Differences of Opinion of Public Information and Speculative Trading in Stocks and Options

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Key Words: High Confidence, Low Confidence, Differential Interpretation of Public Information

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Abstract

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1 Introduction

Trading in exchange-listed securities, such as stocks and their options, is extremely active. In 2000, the average daily trading volume in the NYSE reached 1.04 billion shares for 43.9 billion US dollars. Trading volume in options is also huge. "Options trading is now the world's biggest business, with an estimated daily turnover of over 2.5 trillion US dollars and an annual growth rate of around 14%".[1] Given such a high trading volume, the following question arises naturally: What drives investors' trading in the securities market and the associated options market?

Empirically, Kandel and Pearson (1995) demonstrate that the trading volume of a stock can be positive even if the stock price does not change, and Huberman and Regev (2001) find that a reinterpretation of an old public news about a firm generates high trading volume not only for the concerned firm but also for other related firms. These studies present challenges to the extant theoretical literature in which the trading volume of a stock is typically related to its own price change and the trading volume involving multiple stocks is not considered. In addition, the theoretical literature on differences of opinion has analyzed neither the investors' perceived welfare gains in trading stocks and options, nor the difference between the volatility implied by the option price and the realized volatility of the stock price and their relations to the expected stock returns.

This paper analyzes the perceived value of options trading, the effects of options introduction on the trading volume of the underlying stock, as well as the difference between the two volatilities and their relations to the expected stock returns. In our model, investors have heterogeneous beliefs even when they observe the same public signal. The Milgrom-Stokey (1982) no-trade theorem does not apply as investors interpret public information differently from one another, and consequently, the beliefs about public signals do not satisfy the essentially concordance condition.

More specifically, in our model, investors have CARA utility and believe that the stock payoff distribution is normal. After the first round of trade, there are new public signals about the final stock payoff arriving at the market. Investors have the opportunity to trade again in the market. These new public signals create differences of opinion across investors because investors interpret them differently. Following Daniel, Hirshleifer and Subrahmanyam (1998) and Hong and Stein (2003), we assume that investors disagree on the quality of the signals and that they all believe that they know the precisions of the signals and update their expectations of the final

[1] From *Building the Global Market: A 4000 Year History of Options* by Edward Swan (2000).

stock payoff accordingly.[2] Because an investor's posterior expectation of the stock payoff is the precision weighted average of prior expectation and the signal, an investor who overweighs the precision of the public signal will update upward more than the average investor. We find that an investor's perceived value of additional rounds of trading is positive and increases with the absolute value of the difference between his perceived precision of the public signal and the average perceived precision of all investors. We also find that the trading volume of the underlying stock is proportional to the changes in its price. Moreover, if investors disagree on a public signal once, they will continue to trade even if they agree on the signals in all future periods.

When options are introduced, we show that investors who have higher conditional volatility (lower precision) about the stock payoff take long positions in options to synthesize convex payoffs, whereas investors who have lower conditional volatility short options to achieve concave payoffs.[3] We further show that the trading volume of the stock is related not only to its price change but also to lagged price changes, which implies that the trading volume is positive even when the stock price does not change. Both multiple trading and trading in options increase investors' perceived expected utility. The gains from trading in options are greater than those from trading in the underlying stock for high trading frequency. In particular, contrary to other models with dynamic trading [e.g., Brennan and Cao (1996)], the gains of trading options in our model cannot be approximated by frequent trading. Somewhat surprisingly, even when the differences of opinion become arbitrarily small [e.g., in the order of $1/\sqrt{T}$ with T going to infinity], and trading takes place arbitrarily frequently, the perceived gains in trading stocks approach zero but those in trading options remain strictly positive. In addition, we find that even if a public signal is arbitrarily noisy, the investors' perceived gains from options trading remain finite whereas those from stock trading converge to zero. In the presence of differences of opinion, it is demonstrated that options are not redundant assets and that it is important to introduce options in a trading model.[4]

While each investor is better off in his own view with respect to more frequent trading or the addition of options, others may view his expected gains differently. Considering two types of investors, we analyze investor i's gains in expected utility from investor j's point of view. We show that from investor j's point of view, investor i may not benefit from more frequent trading

[2]Note that observations of signals do not affect investors' view of the precisions because precisions are deterministic.

[3]Leland (1980) shows that investors with higher expected returns buy portfolio insurance, but he does not consider the case of different volatilities.

[4]For another model in which options are not redundant assets, see, for example, Liu and Pan (2003).

or the introduction of options.

Interestingly, the introduction of certain options can make the market complete.[5] Consequently, we show that the prices of all option claims on the underlying stock satisfy the "risk-neutral" pricing property of the Black and Scholes (1973) model and that the prices of all assets are determined as if there existed a representative agent. The representative agent's belief is equal to the average of all investors' beliefs. The implied volatility from the option price is the conditional volatility of the stock payoff estimated by this representative agent. Unless the representative agent's belief happens to be the correct one, the implied volatility will not be the realized volatility of the stock price. In other words, when investors are heterogenous, one may not back out the correct volatility of a stock from its option price. If the implied volatility is greater than the realized volatility or if the representative agent has a higher precision about the signals than the true precision of the signals, then the expected stock returns are positively autocorrelated over time. If the implied volatility is lower than the realized volatility or if the representative agent has a lower precision than the true precision of the signals, then the expected stock returns are negatively autocorrelated. We find that both the implied volatility and the realized volatility have explanatory power for the expected stock returns.

Our model also has the following empirical implications regarding trading volumes of the stock and options. First, trading volumes in options should be higher around the dates of public events, such as earnings announcements, mergers and acquisitions, and bond rating changes, because public information generates differences of opinion, which drives options trading. Second, when there are more differences of opinion about a stock's payoff, trading volumes in both the stock and its options should be higher because investors' demands on options depend on their beliefs about the volatility of the stock payoff.[6] Third, trading volumes are higher for stocks with options because investors use underlying stocks to hedge their positions in the options market. Fourth, trading volume for optioned stocks should be more responsive to concurrent and lagged price changes than non-optioned stocks due to additional hedging demands associated with options. In particular, for non-optioned stocks, one should find lower trading volume following public signals when the stock price does not change compared to optioned stocks.

