

# Estimation of continuous-time models with an application to equity volatility dynamics

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# Estimation of continuous-time models with an application to equity volatility dynamics

## **Abstract**

The treatment of this article renders closed-form density approximation feasible for univariate continuous-time models. Implementation methodology depends directly on the parametric form of the drift and the diffusion of the primitive process and not on its transformation to a unit-variance process. Offering methodological convenience, the approximation method relies on numerically evaluating one-dimensional integrals and circumvents existing dependence on intractable multidimensional integrals. Density-based inferences can now be drawn for a broader set of models of equity volatility. Our empirical results provide insights on crucial outstanding issues related to the rank-ordering of continuous-time stochastic volatility models, the absence/presence of nonlinearities in the drift function of equity volatility, and the desirability of pursuing more flexible diffusion function specifications.

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Whether the end-goal is martingale pricing or maximum-likelihood estimation, theory invariably requires the knowledge of the transition density of the economic forcing variable which is generally unamenable to closed-form characterization. In this sense the lack of analyticity of the density function has hampered empirical testing and the validation of alternative hypotheses about continuous-time models. To remedy this deficiency, Aït-Sahalia (1999, 2002) has proposed a method to approximate the transition density in a one-dimensional diffusion setting. Given its potential utility to the researcher in applied and theoretical work, the purpose of this article is to expand on the analytical density approach of Aït-Sahalia (1999, 2002) and our treatment renders the original method feasible for a substantially larger class of one-dimensional models. Armed with this modification we empirically implement the density approximation method to study the plausibility of general models of equity volatility. Density-based inferences allow us to disentangle issues connected with (i) the rank-ordering of continuous-time volatility models, (ii) the presence of nonlinearities in the drift function, and (iii) the desirability of adopting more flexible diffusion specifications.

The motivation for our analysis to expand on Aït-Sahalia (1999, 2002) derives from two considerations. First, in the context of one-dimensional diffusions  $dX_t = \mu[X_t] dt + \sigma[X_t] dW_t$ , for economic-variable  $X_t$ , extant density approximations crucially hinge on transforming  $X_t$  to a unit-variance process via  $\int^X \frac{du}{\sigma[u]}$  and then on inverting  $\int^X \frac{du}{\sigma[u]}$ . This requirement has proved analytically challenging for some interesting economic models (see Bakshi and Ju (2005)). While developing the likelihood function of arbitrary multivariate diffusions, Aït-Sahalia (2003) has constructed a Taylor series solution of the expansion coefficients. Such a procedure is capable of delivering a closed-form density approximation even when the multivariate diffusion is not reducible. There is a potential tradeoff between the fully closed-form irreducible method and our approach, which will be articulated in the ensuing discussions. Second, in the enhanced-method of Aït-Sahalia (1999, 2002), the recursively defined coefficients that fulfill the forward/backward equation have a multidimensional integral dependence and are seldom tractable outside of the constant elasticity of variance diffusion class. The framework of this paper overcomes both hurdles associated with implementing Aït-Sahalia (1999, 2002). Broadening the appeal of the methodology we show that the density approximation can be derived without reducing the primitive process to a unit-variance process and without analytically integrating and inverting  $\int^X \frac{du}{\sigma[u]}$ . The contribution of our approach also lies in determining the recursively defined expansion coefficients that exhibit at most a single integral dependence and consequently affords tractability. An advantage of this new approach is that it causes the density approximation to be virtually analytical for continuous-time models with nonlinear drift and diffusion functions of the general type analyzed in Aït-Sahalia (1996).

Market index volatility is one of the most fundamental variables determined in financial markets and is a particularly relevant input into option pricing, risk management systems, and volatility-based contingent claims (i.e., variance futures and options on volatility). Despite the flurry of recent modeling efforts (see Andersen, Bollerslev, Diebold, and Labys (2003), Heston (1993), and Jones (2003)) there is yet no consensus on the dynamic evolution of equity volatility in continuous-time. Exploiting the closed-form density approximation, our empirical analysis of daily market volatility provides evidence for a volatility process that has substantial nonlinear mean-reverting drift underpinnings.

Supporting a strand of drift specifications taking the parametric form  $\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}$ , the finding of statistically significant  $\alpha_2 < 0$  and  $\alpha_3 > 0$  indicates drift-reversals at both high and low ends of the equity volatility spectrum. Volatility processes omitting a role for nonlinear diffusion coefficient  $\sigma[X] \equiv \sqrt{\beta_1 X_t + \beta_2 X_t^{\beta_3}}$  with  $\beta_3 > 2$  are structurally flawed and destined for unsatisfactory empirical performance. Thus, the inconsistency of affine volatility models (i.e., Heston (1993)) can be attributed to misspecified drift and diffusion coefficients. Overall, a equity market variance specification with nonlinear drift and diffusion function delivers the most desirable goodness-of-fit statistics and this new empirical result has wide-ranging implications for pricing and trading of risks associated with equity and volatility derivatives.

The rest of the paper is divided into four parts. Section 1 discusses the enhanced density approximation method of Ait-Sahalia (1999, 2002) and develops results aimed at simplifying the multidimensional structure of the expansion coefficients up to fourth-order. Our characterizations are derived entirely in terms of the drift and diffusion of the underlying primitive process. Section 2 presents the density approximation for the encompassing model of Ait-Sahalia (1996). Section 3 is devoted to empirically evaluating continuous-time models of equity volatility. Section 4 summarizes our contributions and concludes.

## 1 Maximum-likelihood estimation of continuous-time models

Consider a one-dimensional diffusion process for a state variable  $X_t$ :

$$dX_t = \mu[X_t; \theta] dt + \sigma[X_t; \theta] dW_t, \tag{1}$$

where  $\mu[X_t; \theta]$  and  $\sigma[X_t; \theta]$  are respectively the coefficients of drift and diffusion, and  $\theta$  represents the unknown parameter vector in an open bounded set  $\Theta \subset R^d$ . The maximum likelihood estimation of  $\theta$  using discretely observed data requires the underlying transition density.

To facilitate empirical testing using density methods, Ait-Sahalia (1999, 2002) has developed

two analytical density approximations. Of particular interest are the *enhanced formulae* in Ait-Sahalia (1999, 2002) which correspond to the limit where the order of the Hermite polynomials converges to infinity and is derived by forcing the coefficients to fulfill the Fokker-Plank-Kolmogorov partial differential equation. The contribution of this section is to propose a modification to the enhanced-method and shows that the resulting density approximation applies to a broader class of  $\mu[X_t; \theta]$  and  $\sigma[X_t; \theta]$ .

### 1.1 Enhanced-method in Ait-Sahalia (1999, 2002)

Ait-Sahalia (1999, 2002) constructs a unit-variance process  $Y_t$  defined by:

$$Y_t \equiv \gamma[X; \theta] = \int^X \frac{du}{\sigma[u; \theta]}. \quad (2)$$

Letting  $\gamma^{-1}[y; \theta]$  be the inverse function of  $\gamma[X; \theta]$ , the drift of  $dY_t$  is:

$$\mu_Y[y; \theta] = \frac{\mu[\gamma^{-1}[y; \theta]; \theta]}{\sigma[\gamma^{-1}[y; \theta]; \theta]} - \frac{1}{2} \frac{\partial \sigma}{\partial x} [\gamma^{-1}[y; \theta]; \theta]. \quad (3)$$

Denoting  $\phi[z] \equiv e^{-z^2/2}/\sqrt{2\pi}$  and  $\Delta$  as a discrete time interval, Ait-Sahalia (1999, 2002) shows that the density of  $Y_t = y$  can be approximated up to the  $K$ -th term by

$$p_Y^{(K)}[\Delta, y|y_0; \theta] = \Delta^{-1/2} \phi\left[\frac{y - y_0}{\Delta^{1/2}}\right] \exp\left(\int_{y_0}^y \mu_Y[w; \theta] dw\right) \sum_{k=0}^K c_k[y|y_0; \theta] \frac{\Delta^k}{k!}, \quad (4)$$

with  $c_0[y|y_0, \theta] \equiv 1$ . For  $j \geq 1$ , the recursive coefficients,  $c_j[y|y_0; \theta]$ , can be derived by solving:

$$c_j[y|y_0; \theta] = j (y - y_0)^{-j} \int_{y_0}^y (w - y_0)^{j-1} \left( \lambda[w] c_{j-1}[w] + \frac{1}{2} \frac{\partial^2 c_{j-1}[w|y_0; \theta]}{\partial w^2} \right) dw, \quad (5)$$

where  $\lambda[y; \theta]$  is defined in terms of the drift function of  $dY_t$  as,

$$\lambda[y; \theta] \equiv -\frac{1}{2} \left( \mu_Y^2[y; \theta] + \frac{\partial \mu_Y[y; \theta]}{\partial y} \right). \quad (6)$$

The transition density of  $X_t$  is then obtained through the Jacobian formula as:

$$p_X^{(K)}[\Delta, x|x_0; \theta] = (\sigma[x; \theta])^{-1} p_Y^{(K)}[\Delta, \gamma[x] | \gamma[x_0]; \theta]. \quad (7)$$

There are several aspects of this methodology that need discussion. For the expansion in (4) to converge, the  $X_t$  process (1) must first be transformed to be sufficiently Gaussian. Based on

the theoretical models adopted by Ait-Sahalia (1999) it is clear that when  $Y_t$  and  $\gamma^{-1}[y; \theta]$  are in analytical closed-form, then  $\mu_Y[y; \theta]$  and  $\lambda[y; \theta]$  are completely analytical and the reduction to a unit-variance diffusion is feasible. Both Brandt and Santa-Clara (2002) and Durham (2003) have argued that - for a broad class of continuous-time models - the reduction step from  $X_t$  to  $Y_t$  is restrictive and can curb the appeal of the density approximation in empirical applications.

Recognizing this disparity between theory and implementation, Bakshi and Ju (2005) relax the requirement that both  $Y_t$  and  $\gamma^{-1}[y; \theta]$  be known analytically, and explain how the Hermite expansions in the basic-approach of Ait-Sahalia (2002) can be reformulated so that only the numerical value of  $Y_t$  is needed. While the Bakshi and Ju (2005) refinement is appealing because it makes the density approximation possible for a wide class of  $\sigma[X]$ , there are reasons to believe that the method in (4) is superior if it can be transformed to make it apply to an equally wide class. The accuracy tests in Bakshi and Ju (2005) indicate that the approximation based on (4) is accurate with expansion coefficients as few as three, and is substantially more reliable (see also our comparisons to come in Table 1). The Hermite approach relies on expanding the density of  $Y_t$  around a standard normal while the expansion in (4) forces the density function to satisfy the Kolmogorov forward and backward equations to the order  $\Delta^K$ , resulting in greater accuracy.

