b_{\Psi^i}^T \bar{\Phi}^{iT}\right] + V_t^i \\
&+V_W^i\left[rW^i + a_Q A^i\Psi^i - R_a\bar{\Phi}_t^i A^i b_{\Psi^i} {b_Q^i}^T\right] + \frac{1}{2}V_{WW}^i b_Q^i {b_Q^i}^T {A^i}^2\Bigg\} = 0.
\end{aligned}$$

Substituting Eq. (A43) into the above equation and using Eq. (A37), we can show that the manager's Bellman equation is independent of Ψ^i at $A_t^i = A_t^{i*}$. Given that $f_1(t) = a(t)$, the above equation becomes independent of W_t^i. The manager's Bellman equation is then satisfied with a proper choice of $f_2(t)$.

We next verify that the investor's optimal policy satisfies the manager's FOC. The FOC of the manager's Bellman equation yields

$$\left[a(t)^2 R_a b_Q^i {b_Q^i}^T\right] A_t^i = a(t)a_Q\Psi^i - R_a\bar{\Phi}^i\Psi^i {b_Q^i}^T.$$

Substituting Eq. (A37) into the above FOC leads to

$$A_t^i = \frac{a(t)\left[a_Q - 2\frac{R_aR_p}{R_a+R_p} b_Q^i b_{\Psi^i}^T V(t)\right]}{k + a^2(t)\frac{R_aR_p}{R_a+R_p} b_Q^i {b_Q^i}^T}\Psi^i,$$

which is indeed the investor's optimal policy.

To show the contract form (**??**) in Theorem 1, we note that

$$\int_0^T a(t)A_t^*\Omega_{Q\Psi}d\Psi_t = \int_0^T a(t)A_t^*\Omega_{Q\Psi}[a_\Psi^i\Psi_t^i dt + b_\Psi^i dB_{\Psi^i}]$$
$$= \int_0^T a(t)A_t^*[dQ^i - a_Q^i\Psi_t^i dt],$$

where

$$\int_0^T a(t)A_t^* dQ^i = \int_0^T a(t)[dW_t - rW_t dt]$$
$$= (1-\frac{\gamma}{r})W_T + \frac{\gamma}{r}\int_0^T [dW_t - rW_t dt]$$
$$= (1-\frac{\gamma}{r})W_T + \frac{\gamma}{r}\int_0^T A_t^* dQ^i.$$

Equation (**??**) can then be obtained in a straightforward manner. The proof for the implementability of this contract in the case of $k^i(t) = 0$ follows the same strategy as for the general case and is thus omitted. The point is that when $k^i(t) = 0$, there exist multiple solutions for the manager's and the investor's FOCs.[2] We next show that our optimal contract achieves optimal risk sharing and that there exists a simple linear contract form in the absence of a cost function for the manager.

The investor's Bellman Eq. (A42) can also be written as:

$$\begin{aligned}&\dot{V} + \frac{a(t)}{2}\left[a_Q^T H + H^T a_Q\right] + \left[V^T a_\Psi + a_\Psi^T V\right] + \\ &\frac{R_p^2}{2(R_a+R_p)\Delta^2}\left[a(t)H^T b_Q + 2V^T b_{\Psi^i}\right]\left[a(t)H^T b_Q + 2V^T b_{\Psi^i}\right]^T - \frac{1}{2}kH^T H - \\ &-\frac{R_p}{2\Delta^2}\left[a(t)H^T b_Q + 2V^T b_{\Psi^i}\right]\left[a(t)H^T b_Q + 2V^T b_{\Psi^i}\right]^T + m_{11}^N tr\left[V^T b_{\Psi^i} b_{\Psi^i}^T\right] = 0. \qquad \text{(A44)}\end{aligned}$$

This can be simplified as:

$$\begin{aligned}&\dot{V} + \frac{a(t)}{2}\left[a_Q^T H + H^T a_Q\right] + \left[V^T a_\Psi + a_\Psi^T V\right] + \\ &\frac{R_p R_a}{2(R_a+R_p)\Delta^2}\left[a(t)H^T b_Q + 2V^T b_{\Psi^i}\right]\left[a(t)H^T b_Q + 2V^T b_{\Psi^i}\right]^T - \frac{1}{2}kH^T H \\ &+m_{11}^N tr\left[V^T b_{\Psi^i} b_{\Psi^i}^T\right] = 0, \qquad \text{(A45)}\end{aligned}$$

This Bellman equation is equivalent to that of a single agent who has a risk-aversion of $\frac{R_a R_p}{R_a+R_p}$ and who maximizes the utility over his terminal wealth.[3] Therefore, our contract is an optimal risk-

[2]See also Ou-Yang for details.