We then extend the model to a multiple stock setting and show that the trading volume of a stock depends not only on its own stock price change but also on the price changes of related

[5] Yet trading occurs in every period as investors interpret new signals differently in every period.

[6] One proxy for differences of opinion is the dispersion of beliefs among financial analysts. It would be informative to determine whether trading volume in options is higher for stocks with more dispersion in financial analysts' forecasts. Another proxy is to use open interest in futures markets as a measure for differences of opinion. Bessembinder, Chan, and Seguin (1996) find that trading volume in stock index futures and NYSE are correlated with open interest in the indexs futures market.

stocks. It is also shown that even if there are no differences of opinion or no signals about a stock's payoff, there may still be trading in that stock due to differences of opinion about the payoffs of other related stocks. These results may shed light on the empirical findings of Kandel and Pearson (1995) and Huberman and Regev (2001). For example, the Kandel-Pearson result that the trading volume of a stock can be positive even if its price does not change arises in our multi-stock model as well as in our one-stock model with options. Again, the equilibrium asset prices equal to the prices that would arise in a representative-agent economy,[7] and the representative agent's belief is equal to the simple average belief of all investors. We show that the expected asset returns for the representative agent follow the Capital Asset Pricing Model (CAPM) and that this agent holds the market portfolio, which is on his efficient frontier. Suppose that the data represents the true belief of the stock payoff. Our result suggests that the CAPM will not be the correct description of the data unless the average belief happens to be the correct one.

This paper is related to Harris and Raviv (1993) and Kandel and Pearson (1995), both use differences of opinion to generate trades for a stock in the absence of options. They show that with two types of investors, differences of opinion can generate trading patterns consistent with stylized empirical evidence. In particular, Harris and Raviv show that trading volume is positive when there is a change in stock prices, whereas Kandel and Pearson show that trading can occur even when stock prices remain the same. The main difference between our work and those studies is that we focus on the trading volume of options as well as on the trading volume of a stock in a multiple stock environment, whereas they consider the relationship between the trading volume and the price change of the stock. Moreover, we analyze the value of both additional trading sessions and options and show that the gains from options trading cannot be approximated by frequent trading in the underlying stock. Our model also makes different predictions regarding the trading volume of the underlying stock. For example, Harris and Raviv predict that trading can occur only when investors have different interpretations about public signals in every period while we require that investors interpret signals differently in only one period to generate sustained trading. The presence of many types of investors generates additional empirical implications. Kandel and Pearson consider a two-period model and do not analyze the dynamic changes of trading volume of the underlying stock. Other studies that employ differences of opinion to generate trades include Harrison and Kreps (1978), Jarrow (1980), Varian (1989), DeTemple and Murphy (1994), Kraus and Smith (1996), Morris (1996),

[7]Our aggregation result with differences of opinion is similar to those of DeMarzo and Skiadas (1998) and Biais, Bossaerts, and Spatt (2003) with asymmetric information.

Biais and Bossaerts (1998), Odean (1998), Zapatero (1998), Basak (2000), Viswanathan (2001), Brav and Heaton (2002), Duffie, Garleanu, and Pederson (2002), Kyle and Lin (2002), Buraschi and Jiltsov (2003), David (2003), Hong and Stein (2003), Qu, Starks, and Yan (2003), and Scheinkman and Xiong (2003).

In short, the previous theoretical models on differences of opinion have considered neither the welfare implications of introducing options nor multiple stocks, and have not related the implied volatility to the expected stock return. Moreover, those models have not established specific relations both between trading volumes in options and differences of opinion and between the introduction of options and trading volumes in the underlying stock.

Our paper is also related to studies that employ noise traders/random endowments and asymmetric information to generate trades. These include Pfleiderer (1984), Kyle (1985), Admati and Pfleiderer (1988), Brown and Jennings (1989), Grundy and McNichols (1989), Kim and Verrecchia (1991), Holden and Subrahmanayam (1992), Spiegel and Subrahmanayam (1992), Back (1993), Foster and Viswanathan (1993), Shalen (1993), Biais and Hillion (1994), Wang (1994), He and Wang (1995), Brennan and Cao (1996), and Easley, O'Hara, and Srinivas (1998). In particular, Brennan and Cao consider an equilibrium model in the presence of options. They focus on the impact of introducing options on investors' welfare rather than on trading volume in the options market. They show that with the introduction of an appropriate option security, Pareto efficiency can be achieved in only a single round of trading,[8] and, as a result, investors will no longer trade in either the underlying stock or the option security in future rounds. To generate additional trading with the arrival of new public information, Brennan and Cao and all other works under asymmetric information rely on the introduction of additional noise/liquidity trading. A potential problem with this approach is that the argument to explain the trading volume is circular-it essentially requires new exogenous supply shocks to the stock to generate trading volume. In this sense, trading is imposed onto the economy rather than endogenously generated. For example, to generate trading around the earnings announcement dates, these studies need noise traders for the equilibrium to be partially revealing. However, Kandel and Pearson (1995) find no evidence that noise trading is particularly high around earnings announcements, and Pan and Poteshman (2003) find no asymmetrically informed trading in the index options market. On the other hand, trading in our model is driven endogenously by differential interpretation of public signals without the need to introduce exogenous noise

[8] Back (1993), Biais and Hillion (1994), and Easley, O'Hara, and Srinivas (1998) also analyze the effects of options when information is asymmetric. DeTemple and Selden (1991) analyze the effects of options under symmetric information.

traders.