The second feature of the methodology concerns the determination of the recursively defined  $c_j[y|y_0; \theta]$ . From the form of (5) it may be observed that  $c_1[y|y_0; \theta]$  and  $c_2[y|y_0; \theta]$  can be derived by solving one-dimensional integrals and two-dimensional integrals respectively, and higher-dimensional integrals are involved in implementing the density approximation with  $c_3[y|y_0; \theta]$  and beyond. For example, Ait-Sahalia (1999) has solved the density function for models satisfying  $dX_t = (\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}) dt + \sigma X_t^\rho dW_t$  where each  $c_j[y|y_0; \theta]$  is fully analytical. However, the expressions for leading terms  $c_j[y|y_0; \theta]$  are still unknown under a general class of  $\mu[X_t; \theta]$  and  $\sigma[X_t; \theta]$ . Suppose  $dX_t = (\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}) dt + \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}} dW_t$ . For this model class the multidimensional integrals embedded in  $c_j[y|y_0; \theta]$  remain unsolved, thereby eluding closed-form representations for  $\{c_1[y|y_0; \theta], c_2[y|y_0; \theta], c_3[y|y_0; \theta], \dots, c_K[y|y_0; \theta]\}$ . The lack of analyticity of  $c_j[y|y_0; \theta]$  is problematic as it has impaired the density-based inference of models with arbitrary  $\mu[X; \theta]$  and  $\sigma[X; \theta]$ .

Citing this reason, Durham and Gallant (2002), and Brandt and Santa-Clara (2002) have argued in favor of simulation-based methods. However, simulation methods can be cumbersome and computationally expensive. Our objective is to characterize transition densities for a broad parametric class of diffusion processes and exploit them for empirical testing and model selection.

## 1.2 Central elements of the modification

The proposed approximation method, outlined below, directly exploits the form of  $\mu[X_t; \theta]$  and  $\sigma[X_t; \theta]$  and bypasses closed-form reliance on  $\mu_Y[y; \theta]$  and  $\int^{X_t} \frac{du}{\sigma[u; \theta]}$  in practical applications. Our analytical contributions also ensure that each  $c_j[y|y_0; \theta]$  contains at most one-dimensional integrals rather than a set of complex multidimensional integrals.

### 1.2.1 Circumventing the reliance of the approximation on $\mu_Y[y; \theta]$

Models for  $\mu[X; \theta]$  and  $\sigma[X; \theta]$  contained in Ait-Sahalia (1999) stress that when  $Y_t$  and  $\gamma^{-1}[Y; \theta]$  are analytical so are  $\mu_Y[Y]$  and  $\lambda[Y]$ . For this family of continuous-time models, equations (4), (5), and (7) constitute a conceptually simple and accurate approximation method. However, when  $Y_t$  and/or  $\gamma^{-1}[Y; \theta]$  do not admit closed-form representation, it is expedient to reexpress all required density approximation components in terms of  $\mu[X; \theta]$  and  $\sigma[X; \theta]$  of the original  $X_t$  process.

**Proposition 1** Let  $\sigma'[X] \equiv \frac{\partial \sigma[X]}{\partial X}$  and define the function,

$$f[X] \equiv \frac{\mu[X]}{\sigma[X]} - \frac{\sigma'[X]}{2} \quad (8)$$

by analogy with (3). The following density approximation components can be obtained in terms of  $\mu[X]$  and  $\sigma[X]$  of the original process  $X_t$  (suppressing the dependence on  $\theta$ ):

$$\lambda[y] = -\frac{1}{2} \left( f^2[x] + f'[x] \sigma[x] \right), \quad (9)$$

$$\lambda'[y] \equiv \frac{\partial \lambda[y]}{\partial y} = -\frac{1}{2} \sigma[x] \left( f^2[x] + f'[x] \sigma[x] \right)', \quad (10)$$

$$\lambda''[y] \equiv \frac{\partial^2 \lambda[y]}{\partial y^2} = -\frac{1}{2} \sigma[x] \left( \left( f^2[x] + f'[x] \sigma[x] \right)' \sigma[x] \right)', \quad (11)$$

$$y - y_0 = \int_{x_0}^x \frac{du}{\sigma[u]}, \quad (12)$$

and,

$$\int_{y_0}^y \mu_Y[w] dw = \int_{x_0}^x f[u] \frac{du}{\sigma[u]}, \quad (13)$$

$$\int_{y_0}^y \lambda[w] dw = -\frac{1}{2} \int_{x_0}^x \left( f^2[u] + f'[u] \sigma[u] \right) \frac{du}{\sigma[u]}, \quad (14)$$

$$\int_{y_0}^y \lambda^2[w] dw = \frac{1}{4} \int_{x_0}^x \left( f^2[u] + f'[u] \sigma[u] \right)^2 \frac{du}{\sigma[u]}. \quad (15)$$

**Proof:** Make the change of variable  $x = \gamma^{-1}[y]$ . Based on the definition of  $\gamma[X]$  given in (2), we have  $dy = dx/\sigma[x]$ . From the definition of  $\lambda[y]$  in (6) and the new variable  $x$  we get (9). The chain rule of differentiation implies  $\lambda'[y] = \frac{\partial \lambda[y]}{\partial x} \frac{\partial x}{\partial y}$  in (10). Similarly, we obtain (11). Noting that,

$$\begin{aligned}\mu_Y[y] &= \mu[\gamma^{-1}[y]]/\sigma[\gamma^{-1}[y]] - \frac{\partial \sigma}{\partial x}(\gamma^{-1}[y])/2 \\ &= \mu[x]/\sigma[x] - \sigma'[x]/2\end{aligned}$$

and using  $dy = dx/\sigma[x]$ , we have the expression in (13). Using (9) and  $dy = dx/\sigma[x]$  leads to (14) and (15).  $\square$

### 1.2.2 Reduction of $c_j[y|y_0; \theta]$ to a set of one-dimensional integrals

**Proposition 2** *For the recursively defined coefficients  $c_j[y|y_0; \theta]$  in (5) and  $\lambda[y; \theta]$  defined in (6), we may derive the higher-order  $c_j[y|y_0; \theta]$  analytically with only one-dimensional integral dependence (suppressing the  $y_0$  and  $\theta$  arguments):*

$$c_1[y] = \frac{1}{y - y_0} \int_{y_0}^y \lambda[w] dw, \quad (16)$$

$$c_2[y] = c_1^2[y] + \frac{1}{(y - y_0)^2} (\lambda[y] + \lambda[y_0] - 2c_1[y]), \quad (17)$$

$$\begin{aligned}c_3[y] &= c_1^3[y] + \frac{3}{(y - y_0)^2} (c_1[y](\lambda[y] + \lambda[y_0]) - 3c_2[y]) + \\ &\quad \frac{3}{(y - y_0)^3} \left( \frac{\lambda'[y] - \lambda'[y_0]}{2} + \int_{y_0}^y \lambda^2[w] dw \right),\end{aligned} \quad (18)$$

and

$$\begin{aligned}c_4[y] &= c_1^4[y] + \frac{3}{(y - y_0)^2} (2\lambda[y]c_2[y] - 8c_3[y] + 2\lambda[y_0]c_1^2[y]) + \\ &\quad \frac{12c_1[y]}{(y - y_0)^3} \left( \frac{\lambda'[y] - \lambda'[y_0]}{2} + \int_{y_0}^y \lambda^2[w] dw \right) + \\ &\quad \frac{3}{(y - y_0)^4} (3\lambda^2[y] + 5\lambda^2[y_0] + 4\lambda[y]c_1[y] - 12c_2[y] + \lambda''[y] + \lambda''[y_0]).\end{aligned} \quad (19)$$

**Proof:** See Appendix A.  $\square$

To obtain coefficient  $c_1[y]$  involves computing two simple integrals:  $y - y_0 = \int_{x_0}^x \frac{du}{\sigma[u]}$  and  $\int_{y_0}^y \lambda[w] dw$ . Once  $c_1[y]$  is obtained,  $c_2[y]$  follows immediately and  $c_3[y]$  merely requires  $\int_{y_0}^y \lambda^2[w] dw$ . With  $c_3[y]$  known,  $c_4[y]$  can be easily recovered and involves no further integrals. Viewed from this perspective of solving one-dimensional integrals, the density approximation with  $K = 4$  constitutes an efficient method. Thus, Proposition 2 achieves the crucial task of reducing the

recursively defined multidimensional integrals in  $c_j[y]$  to those involving only the one-dimensional integrals outlined in Proposition 1.

With the relevant components in the density approximation expressed directly in terms of the original state variable  $X_t$ , its drift  $\mu[X]$  and diffusion function  $\sigma[X]$ , the method under consideration can be applied to any selected scalar diffusion processes. The density approximation becomes

$$p_X^{(K)}[\Delta, x|x_0; \theta] \approx \frac{\Delta^{-1/2}}{\sigma[X; \theta]} \phi \left[ \frac{1}{\Delta^{1/2}} \int_{x_0}^x \frac{du}{\sigma[u]} \right] \exp \left( \int_{x_0}^x f[u] \frac{du}{\sigma[u]} \right) \sum_{k=0}^4 c_k[\gamma[x]|\gamma[x_0]; \theta] \frac{\Delta^k}{k!}, \quad (20)$$

where  $\{c_1[y|y_0; \theta], \dots, c_4[y|y_0; \theta]\}$  are presented in (16)-(19). Although some integrals still remain to be determined in our formulation in (13)-(15) they are solely required for their numerical values. Compared with the recursively defined multidimensional integrals, the simplified  $c_j[y]$ 's are easier to evaluate and this connection will be highlighted in the context of the general model of Ait-Sahalia (1996). To guarantee that the density remains positive, the approximation for log-density is used:  $\log \left( p_X^{(K)}[\Delta, x|x_0; \theta] \right) \approx -\log(2\pi\sigma^2[X; \theta]\Delta) / 2 - \left( \int_{x_0}^x \frac{du}{\sigma[u]} \right)^2 / (2\Delta) + \int_{x_0}^x f[u] \frac{du}{\sigma[u]} + \sum_{k=0}^4 \mathcal{C}_k[\gamma[x]|\gamma[x_0]; \theta] \frac{\Delta^k}{k!}$ , where  $\mathcal{C}_1 \equiv c_1$ ,  $\mathcal{C}_2 \equiv c_2 - c_1^2$ ,  $\mathcal{C}_3 \equiv c_3 - 3c_2 c_1 + 2c_1^3$ , and  $\mathcal{C}_4 \equiv c_4 - 4c_3 c_1 - 3c_2^2 + 12c_2 c_1^2 - 6c_1^4$ .

Ait-Sahalia (2003) has extended the density approximation method in Ait-Sahalia (1999, 2002) to higher-dimensional diffusion processes. It must be emphasized that the problem of determining  $c_j[y|y_0; \theta]$  is substantially harder in the multivariate setting and we have been unable to work through the multivariate counterparts of Proposition 1 and Proposition 2. We direct the reader to Ait-Sahalia and Kimmel (2003) for closed-form likelihood expansions under affine multifactor models of the term structure of interest rates.