[3]See Appendix E for the Bellman equation for a single agent who trades for his own account without the delegation portfolio management problem.

sharing allocation based upon the risk tolerance of the manager and of the investor as in Wilson (1968).

As a robustness check, we examine the contract form in a special case in which $R_a = 0$. In this case, our general contract form reduces to

$$\begin{aligned} S_T &= \mathcal{E}_0 + W_T^i - W_0^i a(0) + \int_0^T \frac{1}{2} k A_t^2 dt \\ &+ \int_0^T \left[a(t) A b_Q + 2\psi^T V^T b_\psi \right]^T dt - \int_0^T a(t) a_Q \psi A dt + \int_0^T \Phi b_\psi dB, \end{aligned}$$

and the investor's Bellman equation becomes:

$$\psi^T \dot{V} \psi + a(t) \psi^T a_Q^T A + 2\psi^T V^T a_\psi \psi - \frac{1}{2} k A^2 + tr \left[V^T b_\psi b_\psi^T \right] = 0.$$

From this simplified Bellman equation, we obtain

$$\psi^T \dot{V} \psi + 2\psi^T V^T d\psi + \frac{1}{2} tr \left[2V^T b_\psi b_\psi^T \right] = \frac{1}{2} k A^2 - a(t) a_Q \psi A dt + 2\psi^T V b_\psi dB = d\left(\psi^T V \psi \right).$$

Therefore, the optimal contract is given by

$$S_T^i = \mathcal{E}_0 + W_T^i - W_0 a(0) - \psi_0^T V_0 \psi_0 = W_T^i + \text{constant},$$

which means that the investor sells the entire fund to the manager for a constant payment.

We next show that in the absence of a cost function, the linear function of the terminal value of the portfolio W_T is an optimal contract. Recall that the optimal fee structure and the investor's Bellman equation are given by

$$\begin{aligned} S_T^i &= \mathcal{E}_0 + \int_0^T \left(\frac{1}{2} k A^2 + \gamma W \right) dt + \left(\frac{R_p}{R_a + R_p} \right)^2 \frac{R_a}{2} \int_0^T \left[a(t) H^T b_Q + 2V^T b_{\Psi^i} \right] \left[a(t) H^T b_Q + 2V^T b_{\Psi^i} \right]^T dt \\ &+ \frac{R_p}{R_a + R_p} \left[\int_0^T a(t) A b_Q dB + \int_0^T \psi V b_\psi dB \right], \end{aligned}$$

and

$$\begin{aligned} &\dot{V} + \frac{a(t)}{2} \left[a_Q^T H + H^T a_Q \right] + \left[V^T a_\Psi + a_\Psi^T V \right] + \\ &\frac{R_p R_a}{2(R_a + R_p)\Delta^2} \left[a(t) H^T b_Q + 2V^T b_{\Psi^i} \right] \left[a(t) H^T b_Q + 2V^T b_{\Psi^i} \right]^T \\ &- \frac{1}{2} k H^T H + m_{11}^N tr \left[V^T b_{\Psi^i} b_{\Psi^i}^T \right] = 0, \end{aligned} \tag{A46}$$

respectively.

Multiplying the above Bellman equation by $\frac{R_p}{R_a+R_p}$, we have

$$\frac{R_p}{R_a+R_p}\dot{V}+\frac{a(t)}{2}\frac{R_p}{R_a+R_p}\left[a_Q^T H+H^T a_Q\right]+\frac{R_p}{R_a+R_p}\left[V^T a_\Psi+a_\Psi^T V\right]+$$
$$\frac{R_p^2 R_a}{2(R_a+R_p)^2\Delta^2}\left[a(t)H^T b_Q+2V^T b_{\Psi^i}\right]\left[a(t)H^T b_Q+2V^T b_{\Psi^i}\right]^T$$
$$-\frac{1}{2}\frac{R_p}{R_a+R_p}kH^T H+m_{11}^N\frac{R_p}{R_a+R_p}tr\left[V^T b_{\Psi^i}b_{\Psi^i}^T\right]=0. \tag{A47}$$