The rest of this paper is organized as follows. Section 2 considers the benchmark case in the absence of options. Section 3 develops the full model of trading with options. Section 4 discusses the trading volumes in stocks and options. Section 5 develops a multi-stock model and Section 6 concludes the paper. The appendix contains the proofs.

2 Differences of Opinion and Trading in the Stock

In this section we consider trading in the stock due to differences of opinion in the absence of options. It is assumed that the financial market consists of a continuum of investors, each indexed by i where $i \in [0, 1]$. At time 0 each investor is endowed with x^i units of the stock and, to avoid unnecessary notation, we assume that individual endowments of the riskless bond are zero. Without loss of generality, the riskless interest rate is taken as zero. The risky stock pays off at time 1 an amount u, where u is normally distributed with mean $\bar{u}$ and precision h. The per capita supply of the stock is positive and denoted as x. Investor i has a negative exponential utility function defined over time 1 wealth, $U(W^i) = -\exp(-W^i/\tau)$, where τ represents the risk tolerance of investors.

As an introduction, we first consider the basic single-period model.

2.1 A Static One-Period Model

Let P_0 be the price of the stock and D_0^i be the demand of investor i for the stock. Investor i's time 1 wealth is given by $W^i = P_0 x^i + (u - P_0)D_0^i$. It is well known that in this setting [Sharpe (1964)] a linear equilibrium exists in which

$$P_0 = \bar{u} - \frac{x}{\tau h}, \tag{1}$$

$$D_0^i = \tau h[\bar{u} - P_0] = x. \tag{2}$$

Equation (1) expresses the equilibrium price as the expected payoff less a risk premium that depends on the per capita stock supply x. We assume that the values of $\bar{u}$, x, τ, and h are such that P_0 is positive.

The expected utility of investor i conditional on his endowment is given by

$$\mathrm{E}U^i = -\exp\left[\frac{-\bar{u}x^i}{\tau} + \frac{(x^i)^2}{2\tau^2 h} - \frac{(x^i - x)^2}{2\tau^2 h}\right]. \tag{3}$$

Investor i's wealth at time 1, W^i, is a linear function of the stock payoff u and may be written as

$$W^i(u) = P_0 x^i + x(u - P_0). \tag{4}$$

The marginal rate of substitution for investor i between wealth contingent on $u = u_l$ and $u = u_k$ is given by

$$\begin{aligned} M^i_{kl} &= \frac{\exp\{-h(u_k - \bar{u})^2/2 - W^i(u_k)/\tau\}}{\exp\{-h(u_l - \bar{u})^2/2 - W^i(u_l)/\tau\}} \\ &= \exp\left\{-\frac{1}{2}h(u_k - u_l)(u_k + u_l - 2P_0)\right\}. \end{aligned} \tag{5}$$

Because the marginal rate of substitution is the same for all investors, this equilibrium is Pareto efficient.

2.2 A Dynamic Model

In this subsection we extend the single-period model to allow for additional market sessions between time 0 and time 1, at which point the stock payoff is realized and consumption occurs. Immediately before each market session, a public signal about the stock payoff arrives. Note that the one-period equilibrium allocation is Pareto efficient. According to Milgrom and Stokey (1982), there should be no more trading after the first round when new information about the final stock payoff becomes publicly available. However, the Milgrom and Stokey theorem holds only when investors have essentially concordant beliefs about the public information. When investors' beliefs are not essentially concordant, trading among investors can occur with the arrival of public information.

2.2.1 Equilibrium Price and Demand

Specifically, consider a setting in which information about the final payoff u is made available gradually by a series of public signals $y_t = u + \eta_t$ at time $\tau_t = t/T, t = 1, \ldots, T-1$, where η_t is independently and normally distributed. To generate option trading in a tractable manner, we assume that all investors believe that η_t has a mean 0, but they disagree on the precision of η_t. This assumption is adapted from Harris and Raviv (1993), Kandel and Perason (1995), Daniel, Hirshleifer, and Subrahmanayam (1998), and Hong and Stein (2003), who assume that investors have differential confidence about signals. For example, IBM makes a public announcement about its earnings. Some investors believe that this announcement is more accurate than others. In general, different investors have different beliefs about the accuracy of public signals. In particular, investor i believes that η_t has a mean zero and a precision n_t^i. Let $n_t \equiv \int_0^1 n_t^i$ denote the average precision of the public signal. We define the concepts of high confidence and low confidence as follows.

Definition 1 *Let $\rho_t^i \equiv n_t^i/n_t$. When $\rho_t^i > 1$, we define that investor i has high confidence about the public signal at time t. When $\rho_t^i < 1$, we define that investor i has low confidence about the public signal at time t.*

After each signal the market opens for trading, and at time 1, the payoff of the stock is realized and consumption occurs. Let P_t denote the price of the stock at time τ_t. Trader i's optimal demand for the stock at time τ_t is denoted by D_t^i. A dynamic equilibrium is described in the following theorem. Its proof and all other proofs are given in the appendix.

Theorem 1 *In an economy with T trading sessions, there exists a dynamic equilibrium in which prices, and demands for the stock, are given by*

$$P_t = \mu_t - x/(\tau K_t), \tag{6}$$

$$D_t^i = \tau K_t^i[\mu_t^i - P_t] = \tau \left[h\bar{u} + \sum_{j=0}^{t} n_j^i y_j - K_t^i P_t \right], \tag{7}$$

where

$$\mu_t \equiv \frac{h\bar{u} + \sum_{j=0}^t n_j y_j}{h + \sum_{j=0}^t n_j}, \quad \mu_t^i \equiv \frac{h\bar{u} + \sum_{j=0}^t n_j^i y_j}{h + \sum_{j=0}^t n_j^i},$$

$$K_t^i \equiv (\text{Var}_t^i[u])^{-1} = h + \sum_{j=0}^{t} n_j^i, \text{ and } \quad K_t \equiv \int_0^1 K_t^i di = h + \sum_{j=0}^{t} n_j.$$

Here μ_t and μ_t^i denote the conditional expectations of the stock final payoff u for the average investor and investor i at time τ_t, respectively. K_t^i denotes the conditional precision of u for investor i and K_t denotes the average precision of all investors.