### 1.2.3 Density approximation when $x - x_0$ and/or $y - y_0$ is small

For some applications (say, the spot interest rates), the difference between the two adjacent observations ( $x_0$  and  $x$ ) can be small. In such applications, Proposition 3 derives the corresponding closed-form Taylor series approximation of the relevant integrals and renders the method totally analytical.

**Proposition 3** *Let  $E \equiv x - x_0$ ,  $D \equiv y - y_0$ , and define the successive partial derivative entities:*

$$\nu_i \equiv \partial^i (1/\sigma[X_0]) / \partial X_0^i, \quad (21)$$

$$\varphi_i \equiv \partial^i (f[X_0]/\sigma[X_0]) / \partial X_0^i, \quad (22)$$

$$\lambda_i \equiv \partial^i \lambda[y_0] / \partial y_0^i. \quad (23)$$

When  $x - x_0$ , equivalently  $y - y_0$ , is small, we have the following Taylor expansions of the integrals  $\int_{x_0}^x \frac{du}{\sigma[u]}$  and  $\int_{x_0}^x f[u] \frac{du}{\sigma[u]}$ :

$$D = y - y_0 = \int_{x_0}^x \frac{du}{\sigma[u]} = \nu_0 E + \frac{\nu_1 E^2}{2} + \frac{\nu_2 E^3}{6} + \frac{\nu_3 E^4}{24} + \frac{\nu_4 E^5}{120} + \frac{\nu_5 E^6}{720}, \quad (24)$$

$$\int_{y_0}^y \mu_Y[w] dw = \int_{x_0}^x f[u] \frac{du}{\sigma[u]} = \varphi_0 E + \frac{\varphi_1 E^2}{2} + \frac{\varphi_2 E^3}{6} + \frac{\varphi_3 E^4}{24} + \frac{\varphi_4 E^5}{120} + \frac{\varphi_5 E^6}{720}, \quad (25)$$

and  $c_1[y|y_0; \theta]$ ,  $c_2[y|y_0; \theta]$ ,  $c_3[y|y_0; \theta]$ , and  $c_4[y|y_0; \theta]$  are presented in (62)-(65) of Appendix B.

Owing to the results in (24)-(25) and (62)-(67), the approximation for the transition density (20) is now completely analytical. The partial derivatives,  $\nu_i$ ,  $\varphi_i$  and  $\lambda_i$ , are compact and can be coded in any standard programming language for the estimation of a wide class of continuous-time models. The tractability of  $c_j[y|y_0; \theta]$  and the methodological dependence on  $\mu[X; \theta]$  and  $\sigma[X; \theta]$  are at the core of the analytical density approximation.

Upon further reflection, the coefficients in our Taylor series in Proposition 3 correspond to those in Ait-Sahalia (2003) when the irreducible multivariate method is applied to the univariate case. To be exact, Proposition 3 can be interpreted as the *explicit* Taylor expansion solution of the system of equations in Proposition 2 in Ait-Sahalia (2003) for univariate diffusions with arbitrary  $\mu[X]$  and  $\sigma[X]$ . Realize, however, that in the irreducible multivariate modeling case, the expansion coefficients do not generally afford the accuracy of integral representations and must be approximated by Taylor series in  $x - x_0$  irrespective of whether  $x - x_0$  is small or not. On the other hand, the integral equation (5) and the reduced coefficients in our Proposition 2 hold for any  $x$  and  $x_0$  and can be evaluated through efficient integration routines. Thus, it may be preferable to use our Proposition 2 to obtain the expansion coefficients,  $c_j[y|y_0; \theta]$ , when  $x - x_0$  is not very small.

Proposition 3 becomes useful only when  $x - x_0$  is very small (e.g.,  $x = x_0$ ). The reason is that when  $x - x_0 = 0$  (equivalently  $y - y_0 = 0$ ), the coefficients in our Proposition 2 involve 0/0 and when  $x - x_0$  is very small, the division of a very small number by another very small number can occur. In the present method, the development in Proposition 3 is intended to handle a tricky numerical situation while in the irreducible multivariate setting one must resort to Taylor approximations. Thus, it must be appreciated that there is a potential tradeoff between the fully closed-form irreducible method and the present method which retains simple one-dimensional integrals in Proposition 2 and must be evaluated numerically.

## 2 Density approximation for a parametric model class

Spurred by our characterizations in Proposition 1 through Proposition 3, this section applies the density approximation (20) to the following eight-parameter encompassing class of one-dimensional processes due to Ait-Sahalia (1996):

$$\mu[X; \theta] = \alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}, \quad (26)$$

$$\sigma[X; \theta] = \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}}, \quad (27)$$

subject to technical conditions (24a) through (24d) in Ait-Sahalia (1996). Here,

$$\theta \equiv (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3). \quad (28)$$

The distinctive property of this continuous-time process is that  $\mu[X; \theta]$  in (26) accommodates a nonlinear drift, and  $\sigma[X; \theta]$  in (27) implies stochastic elasticity of variance. Given these features, this model is labeled as SEV-ND. Under appropriate restrictions on  $\theta$ , the SEV-ND model subsumes several theoretically appealing models for  $X_t$  that display (i) constant elasticity of variance (hereby CEV) with nonlinear, linear and constant drift, respectively and (ii) the affine model (hereby AFF), as in:

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$$\text{SEV-ND: } dX_t = \left( \alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1} \right) dt + \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}} dW_t$$

$$\text{SEV-LD: } dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}} dW_t$$

$$\text{SEV-CD: } dX_t = \alpha_0 dt + \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}} dW_t$$

$$\text{CEV-ND: } dX_t = \left( \alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1} \right) dt + \beta_2 X_t^{\beta_3} dW_t$$

$$\text{CEV-LD: } dX_t = (\alpha_0 + \alpha_1 X_t) dt + \beta_2 X_t^{\beta_3} dW_t$$

$$\text{CEV-CD: } dX_t = \alpha_0 dt + \beta_2 X_t^{\beta_3} dW_t$$

$$\text{AFF: } dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta_0 + \beta_1 X_t} dW_t$$


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As argued earlier, to apply the enhanced-method of Ait-Sahalia (1999, 2002) to the process in (26)-(27) requires  $Y_t \equiv \int^{X_t} \frac{du}{\sigma[u; \theta]} = \int \frac{du}{\sqrt{\beta_0 + \beta_1 u + \beta_2 u^{\beta_3}}}$ , which has no known closed-form analytical representation. Thus, leaving aside the additional issue of multiple numerical integration determination of higher-order  $c_j[y|y_0; \theta]$ , none of the models in the SEV class are amenable to a density characterization under the approach of Ait-Sahalia (1999, 2002).

Returning to our methodology, we determine the components of  $p_X^{(K)}[\Delta, x|x_0; \theta]$  in (20) by defining  $V_0 \equiv \beta_0 + \beta_1 X_0 + \beta_2 X_0^{\beta_3}$ . Clearly,  $\nu_0^2 V_0 = 1$  where  $\nu_0 \equiv 1/\sqrt{\beta_0 + \beta_1 X + \beta_2 X^{\beta_3}}$ . Straightforward successive differentiation of  $\nu_0^2 V_0 = 1$  with respect to  $X_0$  yields the first five derivatives of  $\nu_0$  with respect to  $X_0$ :

$$\nu_1 = -\nu_0 V_1 / (2V_0), \quad (29)$$

$$\nu_2 = -(3\nu_1 V_1 + \nu_0 V_2) / (2V_0), \quad (30)$$

$$\nu_3 = -(5\nu_2 V_1 + 4\nu_1 V_2 + \nu_0 V_3) / (2V_0), \quad (31)$$

$$\nu_4 = -(7\nu_3 V_1 + 9\nu_2 V_2 + 5\nu_1 V_3 + \nu_0 V_4) / (2V_0), \quad (32)$$

$$\nu_5 = -(9\nu_4 V_1 + 16\nu_3 V_2 + 14\nu_2 V_3 + 6\nu_1 V_4 + \nu_0 V_5) / (2V_0), \quad (33)$$

where  $V_1 = \beta_1 + \beta_2 \beta_3 X_0^{\beta_3-1}$ ,  $V_2 = \beta_2 \beta_3 (\beta_3 - 1) X_0^{\beta_3-2}$ ,  $V_3 = (\beta_3 - 2)V_2/X_0$ ,  $V_4 = (\beta_3 - 3)V_3/X_0$ , and  $V_5 = (\beta_3 - 4)V_4/X_0$  are the partial derivatives of  $V_0$  with respect to  $X_0$ .

Proceeding, as in Proposition 3, we obtain  $\varphi_i = \partial^i (f[X_0]/\sigma[X_0]) / \partial X_0^i$ . Recognize that  $\varphi_0 = f[X_0]/\sigma[X_0] = \mu[X_0]/V_0 - V_1/(4V_0) = U_0/V_0$ , where  $U_0 = \mu[X_0] - V_1/4 = \alpha_0 - \beta_1/4 + \alpha_1 X_0 + \alpha_2 X_0^2 + \alpha_3/X_0 - \beta_2 \beta_3 X_0^{\beta_3-1}/4$ . Thus,  $\varphi_0 V_0 = U_0$ . Successive differentiation of this equation produces the first five derivatives of  $\varphi_0$  with respect to the state variable  $X_0$ ,

$$\varphi_1 = (U_1 - \varphi_0)V_1/V_0, \quad (34)$$

$$\varphi_2 = (U_2 - 2\varphi_1 V_1 - \varphi_0 V_2)/V_0, \quad (35)$$

$$\varphi_3 = (U_3 - 3\varphi_2 V_1 - 3\varphi_1 V_2 - \varphi_0 V_3)/V_0, \quad (36)$$

$$\varphi_4 = (U_4 - 4\varphi_3 V_1 - 6\varphi_2 V_2 - 4\varphi_1 V_3 - \varphi_0 V_4)/V_0, \quad (37)$$

$$\varphi_5 = (U_5 - 5\varphi_4 V_1 - 10\varphi_3 V_2 - 10\varphi_2 V_3 - 5\varphi_1 V_4 - \varphi_0 V_5)/V_0, \quad (38)$$

where  $U_1 = \alpha_1 + 2\alpha_2 X_0 - \alpha_3/X_0^2 - V_2/4$ ,  $U_2 = 2\alpha_2 + 2\alpha_3/X_0^3 - V_3/4$ ,  $U_3 = -6\alpha_3/X_0^4 - V_4/4$ ,  $U_4 = 24\alpha_3/X_0^5 - V_5/4$ , and  $U_5 = -120\alpha_3/X_0^6 - V_6/4$  are the partial derivatives of  $U_0$  with respect to  $X_0$  and  $V_6 = (\beta_3 - 5)V_5/X_0$ . For  $\mu[X_0]$  and  $\sigma[X_0]$  governed via (26)-(27), the recursive nature of  $\varphi_i$  determines (25) of Proposition 3.