Further manipulation leads to:

$$\int_0^T d\left[\psi\frac{R_p}{R_a+R_p}V\psi\right]-2\frac{R_p}{R_a+R_p}\int_0^T\psi V b_\psi dB-\frac{R_p}{2(R_a+R_p)}\int_0^T kA^2dt+\int_0^T a(t)\frac{R_p}{R_a+R_p}\psi^T a_Q^T A$$
$$-\int_0^T\frac{R_p^2R_a}{2(R_a+R_p)^2}\psi^T\left[a(t)H^T b_Q+2V^T b_{\Psi^i}\right]\left[a(t)H^T b_Q+2V^T b_{\Psi^i}\right]^T\psi dt=0.$$

This gives rise to a reduced contract form S_T^i:

$$\begin{aligned}S_T^i &= \mathcal{E}_0+\frac{R_a}{2(R_a+R_p)}\int_0^T kA^2dt+\int_0^T\gamma W dt+\int_0^T d\left[\psi\frac{R_p}{R_a+R_p}V\psi\right]\\ &+ \int_0^T a(t)\frac{R_p}{R_a+R_p}\psi^T a_Q^T A dt+\frac{R_p}{R_a+R_p}\int_0^T a(t)Ab_Q dB\\ &= \mathcal{E}_0+\frac{R_p}{2(R_a+R_p)}\int_0^T kA^2dt+\int_0^T\gamma W dt-\psi_0\frac{R_p}{R_a+R_p}V_0\psi_0+\frac{R_p}{R_a+R_p}\int_0^T a(t)\left[dW-rWdt\right].\end{aligned}$$

When k = 0 and $\gamma=0$, this S_T^i reduces to a linear function of W_T:

$$S_T^i = \text{constant}+\frac{R_p}{R_a+R_p}W_T.$$

D Numerical Procedures in Optimal Linear Contracting

D.1 The Manager's Problem

Given the linear contract form, $S_T=\alpha+\beta W_T$, the manager's problem is

$$\sup_{A_t}-\frac{1}{R_a}E_0\exp\left\{-R_a\left[\alpha+\beta W_T-\int_0^T cdt\right]\right\}.$$

As in the optimal risk sharing case, we define the manager's value function as $V^M=\sup_{A_t}-\frac{1}{R_a}E_t\exp\left\{-R_a\left[\alpha+\beta W_T-\int_t^T cdu\right]\right\}$. The manager's Bellman equation is then given by

$$\sup_{A_t}\left\{-V^M(-\frac{1}{2}kA^2-\gamma W_t)+V_t^M+V_W^M(rW_t+a_QA\Psi)+\frac{1}{2}V_{WW}^M b_Q b_Q^T A^2\right.$$
$$\left.+V_\Psi^M a_\Psi\Psi+\frac{1}{2}tr(V_{\Psi\Psi}^M b_\Psi b_\Psi^T)+b_Q b_{Psi}^T V_{W\Psi}^M A\right\}=0.$$

Conjecture that the manager's value function is given by $V^M = -\frac{1}{R_a}\exp\{-R_a[\alpha + a(t)W_t + \Psi^T\Omega_t\Psi]\}$, with the boundary conditions being $a(T) = \beta$ and $\Omega = 0_{N\times N}$. α is determined by the manager's participation constraint at time 0:

$$V_0^M = E_0\{-\frac{1}{R_a}\exp[-R_a\alpha + a(0)W_0 + \Psi_0^T\Omega_0\Psi_0]\} = -\frac{1}{R_a}\exp(-R_a\mathcal{E}_0),$$

which relates α to β.

It can be shown that the FOC of the manager's Bellman equation yields the following expression for A_t:

$$A_t^* = \frac{a(t)R_a(a_Q - 2R_ab_Qb_\Psi\Omega)\Psi}{k + a^2(t)R_ab_Qb_Q^T}, \quad a(t) = (\beta - \frac{\gamma}{r})\mathrm{e}^{r(T-t)} + \frac{\gamma}{r}.$$

The manager's Bellman equation then reduces to

$$\begin{aligned}&\frac{1}{2}kA^2 - \Psi\dot{\Omega}\Psi - aa_QA\Psi + \frac{1}{2}a^2R_ab_Qb_Q^TA^2 - 2\Psi^T\Omega a_\Psi\Psi\\ &-tr(\Omega b_\Psi b_\Psi^T) + 2R_a\Psi^T\Omega b_\Psi b_\Psi^T\Omega\Psi + 2b_Qb_\Psi aR_a\Omega\Psi A = 0,\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{2}kH^TH - \dot{\Omega} - aH^Ta_Q + \frac{1}{2}a^2R_aH^THb_Qb_Q^T - 2\Omega a_\Psi\\ &-m_{11}tr(\Omega b_\Psi b_\Psi^T) + 2R_a\Omega b_\Psi b_\Psi^T\Omega + 2R_aaH^Tb_Qb_\Omega\Omega = 0.\end{aligned} \tag{A48}$$