Because investor i is risk averse, his demand for the risky stock increases with his precision about the signal as well as his conditional mean of the final payoff, as in a typical mean-variance framework. An interesting feature of the equilibrium is that the stock price depends on the average investor's conditional mean and conditional precision of the stock payoff. The average investor does not buy or sell in equilibrium and serves as the marginal investor. We next examine the conditions under which trading takes place in equilibrium.

2.2.2 Equilibrium Dynamic Trading

Let $\Delta P_t \equiv P_t - P_{t-1}$ denote the price change in trading session t. In equilibrium, equation (7) yields

$$\Delta D_t^i \equiv D_t^i - D_{t-1}^i = \tau \left(\frac{n_t^i}{K_{t-1}^i} - \frac{n_t}{K_{t-1}} \right) \frac{n_t K_{t-1}}{K_{t-1}^i} \Delta P_t. \tag{8}$$

Notice that investor i's posterior expectation is his precision weighted average of prior expectation and the signal at time t. The ratio $\frac{n_t^i}{K_{t-1}^i}$ represents the relative weight between the prior expectation and the signal. When $\left(\frac{n_t^i}{K_{t-1}^i} - \frac{n_t}{K_{t-1}} \right) = 0$, investor i places the same weight on the public signal as the average investor. As a result, there will be no trading. When $\left(\frac{n_t^i}{K_{t-1}^i} - \frac{n_t}{K_{t-1}} \right) \neq 0$, investor i will trade at trading session t if the stock price changes. We discuss two sufficient conditions that generate trades among investors. In the first case, $K_{t-1}^i = K_{t-1}$ but $n_t^i \neq n_t$ for all i. $\left(\frac{n_t^i}{K_{t-1}^i} - \frac{n_t}{K_{t-1}} \right) = \frac{n_t^i - n_t}{K_{t-1}} \neq 0$. Investors agree on how to interpret information prior to trading session t but disagree at trading session t. The Milgrom-Stokey (1982) theorem no longer holds due to differential interpretation of the public signal at trading session t. In the second case, $K_{t-1}^i \neq K_{t-1}$ but $n_t^i = n_t$. We have $\frac{n_t^i}{K_{t-1}^i} - \frac{n_t}{K_{t-1}} = \frac{n_t}{K_{t-1}^i} - \frac{n_t}{K_{t-1}}$. Investors agree on how to interpret information at trading session t but disagree on prior interpretations. In this case, the Milgrom-Stokey theorem again fails because investors' allocation is no longer Pareto optimal, conditional on investors' holdings at trading session $(t-1)$. This can be shown by examining the marginal rate of substitution after trading session $(t-1)$.

The marginal rate of substitution for investor i between wealth contingent on stock payoffs $u = u_l$ and $u = u_k$ is given by

$$\begin{aligned} M_{(t-1)kl}^i &= \frac{\exp\{-K_{t-1}^i (u_k - \mu_{t-1}^i)^2/2 - D_{t-1}^i u_k/\tau\}}{\exp\{-K_{t-1}^i (u_l - \mu_{t-1}^i)^2/2 - D_{t-1}^i u_l/\tau\}} \\ &= \exp\left\{ -\frac{1}{2} K_{t-1}^i (u_k - u_l)(u_k + u_l - 2P_{t-1}) \right\}. \end{aligned} \tag{9}$$

Because the marginal rate of substitution is investor specific through K_{t-1}^i, the precision of the investor's posterior beliefs, this equilibrium is not Pareto efficient. Consequently, the Milgrom-Stokey result does not apply, even when there is no disagreement about future public signals. In other words, as long as investors disagree on past signals, these previous disagreements will generate trades in future periods even if all investors agree on all future public signals. Thus we have the following result regarding the persistence of trading.

Proposition 1 *If investors disagree on the public signal for only one period and agree on the signals before and after that period, then they will continue trading after that period, but the equilibrium allocation will not be Pareto optimal.*

This proposition indicates that to generate trading, investors need to disagree on the public signals only once. Thereafter, they will trade even if they agree on all new public signals in future periods.

Proposition 1 differs from the result of Harris and Raviv (1993), who show that trade occurs only when investors disagree on the public signal in that period. Our model does not require investors to disagree in every period to generate trading. This difference arises from the fact that they assume that investors are risk neutral and that there are short sales constraints, whereas we assume that investors are risk averse and that there are no portfolio constraints.[9]

We next apply the results of Theorem 1 to calculate the perceived gains in the expected utility from the increased trading opportunities.

2.2.3 Perceived Value of Dynamic Trading

In the following lemma, investor i's expected utility conditional on his endowment but before observing the public signals, $\mathrm{E}U^i(T)$, is expressed as a function of his individual and the market average precisions in each of the T trading sessions.

Lemma 1

$$\mathrm{E}U^i(T) = -\prod_{t=1}^{T} \sqrt{\frac{K_{t-1}^i n_t^i}{K_{t-1}^i n_t^i + (n_t K_t^i - n_t^i K_t)^2 / K_t^2}}\,\mathrm{E}U^i(1), \tag{10}$$

where $\mathrm{E}U^i(1)$ *is the expected utility of investor* i *in a one-period economy in which investors can trade only once at time 0.*

An investor's expected utility in our model can be expressed as the product of the expected utility without any future trade and a term that corresponds to the utility gain from future trading opportunities. This permits a simple calculation of the monetary value of trading

[9] When investors are risk neutral, as in Harris and Raviv (1993), they will take unbounded positions in the absence of portfolio constraints.