Finally, we characterize each  $\lambda_i = \partial^i \lambda[y_0] / \partial y_0^i$  in (66)-(67) as a function of  $X_0$ . Based on the calculations in Appendix C, each required  $\lambda_i$  is:

$$\lambda_0 = H_0, \quad (39)$$

$$\lambda_1 = H_1 S_0, \quad (40)$$

$$\lambda_2 = H_2 S_0^2 + \lambda_1 S_1, \quad (41)$$

$$\lambda_3 = H_3 S_0^3 + 3\lambda_2 S_1 + \lambda_1(S_0 S_2 - 2S_1^2), \quad (42)$$

$$\lambda_4 = H_4 S_0^4 + 6\lambda_3 S_1 + \lambda_2(4S_0 S_2 - 11S_1^2) + \lambda_1(S_0^2 S_3 - 6S_0 S_1 S_2 + 6S_1^3), \quad (43)$$

$$\begin{aligned} \lambda_5 = & H_5 S_0^5 + 10\lambda_4 S_1 + \lambda_3(10S_0 S_2 - 35S_1^2) + \lambda_2(5S_0^2 S_3 - 40S_0 S_1 S_2 + 50S_1^3) + \\ & \lambda_1(S_0^3 S_4 - 8S_0^2 S_1 S_3 + 36S_0 S_1^2 S_2 - 6S_0^2 S_2^2 - 24S_1^4), \end{aligned} \quad (44)$$

$$\begin{aligned} \lambda_6 = & H_6 S_0^6 + 15\lambda_5 S_1 + \lambda_4(20S_0 S_2 - 85S_1^2) + \lambda_3(15S_0^2 S_3 - 150S_0 S_1 S_2 + 225S_1^3) + \\ & \lambda_2(6S_0^3 S_4 - 63S_0^2 S_1 S_3 + 346S_0 S_1^2 S_2 - 46S_0^2 S_2^2 - 274S_1^4) + \lambda_1(S_0^4 S_5 - \\ & 10S_0^3 S_1 S_4 + 6S_0^2 S_1^2 S_3 - 20S_0^3 S_2 S_3 - 240S_0 S_1^3 S_2 + 90S_0^2 S_1 S_2^2 + 120S_1^5), \end{aligned} \quad (45)$$

where  $S_0$  through  $S_7$  are shown in (68)-(75), and  $H_0$  through  $H_6$  are shown in (84)-(90). The analyticity of  $\lambda_i$  can now be used to build  $c_1[y|y_0; \theta]$  through  $c_4[y|y_0; \theta]$  in (62)-(65) and  $\varphi_i$  in (34)-(38) are employed for constructing  $\int_{y_0}^y \mu_Y[w] dw$  expansion in (25). Given the choice of  $\mu[X; \theta]$  and  $\sigma[X; \theta]$ , we obtain the density approximation for SEV-ND through (20). Density functions for other continuous-time models are restricted special cases.

Before moving to empirical investigation, it is instructive to determine the accuracy of the density approximation under the proposed scheme for determining  $c_j[y|y_0; \theta]$  in (20). To illustrate this aspect we pick two candidate continuous-time models by setting  $\beta_0 \equiv 0$  in the AFF model (i.e., the square-root model) and  $\alpha_0 \equiv 0$  in the CEV-LD model. Each candidate stochastic process has an exact density which allows comparison to the approximate density. Guided by Aït-Sahalia (1999) and Bakshi and Ju (2005) we compare the maximum absolute error of the approximate density relative to its exact density counterpart. Table 1 judges the worst possible approximation error by presenting  $\text{MAXE} \equiv \max \left( \left| p^{\text{exact}}[x|x_0] - p^{\text{approx}}[x|x_0] \right| \right)$  and the maximum exact conditional density as  $\max(p^{\text{exact}}[x|x_0])$ . The main finding is that our density approximation is accurate for both  $\Delta = 1/12$  and  $\Delta = 1$  regardless of the underlying stochastic process. Inspection of the results also reveals little value-added by including  $c_5[y|y_0; \theta]$  in  $\sum_{k=0}^K c_k[\gamma[x] | \gamma[x_0]; \theta] \frac{\Delta^k}{k!}$ : the approximation with  $K = 5$  marginally improves over  $K = 4$  in some cases. The reason appears to be that the maximum absolute errors with  $K = 4$  - which are in the order of  $10^{-14}$  - have probably reached machine precision. However, as would be expected, embedding each additional term in the approximation tends to make the method progressively more accurate.

### 3 Evaluating continuous-time volatility models

Starting with Heston (1993), there is a tradition to model equity volatility as a continuous-time stochastic process with mean-reverting drift and square-root volatility. Despite the insight this model has enabled, there is, however, a growing consensus that the square-root specification is

grossly misspecified (see Andersen, Benzoni and Lund (2002), Bakshi, Cao, and Chen (2000), Bates (2000), Elerian, Chib, and Shephard (2001), Eraker, Johannes, and Polson (2003), and Pan (2002), among others). With the exception of Jones (2003), who has provided evidence in favor of CEV models of the volatility process, the lack of closed-form density approximations has impeded progress on the testing of volatility processes beyond square-root. As such, several questions still remain unresolved with respect to the documented rejection of square-root volatility models: (i) Does the drift of the volatility process admit departures from linearity? (ii) Are volatility models with general  $\sigma[X]$  more properly specified from empirical standpoints? (iii) What is the empirical potential of variance processes in the Aït-Sahalia (1996) class (26)-(27)? Issues connected with volatility modeling have bearing on the search for better performing option pricing models, parametric compensation for volatility risk, and the timing of volatility risks.

Before we can address the aforementioned questions we need a suitable proxy for market volatility, which is intrinsically unobservable. Among the various possible choices at the daily frequency, the empirical literature has appealed to (i) GARCH volatility constructed from daily returns, (ii) cumulated squared intraday returns (Andersen, Bollerslev, Diebold, and Labys (2003)), (iii) short-term near-money Black-Scholes implied volatility (Pan (2002)), and (iv) market volatility extracted from S&P 100 index option prices, VIX (Jones (2003)). For reasons outlined in Jones (2003), we adopt the forward-looking VIX volatility measure in our empirical work. Thus, it must be understood at the outset that we are drawing conclusions about the desirability of stochastic volatility models based on the estimated dynamics of the VIX index.

Given to us by Chris Jones, the VIX is sampled over the period of July 1, 1988 to January 10, 2000 (2907 observations) and expressed in decimals. To aid comparisons with the empirical literature we let  $X_t \equiv \text{VIX}_t^2$  for  $\{t = i \Delta \mid i = 0, \dots, n\}$ .

The estimate of  $\theta$  in (28) is based on the log-likelihood function:

$$\max_{\theta} \mathcal{L}[\theta] \equiv \sum_{i=1}^n \log \left\{ p_X[\Delta, X_{i\Delta} | X_{(i-1)\Delta}; \theta] \right\} \quad (46)$$

where the transition density,  $p_X[\Delta, X_{i\Delta} | X_{(i-1)\Delta}; \theta]$ , is approximated via (20) with  $\{c_1[y], \dots, c_4[y]\}$  constructed separately for SEV-ND and its nested variants. The efficiency and optimality of the maximum-likelihood estimator is discussed, among others, in Aït-Sahalia (1999, 2002).

Table 2 displays maximum-likelihood model parameters, the standard errors (in parenthesis), the goodness-of-fit maximized log-likelihood values, and the rank-ordering of stochastic volatility models based on the Akaike Information Criterion (AIC). For each of the models the estimate of  $\beta_0$  in the diffusion function  $\sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}}$  was close to zero with no impact on log-likelihood. For this reason we impose  $\beta_0 \equiv 0$ . In this case, the AFF specification collapses to the square-root

model  $dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta_1 X_t} dW_t$ .

The failure of the square-root model is evident even in the presence of a high  $\beta_1$  value of 0.1827 (i.e., volatility of volatility parameter,  $\sqrt{\beta_1}$ , is 0.4274). One drawback of this specification is that it is insufficiently flexible in fitting higher-order volatility moments: With sample VIX skewness and kurtosis of 2.39, and 9.67 respectively, this process can internalize large movements in the underlying market volatility only at the expense of an implausible level of  $\beta_1$ . Negative and statistically significant  $\alpha_1 = -8.0369$  indicates speedy mean-reversion in the variance process and a model long-run volatility level of  $\sqrt{-\alpha_0/\alpha_1} = 19.77\%$ .

Keeping  $\beta_3$  free in the CEV class rectifies modeling deficiencies of the AFF specification that forces  $\beta_3 \equiv 1/2$ , as seen through a large jump in the log-likelihood. Transitioning from AFF to CEV-LD increases the log-likelihood from 11400.28 to 12090.40, thereby rejecting AFF with a significant log-likelihood ratio statistic. Specifically, we compute the log-likelihood ratio statistic as minus twice the difference between the log-likelihood values of the restricted and the unrestricted models:

$$\mathcal{L}^* \equiv -2 \times (\mathcal{L}[\theta_R] - \mathcal{L}[\theta_U]) \quad (47)$$

which is distributed  $\chi^2 [\dim[\theta_U] - \dim[\theta_R]]$ . That the data favors the CEV class of variance processes over the AFF is further validated through high  $-n/2$  AIC. Misspecification of  $\sigma[X]$  is primarily responsible for the rejection of the affine volatility models.<sup>1</sup>

Analyzing the MLE results across CEV-CD, CEV-LD and CEV-ND provides a number of fundamental insights about the dynamics of market variance. First, the highly significant  $\alpha_2$  and  $\alpha_3$  suggest the presence of nonlinearities in the drift of the variance process. Consider CEV-ND: the estimate of  $\alpha_2$  is -166.72 (standard error of 60.53) and the estimate of  $\alpha_3$  is 0.0031 (standard error of 0.0015). Compared to the linear drift model CEV-LD - which has a log-likelihood of 12090.40 - adding two nonlinear drift parameters raises the log-likelihood to 12094.24. The resulting log-likelihood ratio statistic  $\mathcal{L}^*$  is 7.68, which is bigger than  $\chi^2[2]$  critical value of 6.0 (at 95% confidence level) and the linear drift model is rejected in favor of a nonlinear drift in the variance process.