D.2 The Investor's Problem

Given the manager's trading strategy A^* and the constant term in the contract form α, both of which are functions of β, the investor's problem is to choose an optimal β so as to maximize the expected utility over her terminal wealth:

$$\sup_\beta -\frac{1}{R_p}E_0\exp\{-R_p[(1-\beta)W_T - \alpha]\},$$

$$\text{s.t.} \quad dW_t = rW_tdt + A^*(\beta)dQ_t.$$

Numerically, we may simply compute the above expectation using simulations. But we find that it is more efficient to adopt the following dynamic programming approach solving an ODE.

Define the investor's value function as $J = \sup_\beta -\frac{1}{R_p}E_t\exp\{-R_p[(1-\beta)W_T - \alpha]\}$. Her Bellman equation is then given by

$$J_t + J_W(rW + a_QA^*\Psi) + \frac{1}{2}J_{WW}b_Qb_Q^TA^{*2} + J_\Psi a_\Psi\Psi + \frac{1}{2}tr(J_{\Psi\Psi}b_\Psi b_\Psi^T) + b_Qb_\Psi J_{W\Psi}A^* = 0.$$

Conjecture that J is given by $J = -\frac{1}{R_p}\exp\{-R_p[a_p(t)W_t + \Psi^T\Lambda\Psi - \alpha]\}$, with the boundary conditions being $a_p(T) = (1-\beta)$ and $\Lambda_T = \mathbf{0}$. To eliminate the W_t terms from the Bellman equation, we obtain $a_p(t) = (1-\beta)\mathrm{e}^{r(T-t)}$. Note that the matrix Λ is a function of β. It can be shown that the investor's Bellman equation leads to the following ODE:

$$-\dot{\Lambda} - a_p H^T a_Q + \frac{1}{2}a_p^2 R_p b_Q b_Q^T H^T H - 2R_p \Lambda a_\Psi - tr(\Lambda b_\Psi b_\Psi^T) + 2R_p b_\Psi b_\Psi^T \Lambda + 2a_p R_p H^T b_Q b_Q^T \Lambda = 0.$$

D.3 The Numerical Procedure

(1) Guess a β_0, the manager solves his maximization problem for the trading strategy A^*.

(2) Guess the whole path of matrices Ω_0 and Λ_0 in the value functions.

(3) Solve for the λ's from the ODEs derived from the equilibrium condition, i.e., $A_t^* = \theta_t$.

(4) Given the above price process, we solve the investor's maximization problem for β_1 that maximizes the investor's value function at time 0 while making sure that the manager's and the investor's ODEs (from the respectively Bellman equations) are satisfied. This gives rise to Ω_1 and Λ_1.

(5) Use β_1, Ω_1, and Λ_1 to solve the manager's problem for λ_2's, which determine a new price process.

(6) Keep iterating until

$$\|\Omega_{N+1} - \Omega_N\| < 0.001; \quad \|\Lambda_{N+1} - \Lambda_N\| < 0.001;$$

$$\|\lambda_{N+1} - \lambda_N\| < 0.001; \quad \|\beta_{N+1} - \beta_N\| < 0.001.$$

E Proofs of Theorems 3, 4, and 5

E.1 Proof of Theorem 3

$$\Psi = \begin{pmatrix} 1 \\ \Theta_t \\ D_t \end{pmatrix}; \quad a_\Psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_\Theta & 0 \\ G & 0 & 0 \end{pmatrix}; \quad b_\Psi = \begin{pmatrix} 0 & 0 \\ 0 & b_\Theta \\ b_D & 0 \end{pmatrix}.$$

$$a_Q = \left[\begin{array}{ccc} \dot{\lambda}_{0t} - r\lambda_{0t} + G & \dot{\lambda}_{1t} - (a_\Theta + r)\,\lambda_{1t} & \dot{\lambda}_{2t} - r\lambda_{2t} \end{array} \right]; \qquad b_Q = \left[\begin{array}{cc} b_D(1+\lambda_{2t}) & \lambda_{1t} b_\Theta \end{array} \right].$$