opportunities. Define $\gamma^i(T)$ as the difference (for investor i) between the certainty equivalent wealth of T market sessions and that of one market session, that is,

$$\gamma^i(T) \equiv -\tau \ln[EU^i(T)/EU^i(1)].$$

The certainty equivalent gain, $\gamma^i(T)$, is the maximum amount that investor i would be willing to pay to have T trading sessions rather than simply one trading session at time 0. Comparing the expression for the expected utility in the single-session economy in Subsection 2.1 with that for the T-session economy in Lemma 1, we obtain that $\gamma^i(T)$ is given by

$$\gamma^i(T) = \frac{\tau}{2}\sum_{t=1}^{T} \ln\left[1 + \frac{(n_t K_t^i - n_t^i K_t)^2}{n_t^i K_t^2 K_{t-1}^i}\right]. \tag{11}$$

Expression (11) implies that the gain from additional trading sessions is positive and it increases with the number of additional trading sessions. The magnitude of the utility gains depends on the rate at which the individual and the market average posterior precisions, K_t and K_t^i, evolve between market sessions, as well as the time pattern of the public signal precisions, n_t and n_t^i. The following corollary arises naturally from equation (11).

Corollary 1 *Adding additional trading sessions makes investors believe that they are better off.*

Expression (11) is quite complex. To simplify the exposition, we assume in the rest of the paper that the updating ratio $\rho_t^i \equiv n_t^i/n_t$ is time invariant, that is, $\rho_t^i = \rho^i$ for all t. We next discuss the limiting behavior of the gains from trade as the number of trading sessions goes to infinity. One would have thought that with infinite trading sessions, the perceived gains from trade will go to infinity. We show here that this is not the case or that as the number of trading sessions goes to infinity, the trading gains converge to a finite value.

Let $\Delta(1/K_t) \equiv (1/K_{t-1} - 1/K_t) = n_t/(K_{t-1}K_t)$, then $\Delta(1/K_t) \sim 0\left(\frac{1}{T}\right)$ represents the change in conditional volatility of the average investor.[10] Letting T go to infinity and assuming

[10]Based on the average investor's belief, we have $Var_a[P_t - P_{t-1}] \propto \Delta\left(\frac{n_t}{K_{t-1}K_t}\right) \sim 0\left(\frac{1}{T}\right)$.

that n_t is in the order of $1/T$ and $\Delta(1/K_t)/(1/T)$ is bounded from above, we have

$$\begin{aligned}
\gamma^i(T) &\approx \frac{\tau}{2}\sum_{t=1}^{T}\frac{(n_t h - n_t^i h)^2}{n_t^i K_t^2 K_{t-1}^i} = \frac{\tau(\rho^i-1)^2 h^2}{2\rho^i}\sum_{t=1}^{T}\frac{K_{t-1}}{K_{t-1}^i K_t}\Delta(1/K_t) \\
&= \frac{\tau(\rho^i-1)^2 h^2}{2\rho^i}\sum_{t=1}^{T}\frac{1/K_t}{\rho^i + (1-\rho^i)h(1/K_t + \Delta(1/K_t))}\Delta(1/K_t) \\
&\approx \frac{\tau(\rho^i-1)^2 h^2}{2\rho^i}\int_0^{1/h}\frac{q}{\rho^i + (1-\rho^i)hq}dq \\
&= \frac{\tau}{2\rho^i}\int_0^{1-\rho^i}\frac{v}{(v+\rho^i)}dv = \frac{\tau}{2\rho^i}[1-\rho^i + \rho^i \ln \rho^i], \qquad (12)
\end{aligned}$$

which shows that investor i's certainty equivalent gain is a function of the ratio between his perceived precision and that of the average investor, who has the average precision n_t. There is no trading and thus no gains for the average investor. Equation (12) implies the following corollary.

Corollary 2 *As the number of trading sessions goes to infinity, investor i's trading gains converge to a finite value. The certainty equivalent gain in the limit is a function of the updating ratio ρ^i, as given in equation (12), and is zero when the investor has the average precision $\rho^i = 1$.*

For comparison with the gains of trading options to be discussed later, we note that $\gamma^i(T)$ goes to zero as long as ρ^i converges to 1 or when the differences of opinion are arbitrarily small.

We next introduce options into the economy and study the effects of options on various equilibrium properties.

3 Effects of Options

3.1 Equilibrium Prices and Demands

The financial market is incomplete with one risky stock and one risk-free asset. Breeden and Litzenberger (1978) have shown that the market can be completed by the introduction of a complete set of options. We complete the market by introducing all call options with positive strike prices and all put options with negative strike prices. All options are in zero net supply. Any derivative asset with a twice differentiable price function $f(u)$ can be synthesized using a collection of options:

$$f(u) = f(0) + f'(0)u + \int_{-\infty}^{0} (Z-u)^+ f''(Z)dZ + \int_{0}^{\infty} (u-Z)^+ f''(Z)dZ,$$

where Z denotes the strike prices. The next theorem establishes the existence of an equilibrium with options and describes the demands and prices for both stock and options.

Theorem 2 *Let P_{CZt} denote the price of a call option with strike price Z in trading session t. Let D^i_{CZt} denote investor i's demand density of options at strike price Z, that is, the holdings of call options with strike price Z to $Z+dZ$ is given by $D^i_{CZt}dZ$. Define the price and demand for the put options similarly as P_{PZt} and D^i_{PZt}. Then there exists a dynamic equilibrium in which*

$$P_t = \mu_t - x/(\tau K_t), \tag{13}$$

$$D^i_t = \tau K^i_t[\mu^i_t - P_t] - \tau \left[(1-\rho^i)\frac{K_t^2}{n_{t+1}} + K_t - K^i_t \right] P_t, \tag{14}$$

$$P_{CZt} = (P_t - Z)N\left((P_t - Z)\sqrt{K_t}\right) + \frac{1}{\sqrt{K_t}} n\left((P_t - Z)\sqrt{K_t}\right), \tag{15}$$

$$P_{PZt} = (Z - P_t)N\left((Z - P_t)\sqrt{K_t}\right) + \frac{1}{\sqrt{K_t}} n\left((Z - P_t)\sqrt{K_t}\right), \tag{16}$$