Omitting a role for  $\alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3/X_t$  also worsens the performance of the stochastic

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<sup>1</sup>Demonstrating the computational superiority of the density approximation, the MLE computer code converged rapidly (in less than 1 minute on a 1 GHZ laptop computer) regardless of the drift and diffusion combination. Having a closed-form density approximation can therefore accelerate the speed of estimation several hundred fold relative to simulation-based approaches. We refer the reader to the discussion in Durham and Gallant (2002) and Li, Pearson, and Poteshman (2004) on the efficacy of alternative approaches and Brandt and Santa-Clara (2002) on the difficulty in estimating models using simulation-based methods. With finer Euler discretization and reasonably lengthy MCMC sampler draws, our computation time also measures favorably relative to the Bayesian methodology of Jones (2003).

volatility models versus CEV-ND. The realized value of the test statistic  $\mathcal{L}^*$  is 9.18 which can be compared to the  $\chi^2[3]$  critical value of 7.82. However, the same cannot be argued about linear drift versus constant drift accounting for the constant elasticity of variance structure  $\sigma[X_t] = \beta_2 X_t^{\beta_3}$ . The log-likelihood is virtually insensitive to the addition of  $\alpha_1 X_t$  to the CEV-CD specification: the small increase of  $\mathcal{L}$  from 12089.65 to 12090.40 is insufficiently large to make  $\mathcal{L}^* = 1.5$  statistically significant with 1 degree of freedom.

Ranging between 1.2732 to 1.2781, the exponent parameter  $\beta_3$  is statistically significant in CEV-CD, CEV-LD, and CEV-ND and several fold relative to the reported standard errors. The magnitude of  $\beta_3$  is comparable but slightly higher than that reported by Jones (2003). An overarching conclusion is that  $\beta_3 > 1$  is needed to match the time-series properties of the VIX index with the CEV models (the one-sided t-test rejects the null hypothesis of  $\beta_3 \leq 1$ ).

Recall the SEV class of variance processes shares the same drift and volatility structure as models in the CEV class except that the SEV embeds an additional linear-term in the  $\sigma[X]$  specification. The implementation of SEV-CD, SEV-LD, and SEV-ND brings out several incremental facts about the behavior of the volatility processes. Comparing the log-likelihood and model estimates across CEV-CD and SEV-CD, CEV-LD and SEV-LD, and CEV-ND and SEV-ND establishes that the addition of  $\beta_1 X_t$  as in  $\sigma[X_t] = \sqrt{\beta_1 X_t + \beta_2 X_t^{\beta_3}}$  provides additional flexibility in fitting VIX dynamics. In each comparison, the  $\mathcal{L}^*$  statistic ranges between 6.86 and 9.66 which is highly significant given  $\chi^2[1] = 3.84$ . Regardless of the functional form of the drift specification, this result can be interpreted to mean that the shape of  $\sigma[X] = \sqrt{\beta_1 X_t + \beta_2 X_t^{\beta_3}}$  is statistically more attractive than  $\sigma[X] = \beta_2 X_t^{\beta_3}$  in reconciling the path of the VIX index. The estimate of  $\beta_1$  lies between 0.0141 and 0.0168 with a standard error of 0.0050 or 0.0051, making  $\beta_1$  significant across all models in the SEV class.

Since the estimated value of  $\beta_0$  is zero in the SEV class, the volatility function of market variance approaches zero as the variance itself approaches zero. However, the significant positive value of  $\beta_1$  indicates that the volatility of the index variance process approaches zero at a lower rate than that in the CEV specifications. At higher variance levels, the volatility function in the SEV specifications increases faster than that in the CEV specifications. So, the SEV specifications accommodate greater volatility at both low and high variance levels. For the SEV specifications, the linear term  $\beta_1 X$  becomes more pronounced than the nonlinear term  $\beta_2 X^{\beta_3}$  for  $X < (\beta_1/\beta_2)^{\frac{1}{\beta_3-1}}$ . In the region  $X < 0.0136$  in SEV-ND, the linear term  $\beta_1 X$  is more heavily weighted than the nonlinear component  $\beta_2 X^{\beta_3}$  and vice-versa. Therefore,  $\beta_1$  determines the behavior of the volatility function at low variance levels.

Results from SEV-CD, SEV-LD, and SEV-ND strengthen our earlier conclusions from CEV models that support the existence of nonlinearities in the drift function. As seen, the market

variance data prefers SEV-ND over both SEV-CD and SEV-LD, while SEV-CD and SEV-LD are indistinguishable based on the log-likelihoods and log-likelihood ratio statistics. More precisely, the  $\mathcal{L}^*$  is 11.72 between SEV-CD and SEV-ND and 10.48 between SEV-LD and SEV-ND. The estimated  $\alpha_2$  of -166.72 (-209.35) for the CEV-ND (SEV-ND) model is intuitive and guarantees a negative drift function in periods of pronounced market volatility. In the reverse when the market variance is low,  $\mu[X]$  should contain a positive drift which dictates the positivity of  $\alpha_3$  and assures that zero is unattainable. The coexistence of statistically significant  $\alpha_2 < 0$  and  $\alpha_3 > 0$  suggests that mean-reversion and nonlinearity of the drift function are robust phenomena in index volatility markets.

Elasticity parameters obtained from CEV and SEV models exhibit a mutually coherent mapping: the  $\beta_3$  of SEV is slightly more than twice the CEV counterpart. The following interpretation holds for estimated  $\beta_3$  of SEV-ND. With  $\beta_3 = 2.8822$ , the variance function  $\sigma^2[X]$  is convex in variance, that is,  $\frac{\partial \sigma^2[X;\theta]}{\partial X} = \beta_1 + \beta_2 \beta_3 X^{\beta_3-1} > 0$  and  $\frac{\partial^2 \sigma^2[X;\theta]}{\partial X^2} = \beta_2 \beta_3 (\beta_3 - 1) X^{\beta_3-2} > 0$ . The one-sided t-test overwhelmingly rejects the null hypothesis that  $\beta_3 \leq 2$ . This property has the implication that a well-performing variance process should have its variance function increasing at a rate faster than the level of market variance. Continuous-time volatility models violating such a property are likely headed for inconsistencies in the empirical dimension.

Because the most general specification, SEV-ND, exemplifies nonlinearity in the drift and diffusion function of the type described in (26)-(27), it rank-orders first with  $-n/2$  AIC of 12092.07, followed next by SEV-CD and SEV-LD with  $-n/2$  AIC of 12089.21 and 12088.83. Our density-based estimation implies that the linear AFF model displays unsatisfactory goodness-of-fit statistics and has dynamics most inconsistent with the observed movements in the VIX index. While SEV-ND admits a more complex representation, the nonlinearity parameters are vital to generating a more realistic time-evolution of market volatility.

To further analyze the structure of market volatility, Figure 1 plots the drift,  $\mu[X]$ , and diffusion,  $\sigma^2[X]$  function for the SEV-ND model and compares them to the short interest rate counterparts in Figures 4B and 4C of Ait-Sahalia (1996). Accounting for differences in estimation methods and parameter values for  $\theta$  in our Table 2 and Table 4 in Ait-Sahalia (1996), this exercise nonetheless shows that market volatility and interest rate share a closely related parametric structure. First, the shapes of  $\mu[X]$ , and diffusion,  $\sigma^2[X]$  for market volatility and interest rates generally resemble one another. However, the visual impression suggests that the mean-reversion in the market volatility may be relatively stronger, especially in the tails. Second, it appears the exponent parameter,  $\beta_3$ , is instrumental to the volatility specification of both interest-rate and market variance. While Ait-Sahalia (1996) reports  $\beta_3$  close to 2.0, the corresponding estimate from market volatility is 40 percent larger around 2.88. For an economic justification behind the shapes

of  $\mu[X]$  and  $\sigma^2[X]$  for the SEV-ND model, we refer the reader to the discussions in Ait-Sahalia (1996).

## 4 Concluding statements and summary

Density approximation is an issue whenever the researcher needs the state-price density for pricing purposes or for constructing the likelihood function in maximum-likelihood estimation. Building on Ait-Sahalia (1999, 2002) we have provided a method to approximate the transition density, and studied the empirical implications of estimating a much wider class of equity volatility models.

Our theoretical contribution in Proposition 1 and Proposition 2 shows how to express the recursively defined expansion coefficients in terms of one-dimensional integrals. To enhance the appeal of the methodology, the density approximation is derived in terms of the drift and diffusion function of the original state variable. This is done without incurring burdensome integration steps to reduce the state variable dynamics to a unit-variance process. Proposition 3 provides a technical treatment of the case when the necessary integrals can be Taylor series approximated and results in a solution for the expansion coefficients and the transition density. We illustrate the power of the methodology by deriving the density approximation for the general continuous-time model presented in Ait-Sahalia (1996).

Novel to the literature on equity volatility, our empirical results substantiate variance dynamics with nonlinear mean-reversion. Strong statistical evidence exists to support the presence of a nonlinear diffusion coefficient structure: the variance of variance function is composed of a term linear in variance plus a power function term in variance with an exponent that demands a value greater than two. The combined continuous-time variance model with the said properties furnishes reasonable goodness-of-fit statistics and produces superior performance metrics relative to its nested variants.

The method can be adapted for undertaking risk measurements under nonlinear drift and diffusion forcing processes and for approximating the risk-neutral densities for valuing contingent claims when the characteristic function of the state-price density is unavailable (Bakshi and Madan (2000) and Duffie, Pan, and Singleton (2000)). Suppose the log stock price is  $X_t \equiv \log(\mathcal{P}_t)$ . Then Ito's lemma implies  $dX_t = (r - \frac{1}{2}\sigma[e^X; \theta]) dt + \sigma[e^X; \theta] dW_t$ . With an analytical risk-neutral density approximation  $q_X^{(K)}[X]$ , the European option price with strike price  $\mathcal{K}$  and maturity  $T$  can be computed as  $e^{-rT} \int_{\log(\mathcal{K})}^{+\infty} (e^{X_{t+T}} - \mathcal{K}) q_X^{(K)}[X_{t+T}] dX_{t+T}$ . The field of density approximation is amenable to analytical characterizations for estimation and pricing under a large class of non-standard continuous-time models.