Using the optimal policy given in Theorem 1, the investor's Bellman equation (A42), and the equilibrium condition (**??**), we arrive at the ODEs that determine the coefficients λ_{0t}, λ_{1t}, λ_{2t}, and the coefficient matrix V:

$$\dot{\lambda}_{0t} - r\lambda_{0t} + G = 0, \quad \dot{\lambda}_{2t} - r\lambda_{2t} = 0,$$

$$\dot{\lambda}_{1t} - \left(a_\Theta + r + 2\frac{R_a R_p}{R_a + R_p} b_\Theta^2 V(2,2) \right) \lambda_{1t} - \frac{\Delta}{a(t)} = 0,$$

$$\dot{V}(2,2) + a(t)\Gamma_t - 2R_p a(t)\lambda_{1t} V(2,2) b_\Theta^2 - 2a_\Theta V(2,2) - 2R_p V^2(2,2) - \frac{1}{2}\left[k + a^2(t) R_p \left(b_D^2 + \lambda_{1t}^2 b_\Theta^2\right)\right]$$
$$+ \frac{R_p^2}{2\Delta^2(R_a + R_p)} \left[b_D^2 a^4(t)\Gamma_t^2 + \lambda_{1t}^2 b_\Theta^2 \left(a^2(t)\Gamma_t + 2kV(2,2)\right)^2 \right] = 0,$$

where

$$\Gamma_t = \left(2\frac{R_a R_p}{R_a + R_p} \lambda_{1t} b_\Theta^2 V(2,2) + \frac{\Delta}{a(t)} \right); \quad \Delta = k + a^2(t)\frac{R_a R_p}{R_a + R_p}\left[b_D^2 + \lambda_{1t}^2 b_\Theta^2\right].$$

The boundary conditions are $\lambda_{0T} = \lambda_{1T} = \lambda_{2T} = V_T(2,2) = 0$. The solutions to λ_{0t} and λ_{2t} are given by

$$\lambda_{0t} = \frac{G}{r}\left(1 - \mathrm{e}^{r(t-T)}\right), \qquad \lambda_{2t} = 0.$$

λ_{1t} and $V(2,2)$ can be calculated numerically as a set of solutions to the system of above ODEs.

E.2 Proof of Theorem 4

$\Psi^T \equiv (1\ \Theta_t\ G_t^c\ D_t)$. Since the only signal[4] about G is D_t, the solution to this one-dimensional filtering problem is

$$dG_t^c = \frac{f_t}{b_D^2}\left[dD_t - G_t^c dt\right], \quad \frac{df_t}{dt} = -\frac{f_t^2}{b_D^2},$$

with boundary conditions that $G_0^c = G_0$ and $f_0 = b_G^2$. It can be verified that G_t^c is given by

$$G_t^c = \frac{D_t + G_0 I_t}{I_t + t},$$

[4] P_t does not contain any extra information about G in its conjectured form.

where $I_t = \frac{b_D^2}{b_G^2}$. Due to the above linear relationship between G_t^c and D_t, one of them becomes redundant. Therefore, the equilibrium price can be simplified as

$$P_t = \lambda_{0t} + \lambda_{1t}\Theta_t + \lambda_{2t}G_t^c.$$

Note that the price is fully revealing about Θ_t.

The vector of state variables is now given by $\Psi = \begin{pmatrix} 1 \\ \Theta_t \\ G_t^c \end{pmatrix}$. We also have

$$d\tilde{B}_\Psi = \begin{pmatrix} \frac{1}{b_D}\left[dD_t - G_t^c dt\right] \\ dB_\Theta \end{pmatrix},$$

$$a_\Psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_\Theta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b_\Psi = \begin{pmatrix} 0 & 0 \\ 0 & b_\Theta \\ \frac{b_D}{I_t+t} & 0 \end{pmatrix}.$$

$$a_Q = \begin{bmatrix} \dot{\lambda}_{0t} - r\lambda_{0t} & \dot{\lambda}_{1t} - (a_\Theta + r)\lambda_{1t} & \dot{\lambda}_{2t} - r\lambda_{2t} + 1 \end{bmatrix}; \qquad b_Q = \begin{bmatrix} b_D\left(1 + \frac{\lambda_{2t}}{I_t+t}\right) & \lambda_{1t}b_\Theta \end{bmatrix}.$$