$$D^i_{CZt} = D^i_{PZt} = \tau(1-\rho^i)\left[\frac{K_t K_{t+1}}{n_{t+1}} - h\right], \tag{17}$$

where

$$\mu_t \equiv \frac{h\bar{u} + \sum_{j=0}^t n_j y_j}{h + \sum_{j=0}^t n_j}, \quad \mu^i_t \equiv \frac{h\bar{u} + \sum_{j=0}^t n^i_j y_j}{h + \sum_{j=0}^t n^i_j},$$

$$K^i_t \equiv (\mathrm{Var}^i_t[u])^{-1} = h + \sum_{j=0}^{t} n^i_j, \text{ and } \quad K_t \equiv \int_0^1 K^i_t di = h + \sum_{j=0}^{t} n_j.$$

Theorem 2 shows that the options are not redundant securities. In this equilibrium, investors with high confidence ($\rho^i > 1$) take short positions in the options while investors with low confidence ($\rho^i < 1$) take long positions in the options.[11] Intuitively, investors with high confidence perceive a lower volatility for the stock, so they believe that options are overvalued. As

[11]Note that $(K_t K_{t+1}/n_{t+1} - h) > 0$. Hence, the sign of D^i_{CZt} is determined by the sign of $(1-\rho^i)$.

a result, they take short positions on options. Similarly, investors with low confidence perceive options to be undervalued, so they take long positions. Although investors achieve the Pareto optimal allocation, those whose precision about the public signal is different from that of the average investor will trade in the options in every period. In the presence of options, investors will trade in the underlying stock to hedge option positions even if the price of the underlying stock does not change. Indeed, it shall be shown that the trading volume of the underlying stock is positive even if the stock price remains unchanged. Notice that the average investor serves as the representative agent who prices both the stock and the options based on the average belief. This investor does not buy and sell the stock and holds no options in equilibrium.

Options prices are determined in a dynamic equilibrium. Recall that in the Black-Scholes (1973) partial equilibrium model in which the stock price is exogenously given, the option price always increases with the volatility of the stock price. Based on this partial equilibrium result, it has been argued that equity holders may have an incentive to increase the risk of the firm at the expense of the debt holders because the equity of a firm can be viewed as an option due to the limited liability of the equity holders. It has also been argued that a manager whose compensation consists of stock options may have an incentive to increase the risk of the firm. In our model, however, the effect of a higher volatility on the price of options is ambiguous. On the one hand, a higher volatility increases the option price. On the other hand, a higher volatility decreases the stock price, which lowers the option price. This means that the traditional view that managers tend to increase the risk of their firms due to asset substitution effects and their compensation effects may not be correct in an equilibrium model.

With normal stock payoff distribution and CARA utility, the Pareto efficient allocation is a quadratic function of the final payoff of the stock u. In equilibrium, investors use options to synthesize the appropriate payoffs that are quadratic functions of the stock payoff. Consequently, it is not necessary to introduce a continuum of call and put options to complete the market. We next show that a derivative asset with a payoff of $Q(u) = u^2$ can complete the market, yielding the following result.

Theorem 3 *Let P_{Qt} denote the price of the quadratic derivative asset and D^i_{Qt} denote the demand for the quadratic derivative asset by investor i in trading session t. Then there exists a dynamic equilibrium in which*

$$P_t = \mu_t - x/(\tau K_t), \tag{18}$$

$$D^i_t = \tau K^i_t[\mu^i_t - P_t] - 2D^i_{Qt}P_t, \tag{19}$$

$$P_{Qt} = K_t^{-1} + P_t^2, \tag{20}$$

$$D_{Qt}^i = \frac{\tau}{2}(1-\rho^i)\left[\frac{K_t K_{t+1}}{n_{t+1}} - h\right], \tag{21}$$

where K_t^i and K_t are the same as in Theorem 2.

Like Theorem 2, this theorem also shows that investors with high confidence take short positions in options and investors with low confidence take long positions in options. Also, there will be trading volume in the underlying stock even if its price does not change. In one-period models, when markets are complete, there exists a representative investor who prices all assets according to his belief and marginal utility.[12] Interestingly, this result still holds in our dynamic trading model with differential interpretation of public signals, as summarized in the following proposition.

Proposition 2 *When markets are completed by adding either a continuum of call and put options or a single quadratic option, there exists a representative investor with risk tolerance τ and belief $\mathcal{N}(\mu_t, K_t{}^{-1})$, where $\mu_t = K_t{}^{-1}(h\bar{u} + \sum_{j=0}^{t} n_j y_j)$.*

It is indeed quite striking that markets are effectively complete with the introduction of a single derivative, and as a result, the prices of all contingent claims behave as if there existed a representative agent. Trading in options and the underlying stock, however, is active among investors at all trading sessions due to differences of opinion among investors.

We now apply the results obtained in Theorems 2 and 3 to perform various analyses associated with options.

3.2 Implied Volatility and Stock Returns

The belief of the representative agent is given by the simple average belief of $(\mu_t,\ n_t)$. In equilibrium, this agent determines the stock and option prices. Notice that the implied volatility backed out from the option price is based on the representative agent's belief. As a result, unless this agent's belief represents the true distribution of the stock payoff, the implied volatility will not be the realized volatility of the stock price. We next show that the implied volatility may predict the expected stock return.

[12]See Rubinstein (1974), Brennan and Kraus (1978), and Brennan (1979).

Suppose that the representative agent happens to have the true distribution of the stock payoff. The expected (dollar) return of the stock based on this agent's belief is then given by

$$E_t^a[P_{t+1} - P_t] = \frac{x n_{t+1}}{\tau K_t K_{t+1}}, \tag{22}$$

where the superscript "a" refers to the average or representative agent. Notice that the representative agent's expected volatility about the stock (dollar) return $(P_{t+1} - P_t)$ is given by $n_{t+1}/(K_t K_{t+1})$, which also corresponds to the implied volatility of the stock (dollar) return in the option prices. It is clear that a higher implied volatility leads to a higher expected stock return.