## Appendix A: Proof of Proposition 2

To outline a proof of the proposition, consider the determination of  $c_2[y]$ :

$$\begin{aligned} c_2[y] &= \frac{2}{(y-y_0)^2} \int_{y_0}^y (w-y_0) \left( \lambda[w] c_1[w] + \frac{1}{2} \frac{\partial^2 c_1[w]}{\partial w^2} \right) dw \\ &= \frac{2}{(y-y_0)^2} \int_{y_0}^y \lambda[w] \left( \int_{y_0}^w \lambda[z] dz \right) dw + \frac{1}{(y-y_0)^2} \int_{y_0}^y (w-y_0) \frac{\partial^2 c_1[w]}{\partial w^2} dw. \end{aligned} \quad (48)$$

The first part of (48) is equivalent to  $c_1^2[y]$  as proved below:

$$\begin{aligned} \frac{2}{(y-y_0)^2} \int_{y_0}^y \lambda[w] \left( \int_{y_0}^w \lambda[z] dz \right) dw &= \frac{2}{(y-y_0)^2} \int_{y_0}^y \left( \int_{y_0}^w \lambda[z] dz \right) d \left( \int_{y_0}^w \lambda[z] dz \right) \\ &= \frac{1}{(y-y_0)^2} \left( \int_{y_0}^w \lambda[z] dz \right)^2 \Big|_{y_0}^y \\ &= c_1^2[y]. \end{aligned} \quad (49)$$

Continuing, the second component of  $c_2[y]$  reduces, by a repeated application of integration by parts, to:

$$\int_{y_0}^y (w-y_0) \frac{\partial^2 c_1[w]}{\partial w^2} dw = (w-y_0) \frac{\partial c_1[w]}{\partial w} \Big|_{y_0}^y - c_1[w] \Big|_{y_0}^y. \quad (50)$$

Care must be exercised in evaluating the functions at the lower limit  $y_0$ . For example,  $(w-y_0)\partial c_1[w]/\partial w$  does not evaluate to zero at  $w=y_0$ . To this end we note,

$$\frac{\partial c_1[w]}{\partial w} = \frac{\lambda[w] - c_1[w]}{w - y_0}. \quad (51)$$

Consequently it then follows that

$$\int_{y_0}^y (w-y_0) \frac{\partial^2 c_1[w]}{\partial w^2} dw = \lambda[y] - 2c_1[y] - \lambda[y_0] + 2c_1[y_0] = \lambda[y] - 2c_1[y] + \lambda[y_0], \quad (52)$$

where l'Hôpital's rule implies  $c_1[y_0] = \lambda[y_0]$ . Therefore,

$$c_2[y] = c_1^2[y] + \frac{1}{(y-y_0)^2} (\lambda[y] + \lambda[y_0] - 2c_1[y]). \quad (53)$$

Now consider  $c_3[y]$ , which is recursively defined as

$$c_3[y] = \frac{3}{(y-y_0)^3} \int_{y_0}^y (w-y_0)^2 \left( \lambda[w] c_2[w] + \frac{1}{2} \frac{\partial^2 c_2[w]}{\partial w^2} \right) dw. \quad (54)$$

For clarity consider each part of (54) separately. Using (53) and skipping intermediate steps,  $\frac{3}{(y-y_0)^3} \int_{y_0}^y (w-y_0)^2 \lambda[w] c_2[w] dw$  simplifies to:

$$\begin{aligned} & \frac{3}{(y-y_0)^3} \int_{y_0}^y (w-y_0)^2 \lambda[w] \left( c_1^2[w] + \frac{\lambda[w] + \lambda[y_0] - 2c_1[w]}{(w-y_0)^2} \right) dw \\ = & c_1^3[y] + \frac{3}{(y-y_0)^3} \int_{y_0}^y \lambda[w] (\lambda[w] + \lambda[y_0] - 2c_1[w]) dw. \end{aligned} \quad (55)$$

The second part of (54) has an analytical representation using integration by parts:

$$\begin{aligned} & \frac{3}{(y-y_0)^3} \int_{y_0}^y (w-y_0)^2 \left( \frac{1}{2} \frac{\partial^2 c_2[w]}{\partial w^2} \right) dw = \frac{3(w-y_0)^2}{2(y-y_0)^3} \frac{\partial c_2[w]}{\partial w} \Big|_{y_0}^y - \\ & \frac{3(w-y_0)c_2[w]}{(y-y_0)^3} \Big|_{y_0}^y + \frac{3}{(y-y_0)^3} \int_{y_0}^y c_2[w] dw. \end{aligned} \quad (56)$$

It is straightforward to show that

$$\frac{\partial c_2[w]}{\partial w} = -\frac{2c_2[w]}{w-y_0} + \frac{2}{w-y_0} \left( \lambda[w] c_1[w] + \frac{1}{2} \frac{\partial^2 c_1[w]}{\partial w^2} \right), \quad (57)$$

and it holds that

$$\frac{\partial^2 c_1[w]}{\partial w^2} = \frac{\lambda'[w]}{w-y_0} - \frac{2(\lambda[w] - c_1[w])}{(w-y_0)^2}. \quad (58)$$

By a basic application of l'Hôpital's rule at the lower limit  $y_0$

$$\begin{aligned} & \frac{3}{(y-y_0)^3} \int_{y_0}^y (w-y_0)^2 \left( \frac{1}{2} \frac{\partial^2 c_2[w]}{\partial w^2} \right) dw = \frac{3}{(y-y_0)^2} (\lambda[y] c_1[y] - 2c_2[y]) - \\ & \frac{3(\lambda[y] - c_1[y])}{(y-y_0)^4} + \frac{3\lambda'[y]}{2(y-y_0)^3} + \frac{3}{(y-y_0)^3} \int_{y_0}^y c_2[w] dw. \end{aligned} \quad (59)$$

Combining (55) and (59) we finally have

$$\begin{aligned} c_3[y] = & c_1^3[y] + \frac{3}{(y-y_0)^2} (c_1[y] (\lambda[y] + \lambda[y_0]) - 2c_2[y]) + \frac{3}{(y-y_0)^3} \left( \int_{y_0}^y \lambda^2[w] dw + \right. \\ & \left. \int_{y_0}^y c_2[w] dw - \int_{y_0}^y 2\lambda[w] c_1[w] dw + \frac{\lambda'[y]}{2} \right) - \frac{3(\lambda[y] - c_1[y])}{(y-y_0)^4}. \end{aligned} \quad (60)$$

Now

$$\begin{aligned}
\int_{y_0}^y c_2[w]dw &= \int_{y_0}^y \frac{2}{(w-y_0)^2} \int_{y_0}^w (z-y_0) \left( \lambda[z]c_1[z] + \frac{1}{2} \frac{\partial^2 c_1[z]}{\partial z^2} \right) dz dw = \\
&= \int_{y_0}^y \int_{y_0}^y \frac{2}{(w-y_0)^2} (z-y_0) \left( \lambda[z]c_1[z] + \frac{1}{2} \frac{\partial^2 c_1[z]}{\partial z^2} \right) 1(z < w) dz dw = \\
&= 2 \int_{y_0}^y (z-y_0) \left( \lambda[z]c_1[z] + \frac{1}{2} \frac{\partial^2 c_1[z]}{\partial z^2} \right) \left( \frac{1}{z-y_0} - \frac{1}{y-y_0} \right) dz = \\
&= 2 \int_{y_0}^y \left( \lambda[z]c_1[z] + \frac{1}{2} \frac{\partial^2 c_1[z]}{\partial z^2} \right) dz - (y-y_0) c_2[y] = \\
&= 2 \int_{y_0}^y \lambda[z] c_1[z] dz + \frac{\partial c_1[z]}{\partial z} \Big|_{y_0}^y - (y-y_0) c_2[y] = \\
&= 2 \int_{y_0}^y \lambda(z) c_1[z] dz + \frac{\lambda[y] - c_1[y]}{y-y_0} - \frac{1}{2} \lambda'[y_0] - (y-y_0) c_2[y]. \tag{61}
\end{aligned}$$

Plugging (61) into (60) and rearranging

$$\begin{aligned}
c_3[y] &= c_1^3[y] + \frac{3}{(y-y_0)^2} (c_1[y](\lambda[y] + \lambda[y_0]) - 3c_2[y]) + \\
&= \frac{3}{(y-y_0)^3} \left( \frac{\lambda'[y] - \lambda'[y_0]}{2} + \int_{y_0}^y \lambda^2[w] dw \right).
\end{aligned}$$

The proof of  $c_4[y]$  is rather unwieldy and omitted to save on space (available from the authors).

## Appendix B: Expressions for $c_j[y|y_0; \theta]$ in Proposition 3

$$c_1[y|y_0; \theta] = \lambda_0 + \frac{\lambda_1 D}{2} + \frac{\lambda_2 D^2}{6} + \frac{\lambda_3 D^3}{24} + \frac{\lambda_4 D^4}{120} + \frac{\lambda_5 D^5}{720} + \frac{\lambda_6 D^6}{5040}, \tag{62}$$

$$c_2[y|y_0; \theta] = c_1^2[y_0] + \frac{\lambda_2}{6} + \frac{\lambda_3 D}{12} + \frac{\lambda_4 D^2}{40} + \frac{\lambda_5 D^3}{180} + \frac{\lambda_6 D^4}{1008}, \tag{63}$$

$$\begin{aligned}
c_3[y|y_0; \theta] &= c_1^3[y_0] + \left( \frac{\lambda_1^2}{4} + \frac{\lambda_0 \lambda_2}{2} + \frac{\lambda_4}{40} \right) + \left( \frac{\lambda_1 \lambda_2}{2} + \frac{\lambda_0 \lambda_3}{4} + \frac{\lambda_5}{80} \right) D + \\
&= \left( \frac{3\lambda_2^2}{20} + \frac{\lambda_1 \lambda_3}{5} + \frac{3\lambda_0 \lambda_4}{40} + \frac{\lambda_6}{280} \right) D^2 + \left( \frac{5\lambda_2 \lambda_3}{48} + \frac{13\lambda_1 \lambda_4}{240} + \frac{\lambda_0 \lambda_5}{60} \right) D^3 + \\
&= \left( \frac{23\lambda_3^2}{1344} + \frac{11\lambda_2 \lambda_4}{420} + \frac{19\lambda_1 \lambda_5}{1680} + \frac{\lambda_0 \lambda_6}{336} \right) D^4 + \left( \frac{\lambda_3 \lambda_4}{120} + \frac{\lambda_2 \lambda_5}{192} + \frac{13\lambda_1 \lambda_5}{6720} \right) D^5 \\
&+ \left( \frac{43\lambda_4^2}{4320} + \frac{7\lambda_3 \lambda_5}{4320} + \frac{13\lambda_2 \lambda_6}{15120} \right) D^6, \tag{64}
\end{aligned}$$

$$c_4[y|y_0; \theta] = c_1^4[y_0] + \left( \lambda_0 \lambda_1^2 + \lambda_0^2 \lambda_2 + \frac{3\lambda_2^2}{20} + \frac{\lambda_1 \lambda_3}{5} + \frac{\lambda_0 \lambda_4}{10} + \frac{\lambda_6}{280} \right) +$$