Using the optimal policy given in Theorem 1, the investor's Bellman equation (A42), and the equilibrium condition (**??**), we get the ODEs as

$$\dot{\lambda}_{0t} - r\lambda_{0t} = 0, \quad \dot{\lambda}_{2t} - r\lambda_{2t} + 1 = 0,$$

$$\dot{\lambda}_{1t} - (a_\Theta + r)\lambda_{1t} - 2\frac{R_a R_p}{R_a + R_p} b_\Theta^2 V(2,2) = 0,$$

$$\dot{V}(2,2) + a(t)\Gamma_t - 2R_p a(t)\lambda_{1t}V(2,2)b_\Theta^2 - 2a_\Theta V(2,2) - 2R_p V^2(2,2) - \frac{1}{2}\left[k + a^2(t)R_p\left(b_D^2 + \lambda_{1t}^2 b_\Theta^2\right)\right]$$
$$+\frac{R_p^2}{2\Delta^2(R_a+R_p)}\left[b_D^2\left(1 + \frac{1}{I_t+t}\right)^2 a^4(t)\Gamma_t^2 + \lambda_{1t}^2 b_\Theta^2\left(a^2(t)\Gamma_t + 2kV(2,2)\right)^2\right] = 0$$

where

$$\Gamma_t = \left(2\frac{R_a R_p}{R_a + R_p}\lambda_{1t} b_\Theta^2 V(2,2) + \frac{\Delta}{a(t)}\right).$$

The boundary conditions are given by $\lambda_{0T} = \lambda_{1T} = \lambda_{2T} = 0$ and $V_T(2,2) = 0$. λ_{0t} and λ_{2t} are given by

$$\lambda_{0t} = 0, \quad \lambda_{2t} = \frac{1}{r}\left[1 - \mathrm{e}^{-r(T-t)}\right].$$

λ_{1t} and $V(2,2)$ can be solved numerically from the above ODEs.

E.3 Proof of Theorem 5

Define agent i's value function as

$$J^i(t, W, \Psi^i) = E_t^i \left[-\frac{1}{R_a} \exp \left[-R_a [W_T - \frac{1}{2} \int_t^T k^i(t) A_t^{i2} dt] \right] \right].$$

The agent's Bellman equation is then given by

$$\begin{aligned} & \sup_{A_t^i} \Big\{ J R_a c + J_t + J_W \left[rW^i + a_Q A_t^i \Psi^i \right] + \frac{1}{2} J_{WW} A_t^{i2} b_Q^i {b_Q^i}^T \\ + \quad & J_{\Psi^i}^T a_\Psi \Psi^i + \frac{1}{2} tr \left(J_{\Psi^i \Psi^i} b_{\Psi^i} b_{\Psi^i}^T \right) + b_Q^i b_{\Psi^i} J_{W \Psi^i} A_t^i \Big\} = 0 \end{aligned}$$

The FOC with respect to A_t^i yields

$$J R_a k^i(t) A_t^i + J_W a_Q \Psi^i + J_{WW} A_t^i b_Q^i {b_Q^i}^T + b_Q^i b_{\Psi^i} J_{W \Psi^i} = 0$$

Conjecture that the agent's value function $J(\cdot)$ is given by

$$J = -\frac{1}{R_a} \exp \left[-R_a \left(a(t) W^i + {\Psi^i}^T V(t) \Psi^i \right) \right],$$

with the boundary conditions that $a(T) = 1$ and $V(T) = 0$. A_t^i then reduces to

$$A_t^i = H \Psi^i; \quad H = \frac{a(t) \left[a_Q - 2 R_a b_Q^i b_{\Psi^i}^T V(t) \right]}{k^i(T) + a^2(t) R_a b_Q^i {b_Q^i}^T}.$$

Substituting A_t^i and $J(\cdot)$ into the agent's Bellman equation, we obtain

$$\begin{aligned} & \dot{V} + \frac{a(t)}{2} \left[a_Q^T H + H^T a_Q \right] + \left[V^T a_\Psi + a_\Psi^T V \right] + \\ & -\frac{1}{2} H^T H \left[k + a^2(t) R_a b_Q b_Q^T \right] - a(t) R_a \left[V^T b_{\Psi^i} b_Q^T H + H^T b_Q b_{\Psi^i}^T V \right] \\ & -2 R_a b_{\Psi^i} b_{\Psi^i}^T V V^T + m_{11}^N tr \left[V^T b_{\Psi^i} b_{\Psi^i}^T \right] = 0, \end{aligned} \tag{A49}$$

where m_{11}^N denotes an $N \times N$ matrix with the element (1,1) = 1 and all other elements being zero, and with N being the dimensionality of the vector Ψ^i.