It is perhaps more general to assume that one of the agents, i, rather than the representative agent, possesses the true distribution of the stock payoff. The expected stock return based on this agent's belief is then given by

$$\begin{aligned} E_t^i[P_{t+1} - P_t] &= E_t^i\left[\frac{n_{t+1}}{K_{t+1}}(y_{t+1} - P_t)\right] = \frac{n_{t+1}}{K_{t+1}}(\mu_t^i - P_t) \\ &= \frac{n_{t+1}}{K_{t+1}}\left(\frac{h\bar{u} + \rho^i \sum_{j=1}^t n_j y_j}{K_t^i} - P_t\right) \\ &= \frac{n_{t+1}}{K_{t+1}}\left(\frac{h\bar{u} + \rho^i[K_t P_t - h\bar{u} + x/\tau]}{K_t^i} - P_t\right) \\ &= \frac{n_{t+1}h}{K_t^i K_{t+1}}(\rho^i - 1)(P_t - \bar{u}) + \frac{x n_{t+1}}{\tau K_t K_{t+1}}\left[1 + \frac{(\rho^i - 1)h}{K_t^i}\right], \end{aligned} \tag{23}$$

where K_t^i and K_t represent the correct and implied precisions of the stock payoff v, respectively. Again, the implied volatility may have explanatory power for the expected stock return. If $\rho^i > 1$, then the expected stock return is positively related to the prior stock return of $(P_t - P_0)$. $\rho^i > 1$ means that investor i has a higher precision of the signal than the representative or average investor. From investor i's point of view, the representative agent under-reacts to the public news, so he expects the stock price to go up in future periods, yielding a higher expected stock return or positive autocorrelation. In addition, investor i's conditional estimate of the stock payoff μ_t^i increases with $\sum_{j=0}^t n_j^i y_j$. When $(P_t - P_0)$ is higher, the realization of y is likely higher. As a result, investor i's estimate of u is higher because his estimate is proportional to the level of y. In other words, investor i believes that the representative agent underestimates the stock payoff more when y is higher, thus, he expects the stock price to go higher in the future.

Hence, investor i's expected stock return increases with the magnitude of $(P_t - P_0)$. Similarly, we can explain that when $\rho^i < 1$, the expected stock return is negatively related to $(P_t - P_0)$ or the stock returns exhibit negative autocorrelation.

It would be of great importance to determine empirically the sign of $(\rho^i - 1)$ so that we can understand how the prior return predicts future returns. We propose to compare the implied volatility to the realized volatility. If the former is lower than the latter, then it suggests that the representative agent's view $(\rho^a = 1)$ is less confident than that of investor i or $\rho^i > 1$. In this case, investor i expects the stock return to be positively related to the previous stock return $(P_t - \bar{\mu})$. On the other hand, if the implied volatility is higher than the realized volatility, then $\rho^i < 1$ or the representative agent's view is more confident than that of investor i, then the expected stock return is negatively related to the previous stock return.

The implied volatility (IV) of the stock return at trading session t, calculated by the representative agent, is given by

$$IV_t \equiv \mathrm{V}ar^a[P_{t+1} - P_t] = \frac{n_{t+1}}{K_t K_{t+1}}.$$

The realized volatility, calculated by investor i who happens to have the true distribution of the stock payoff, is given by

$$RV_t \equiv \mathrm{V}ar^i[P_{t+1} - P_t] = \frac{n_{t+1}^2 K_{t+1}^i}{K_{t+1}^2 K_t^i n_{t+1}^i}.$$

The difference is then given by

$$IV_t - RV_t = \frac{(\rho^i - 1) n_{t+1} (K_t K_{t+1}^i - h n_{t+1}^i)}{\rho^i K_t K_t^i K_{t+1}^2}.$$

We can express the expected stock return in terms of IV_t and RV_t as

$$E_t^i[P_{t+1} - P_t] = \frac{h}{(K_t K_{t+1}^i - h n_{t+1}^i)} (IV_t - RV_t) \rho^i K_t K_{t+1} (P_t - P_0) + \frac{x}{\tau} \left[RV_t + \frac{(IV_t - RV_t) h K_t}{K_t K_{t+1}^i - h n_{t+1}^i} \right]. \quad (24)$$

When ρ^i is very close to 1 and $K_{t+1} >> n_{t+1}$, we have the following approximation:

$$E_t^i[P_{t+1} - P_t] \approx h(IV_t - RV_t)(P_t - P_0) + \frac{x}{\tau} \left[RV_t \frac{K_t - h}{K_t} + IV_t \frac{h}{K_t} \right]. \quad (25)$$

This relation shows that the expected stock return is related to both the implied volatility and the realized volatility. In addition, the expected return is related to the lagged returns. The relationship between the expected return and the lagged returns are modified by the difference between the implied volatility and the realized volatility. When the implied volatility is higher than the realized volatility, the returns will be positively correlated. When the implied volatility is lower than the realized volatility, the returns will be negatively correlated.

3.3 Value of Options

For comparison with the case without options, we determine the perceived value of adding options to the market, which is given in the following lemma. Let $EU^i_o(T)$ denote the expected utility of investor i in an economy with options and T trading sessions.