$$\begin{aligned}
& \left( \frac{\lambda_3^2}{2} + 2\lambda_0\lambda_1\lambda_2 + \frac{\lambda_0^2\lambda_3}{2} + \frac{\lambda_2\lambda_3}{4} + \frac{3\lambda_1\lambda_4}{20} + \frac{\lambda_0\lambda_5}{20} \right) D + \\
& \left( \frac{11\lambda_1^2\lambda_2}{12} + \frac{3\lambda_0\lambda_2^2}{5} + \frac{4\lambda_0\lambda_1\lambda_3}{5} + \frac{23\lambda_3^2}{336} + \frac{19\lambda_2\lambda_4}{168} + \frac{3\lambda_1\lambda_5}{56} + \frac{\lambda_0\lambda_6}{70} \right) D^2 + \\
& \left( \frac{7\lambda_1\lambda_2^2}{15} + \frac{19\lambda_1^2\lambda_3}{60} + \frac{5\lambda_0\lambda_2\lambda_3}{12} + \frac{13\lambda_0\lambda_1\lambda_4}{60} + \frac{\lambda_3\lambda_4}{20} + \frac{\lambda_0^5\lambda_5}{30} + \frac{\lambda_2\lambda_5}{30} + \right. \\
& \left. \frac{11\lambda_1\lambda_6}{840} \right) D^3 + \left( \frac{13\lambda_2^3}{180} + \frac{3\lambda_1\lambda_2\lambda_3}{10} + \frac{5\lambda_0\lambda_2\lambda_3}{12} + \frac{13\lambda_0\lambda_1\lambda_4}{60} + \frac{23\lambda_0\lambda_3^2}{336} + \right. \\
& \left. \frac{19\lambda_2^2\lambda_4}{420} + \frac{11\lambda_0\lambda_2\lambda_4}{105} + \frac{11\lambda_1\lambda_6}{840} + \frac{23\lambda_4^2}{2880} + \frac{\lambda_0^2\lambda_6}{168} + \frac{19\lambda_0\lambda_1\lambda_5}{420} + \frac{19\lambda_3\lambda_5}{1440} + \right. \\
& \left. \frac{37\lambda_2\lambda_6}{5040} \right) D^4 + \left( \frac{\lambda_2^2\lambda_3}{15} + \frac{157\lambda_1\lambda_3^2}{3360} + \frac{181\lambda_1\lambda_2\lambda_4}{2520} + \frac{\lambda_0\lambda_3\lambda_4}{30} + \frac{79\lambda_1^2\lambda_5}{5040} + \right. \\
& \left. \frac{\lambda_0\lambda_2\lambda_5}{48} + \frac{7\lambda_4\lambda_5}{1800} + \frac{13\lambda_0\lambda_1\lambda_6}{1680} + \frac{137\lambda_3\lambda_6}{50400} \right) D^5 + \left( \frac{9\lambda_2\lambda_3^2}{448} + \frac{391\lambda_2^2\lambda_4}{25200} + \right. \\
& \left. \frac{79\lambda_1\lambda_3\lambda_4}{3600} + \frac{43\lambda_0\lambda_4^2}{10800} + \frac{139\lambda_1\lambda_2\lambda_5}{10800} + \frac{7\lambda_0\lambda_3\lambda_5}{1080} + \frac{71\lambda_5^2}{158400} + \right. \\
& \left. \frac{13\lambda_0\lambda_2\lambda_6}{3780} + \frac{281\lambda_4\lambda_6}{369600} \right) D^6. \tag{65}
\end{aligned}$$

In terms of  $\mu[X_0]$  and  $\sigma[X_0]$ , each required  $\lambda_i$  can be recursively derived as:

$$\lambda_0 = -(f^2[X_0] + f'[X_0]\sigma[X_0])/2, \tag{66}$$

$$\lambda_i = \lambda'_{i-1}\sigma[X_0], \quad i = 1, 2, 3, 4, 5, 6, \tag{67}$$

which completes the characterization of the density approximation.

## Appendix C: Expressions for $\lambda_j$ in the SEV-ND Model

To this end, let

$$S_0 = \sigma[X_0] = \sqrt{\beta_0 + \beta_1 X_0 + \beta_2 X_0^{\beta_3}}. \tag{68}$$

Then  $S_0^2 = \beta_0 + \beta_1 X_0 + \beta_2 X_0^{\beta_3} = V_0$ . Differentiation leads to the following derivatives of  $S_0$  with respect to  $X_0$ :

$$S_1 = V_1/(2S_0), \tag{69}$$

$$S_2 = (V_2 - 2S_1^2)/(2S_0), \tag{70}$$

$$S_3 = (V_3 - 6S_1S_2)/(2S_0), \tag{71}$$

$$S_4 = (V_4 - 8S_1S_3 - 6S_2^2)/(2S_0), \quad (72)$$

$$S_5 = (V_5 - 10S_1S_4 - 20S_2S_3)/(2S_0), \quad (73)$$

$$S_6 = (V_6 - 12S_1S_5 - 30S_2S_4 - 20S_3^2)/(2S_0), \quad (74)$$

$$S_7 = (V_7 - 14S_1S_6 - 42S_2S_5 - 70S_3S_4)/(2S_0), \quad (75)$$

where  $V_7 = (\beta_3 - 6)V_6/X_0$ .

Let  $F_0 = f[X] = \mu[X_0]/\sigma[X_0] - \sigma'[X_0]/2 = U_0/S_0$ . So  $S_0F_0 = U_0$ . Obeying the above successive differentiation rules we arrive at:

$$F_0 = U_0/S_0, \quad (76)$$

$$F_1 = (U_1 - F_0S_1)/S_0, \quad (77)$$

$$F_2 = (U_2 - 2F_1S_1 - F_0S_2)/S_0, \quad (78)$$

$$F_3 = (U_3 - 3F_2S_1 - 3F_1S_2 - F_0S_3)/S_0, \quad (79)$$

$$F_4 = (U_4 - 4F_3S_1 - 6F_2S_2 - 4F_1S_3 - F_0S_4)/S_0, \quad (80)$$

$$F_5 = (U_5 - 5F_4S_1 - 10F_3S_2 - 10F_2S_3 - 5F_1S_4 - F_0S_5)/S_0, \quad (81)$$

$$F_6 = (U_6 - 6F_5S_1 - 15F_4S_2 - 20F_3S_3 - 15F_2S_4 - 6F_1S_5 - F_0S_6)/S_0, \quad (82)$$

$$F_7 = (U_7 - 7F_6S_1 - 21F_5S_2 - 35F_4S_3 - 35F_3S_4 - 21F_2S_5 - 7F_1S_6 - F_0S_7)/S_0, \quad (83)$$

where  $U_6 = 720\alpha_3/X_0^7 - V_7/4$ ,  $U_7 = -5040\alpha_3/X_0^8 - V_8/4$  and  $V_8 = (\beta_3 - 7)V_7/X_0$ .

We can now conveniently write

$$H_0 = -(f^2[X_0] + f'[X_0]\sigma[X_0])/2 = -(F_0^2 + F_1S_0)/2. \quad (84)$$

The first six derivatives of  $H_0$  with respect to  $X_0$  are:

$$H_1 = -(2F_0F_1 + F_2S_0 + F_1S_1)/2, \quad (85)$$

$$H_2 = -(2F_1^2 + 2F_0F_2 + 2F_2S_1 + F_3S_0 + F_1S_2)/2, \quad (86)$$

$$H_3 = -(6F_1F_2 + 2F_0F_3 + F_4S_0 + 3F_3S_1 + 3F_2S_2 + F_1S_3)/2, \quad (87)$$

$$H_4 = -(6F_2^2 + 8F_1F_3 + 2F_0F_4 + F_5S_0 + 4F_4S_1 + 6F_3S_2 + 4F_2S_3 + F_1S_4)/2, \quad (88)$$

$$H_5 = -(20F_2F_3 + 10F_1F_4 + 2F_0F_5 + F_6S_0 + 5F_5S_1 + 10F_4S_2 + 10F_3S_3 + 5F_2S_4 + F_1S_5)/2, \quad (89)$$

$$H_6 = -(20F_3^2 + 30F_2F_4 + 12F_1F_5 + 2F_0F_6 + F_7S_0 + 6F_6S_1 + 15F_5S_2 + 20F_4S_3 + 15F_3S_4 + 6F_2S_5 + F_1S_6)/2. \quad (90)$$

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**TABLE 1: Maximum Absolute Errors of the Density Approximation**

Maximum absolute errors of the approximations are based on  $K = 1, 2, 3, 4, 5$  and the Euler approximation. The approximate density is based on (20) with  $c_j[\gamma[x]|\gamma[x_0];\theta]$  presented in Proposition 2. Entries corresponding to  $\max(p|x_0)$  are the maximum conditional density. Computations involving Panels A and B use  $\alpha_0 = 0.145 \times 0.0732$ ,  $\alpha_1 = -0.145$ ,  $\beta_0 = 0$ , and  $\beta_1 = 0.06521^2$ . For the calculations performed in Panels C and D the initial stock price is fixed at  $x_0 = \$50$ , the initial volatility level is  $\beta_2 x_0^{\beta_3 - 1} = 0.3$ , and  $\alpha_1$  and  $\beta_2$  are allowed to vary. Tabulating the results by changing the elasticity of volatility coefficient  $\beta_3$  provided the same conclusions and, therefore, omitted.

<b>Panel A:</b> $dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta_1 X_t} dW_t, t = 1/12$									
$x_0$	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
$\max(p x_0) (10^1)$	15.0	10.7	8.71	7.55	6.75	6.17	5.71	5.34	5.04
$K = 1 (10^{-3})$	6.46	1.32	0.67	0.36	0.18	0.32	0.15	1.13	2.70
$K = 2 (10^{-6})$	143	10.3	3.90	0.98	0.82	2.71	3.86	2.54	3.96
$K = 3 (10^{-8})$	89.7	4.11	1.34	0.26	0.31	0.15	1.36	2.83	3.26
$K = 4 (10^{-10})$	90.9	5.35	0.36	0.14	0.26	0.33	0.21	0.75	2.06
$K = 5 (10^{-12})$	153	8.27	6.57	5.95	8.55	9.82	14.9	5.94	5.98
Euler	7.17	3.73	2.55	1.93	1.59	1.35	1.17	1.04	0.94
<b>Panel B:</b> $dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta_1 X_t} dW_t, t = 1$									
$x_0$	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
$\max(p x_0) (10^1)$	4.39	3.25	2.69	2.35	2.11	1.94	1.80	1.68	1.59
$K = 1 (10^{-2})$	28.7	7.19	2.61	1.78	0.81	1.35	1.19	4.83	11.1
$K = 2 (10^{-3})$	79.8	4.96	2.23	0.71	0.50	1.35	2.00	1.83	2.45
$K = 3 (10^{-4})$	51.1	4.45	0.78	0.29	0.17	0.20	0.83	1.71	2.25
$K = 4 (10^{-6})$	117	41.7	5.1	0.67	1.44	2.27	1.81	5.38	14.4
$K = 5 (10^{-7})$	154	9.85	4.23	0.61	0.26	0.70	1.70	2.00	3.43
Euler	8.21	4.70	3.42	2.76	2.35	2.07	1.87	1.72	1.60
<b>Panel C:</b> $dX_t = \alpha_1 X_t + \beta_2 X_t^{\beta_3} dW_t, t = 1/12$									
$\alpha_1$	0.04	0.04	0.04	0.06	0.06	0.06	0.08	0.08	0.08
$\beta_2$	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90
$\max(p x_0) (10^{-2})$	9.20	9.22	9.24	9.19	9.20	9.22	9.18	9.19	9.21
$K = 1 (10^{-8})$	1.48	0.55	0.11	15.05	6.26	1.12	47.50	22.69	6.24
$K = 2 (10^{-11})$	4.26	0.17	0.09	3.41	18.33	0.04	19.28	6.56	1.49
$K = 3 (10^{-13})$	1.56	0.32	2.83	2.64	0.36	2.31	7.10	1.24	2.02
$K = 4 (10^{-14})$	2.23	3.22	28.28	1.50	2.12	2.31	1.37	3.43	20.42
$K = 5 (10^{-14})$	2.28	3.22	28.28	1.54	2.12	2.31	1.34	3.42	20.42
Euler ( $10^{-3}$ )	2.77	3.93	5.11	2.73	3.88	5.05	2.69	3.83	5.00
<b>Panel D:</b> $dX_t = \alpha_1 X_t + \beta_2 X_t^{\beta_3} dW_t, t = 1$									
$\alpha_1$	0.04	0.04	0.04	0.06	0.06	0.06	0.08	0.08	0.08
$\beta_2$	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90
$\max(p x_0) (10^{-2})$	2.63	2.68	2.75	2.59	2.63	2.70	2.55	2.58	2.64
$K = 1 (10^{-7})$	8.19	2.51	0.49	65.17	26.24	4.66	206.58	95.23	25.89
$K = 2 (10^{-9})$	28.51	0.85	0.32	19.05	1.51	0.16	125.72	37.89	7.80
$K = 3 (10^{-10})$	13.20	0.05	0.00	17.29	1.46	0.03	43.33	6.92	0.11
$K = 4 (10^{-12})$	63.43	0.24	0.00	55.15	6.52	0.01	49.19	8.88	0.19
$K = 5 (10^{-14})$	456.61	0.57	0.39	0.04	1.16	0.52	325.76	2.23	0.37
Euler ( $10^{-3}$ )	2.82	4.14	5.53	2.68	3.94	5.30	2.54	3.76	5.07

**TABLE 2: Estimation Results for Market Index Variance**

The encompassing model, **SEV-ND**, accommodates stochastic elasticity of variance and a non-linear drift function of the type:  $dX_t = (\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}) dt + \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}} dW_t$ . Nesting the constant drift, CEV-CD, and linear drift, CEV-LD, the constant elasticity of variance with non-linear drift, **CEV-ND**, is  $dX_t = (\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}) dt + \beta_2 X_t^{\beta_3} dW_t$ . The **AFF** model is  $dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sqrt{\beta_0 + \beta_1 X_t} dW_t$ . Throughout  $\beta_0 \equiv 0$  as the estimate of  $\beta_0$  was zero and has no impact on log-likelihood. The market index volatility is proxied by the daily VIX index. The daily data is sampled over the period of July 1, 1988 to January 10, 2000 (2907 observations). We take  $X_t = \text{VIX}_t^2$  and the VIX series is scaled by 100 to convert it into a decimal. The approximate density is analytical and displayed in (20) with  $c_j[\gamma[x]|\gamma[x_0]; \theta]$  presented in Proposition 2. Reported volatility model parameters and standard errors (in parenthesis) are based on maximizing the log-likelihood  $\mathcal{L}[\theta] \equiv \sum_{i=1}^n \log \{p_X[\Delta, X_{i\Delta}|X_{(i-1)\Delta}; \theta]\}$ . The Akaike Information Criterion (AIC) is computed as  $-2/n(\mathcal{L}[\theta] - \dim[\theta])$ . Thus, a more properly specified model has higher  $-\frac{n}{2}\text{AIC}$ . Likelihood ratio test statistic for comparing nested models is  $\mathcal{L}^* \equiv -2 \times (\mathcal{L}[\theta_R] - \mathcal{L}[\theta_U]) \sim \chi^2[df]$ , where df is the number of exclusion restrictions and the 95% criterion values are  $\frac{\chi^2[df]}{\chi^2[df]}$  |  $\begin{matrix} 1 & 2 & 3 & 4 \\ 3.84 & 6.0 & 7.82 & 9.50 \end{matrix}$ .  $\dim[\theta]$  is reported in curly brackets below  $\mathcal{L}$ .

Model	$\mathcal{L}$	$-\frac{n}{2}$ AIC	$\alpha_0$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\beta_1$	$\beta_2$	$\beta_3$
AFF	11400.28 {3}	11397.28	0.3141 (0.0397)	-8.0369 (1.1977)			0.1827 (0.0000)		
CEV-CD	12089.65 {3}	12086.65	0.0664 (0.0114)					4.7825 (0.1365)	1.2781 (0.0214)
CEV-LD	12090.40 {4}	12086.40	0.0941 (0.0255)	-1.4607 (1.1939)				4.7046 (0.1365)	1.2732 (0.0217)
CEV-ND	12094.24 {6}	12088.24	-0.3400 (0.1846)	15.2476 (6.3934)	-166.7249 (60.5296)	0.0031 (0.0015)		4.7645 (0.1381)	1.2766 (0.0218)
GEN-CD	12093.21 {4}	12089.21	0.0654 (0.0117)				0.0142 (0.0050)	47.0436 (10.4937)	2.8302 (0.1183)
GEN-LD	12093.83 {5}	12088.83	0.0920 (0.0267)	-1.3540 (1.2159)			0.0141 (0.0051)	44.9773 (10.0673)	2.8161 (0.1187)
GEN-ND	12099.07 {7}	12092.07	-0.5537 (0.2162)	21.3224 (7.3014)	-209.3476 (68.6506)	0.0051 (0.0018)	0.0168 (0.0050)	53.9726 (12.4970)	2.8822 (0.1226)

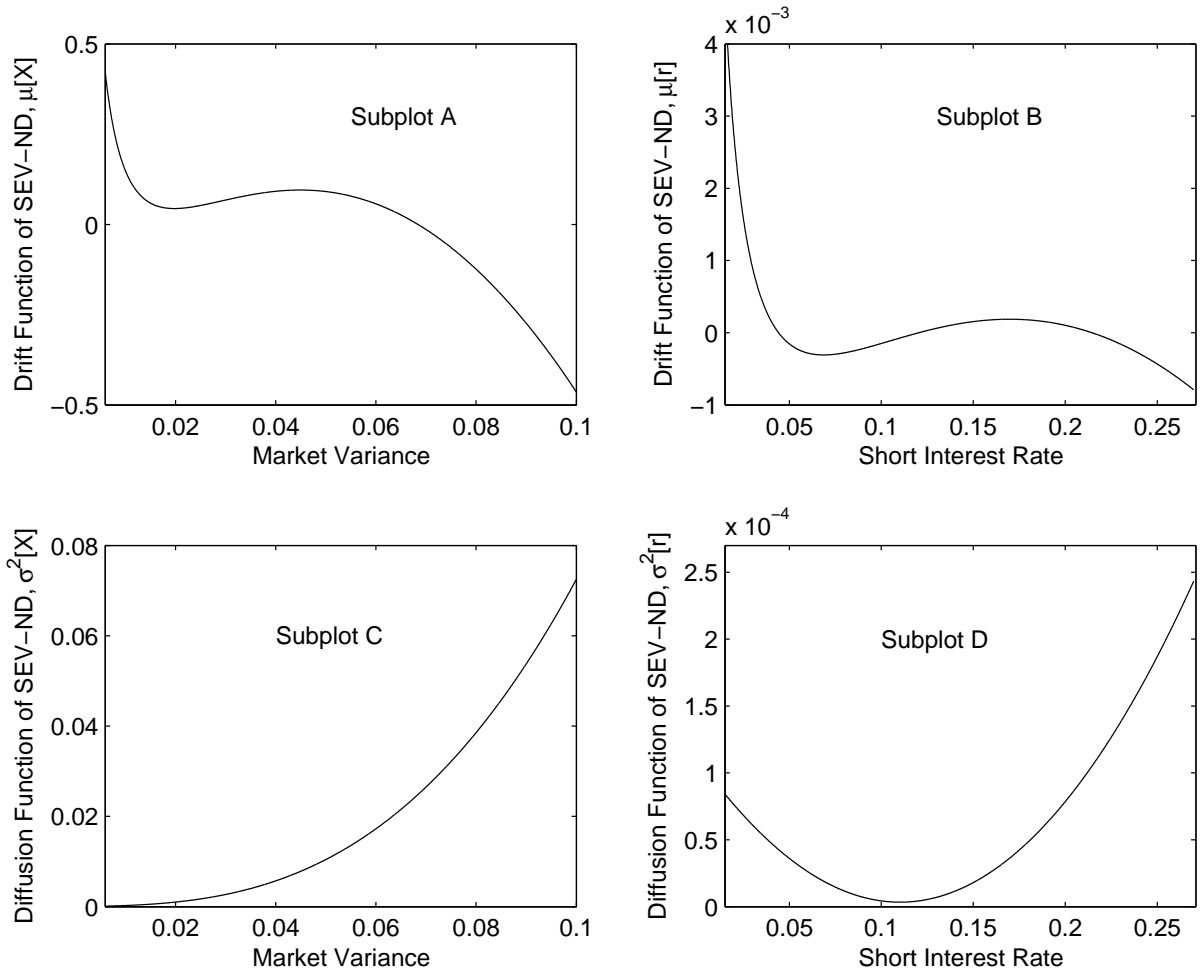


Figure 1: **Drift and diffusion functions for market variance and short interest-rate for the SEV-ND model.** Subplot A and B respectively plot the drift function,  $\mu[.]$ , when the market variance and the short interest-rate each follow the SEV-ND process:  $dX_t = (\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2 + \alpha_3 X_t^{-1}) dt + \sqrt{\beta_0 + \beta_1 X_t + \beta_2 X_t^{\beta_3}} dW_t$ . We plot the corresponding diffusion function,  $\sigma^2[.]$ , in Subplot C and D. All calculations for the market variance process are based on the maximum-likelihood estimation results reported in Table 2. The parameter estimates for the interest-rate process are taken from Table 4 in Aït-Sahalia (1996):  $\alpha_0 = -4.643 \times 10^{-3}$ ,  $\alpha_1 = 4.333 \times 10^{-2}$ ,  $\alpha_2 = -1.143 \times 10^{-1}$ ,  $\alpha_3 = 1.304 \times 10^{-4}$ ,  $\beta_0 = 1.108 \times 10^{-4}$ ,  $\beta_1 = -1.883 \times 10^{-3}$ ,  $\beta_2 = 9.681 \times 10^{-3}$ , and  $\beta_3 = 2.073$ . Thus, our interest-rate Subplot B and D coincide with Figures 4B and 4C in Aït-Sahalia (1996).