Lemma 2 *The certainty equivalent value of trading in the options is given by*

$$\gamma^i_o(T) \equiv -\tau \ln\left(\frac{\mathrm{E}U^i_o(T)}{\mathrm{E}U^i(T)}\right) = \frac{\tau}{2}\sum_{j=0}^{T-1}\left[x^i_j - \ln(1+x^i_j)\right] > 0, \tag{26}$$

where $EU^i(T)$ denotes investor i's expected utility in an economy without options and where x^i_j is given by

$$x^i_j = (\rho^i - 1)\left(1 - \frac{hn_{j+1}}{K_j K_{j+1}}\right). \tag{27}$$

The expression $\frac{\tau}{2}[x^i_t - \ln(1+x^i_t)]$ can be viewed as the gains from options trading in session t. Interestingly, even as n_{t+1} goes to zero, x^i_t does not converge to zero. The reason is that $\rho^i \equiv n^i_{t+1}/n_{t+1}$ is the ratio of n^i_{t+1} to n_{t+1}. Even if both n^i_{t+1} and n_{t+1} are arbitrarily small, their ratio may not be negligible. In other words, we may ignore terms (e.g., $hn_{t+1}/[K_t K_{t+1}]$), which involve n^i_{t+1} and n_{t+1}, but we cannot ignore the term of $(\rho^i - 1)$ in the calculations. Thus, even if a signal becomes arbitrarily noisy, the perceived gains from options trading remain finite. In contrast, from expression (11), the gains from trading the stock alone converges to zero as n_{t+1} goes to zero. This is somewhat surprising because one would expect that when the time interval ($\Delta t \equiv 1/T$) of trading periods goes to zero, the perceived gains of trading options in *each* trading session should also go to zero (in the order of Δt). We have demonstrated here, however, that as long as the differences of opinion $(\rho^i - 1)$ is finite and even if the signal is arbitrarily noisy, the investors' perceived gains in trading options in a single trade remain finite. As a result, options will not be redundant assets because investors believe that trading in options enhances their perceived welfare.

To understand the difference between trading options and stocks, it is helpful to examine the Sharpe ratio for one trading session from investor i's point of view regarding a stock and an option. Notice that from Theorem 3, the excess payoff from the quadratic option at time $t+1$ can be written as

$$P_{t+1}^2 + K_{t+1}^{-1} - P_{Qt} = (P_{t+1} - P_t)^2 + 2P_t(P_{t+1} - P_t) - n_{t+1}/[K_t K_{t+1}].$$

Holding the quadratic asset at time t is equivalent to holding a derivative asset O, with a payoff of $(P_{t+1} - P_t)^2$ at time $t+1$, and $2P_t$ shares of the stock.

To simplify the analysis, we examine the Sharpe ratio of $(P_{t+1} - P_t)^2$ at trading session t for investor i.

$$\begin{aligned} S_{ot}^i &= \frac{\mathrm{E}_t^i[(P_{t+1} - P_t)^2 - n_{t+1}/(K_t K_{t+1})]}{\sqrt{\mathrm{V}ar_t^i[(P_{t+1} - P_t)^2]}} \\ &= \frac{1 - K_t^i n_{t+1}^i K_{t+1}/[K_t n_{t+1} K_{t+1}^i] + n_{t+1}^i K_t^i(\mu_t^i - P_t)^2/K_{t+1}^i}{\sqrt{2 + 4n_{t+1}^i K_t^i(\mu_t^i - P_t)^2/K_{t+1}^i}}. \end{aligned}$$

Here we have used the following relations:

$$\begin{aligned} \mathrm{E}_t^i\left[(P_{t+1} - P_t)^2\right] &= \frac{(n_{t+1})^2}{(K_{t+1})^2}\mathrm{E}_t^i[(y_{t+1} - P_t)^2] = \frac{(n_{t+1})^2}{(K_{t+1})^2}\left[(\mu_t^i - P_t)^2 + \left(\frac{1}{K_t^i} + \frac{1}{n_{t+1}^i}\right)\right] \\ &= \frac{(n_{t+1})^2}{(K_{t+1})^2}\left[(\mu_t^i - P_t)^2 + \frac{K_{t+1}^i}{K_t^i n_{t+1}^i}\right] \end{aligned}$$

and

$$\mathrm{Var}_t^i[(P_{t+1} - P_t)^2] = E_t^i\left[(P_{t+1} - P_2)^4 - E_t^i(P_{t+1} - P_2)^2\right]^2,$$

where the first term is given by

$$\begin{aligned} \mathrm{E}_t^i\left[(P_{t+1} - P_t)^4\right] &= \frac{n_{t+1}^4}{K_{t+1}^4}\mathrm{E}_t^i\left[(\mu_t^i - P_t)^4 + 6(\mu_t^i - P_4)^2\left(\frac{1}{K_t^i} + \frac{1}{n_{t+i}}\right) + 3\left(\frac{1}{K_t^i} + \frac{1}{n_{t+1}}\right)^2\right] \\ &= \frac{n_{t+1}^4}{K_{t+1}^4}\left[4(\mu_t^i - P_t)^2\left(\frac{1}{K_t^i} + \frac{1}{n_{t+1}}\right) + 2\left(\frac{1}{K_t^i} + \frac{1}{n_{t+1}^i}\right)^2\right]. \end{aligned}$$

Surprisingly, the Sharpe ratio for $(P_{t+1} - P_t)^2$ converges to a finite value $(1 - \rho^i)/\sqrt{2}$ as n_{t+1} goes to zero.

For the stock trading at time t, we have

$$\mathrm{E}_t^i[P_{t+1} - P_t] = \frac{n_{t+1}}{K_{t+1}}(\mu_t^i - P_t) \quad \text{and} \quad \mathrm{Var}_t^i[P_{t+1} - P_t] = \frac{(n_{t+1})^2}{(K_{t+1})^2}\frac{K_{t+1}^i}{K_t^i n_{t+1}^i}.$$

Thus, for investor i, the Sharpe ratio for the stock is given by

$$S_t^i \equiv \frac{\mathrm{E}_t^i(P_{t+1} - P_t)}{\sqrt{\mathrm{Var}_t^i[P_{t+1} - P_t)]}} = \frac{\sqrt{\rho^i n_{t+1} K_t^i}(\mu_t^i - P_t)}{\sqrt{K_{t+1}^i}},$$

which approaches zero in the order of $\sqrt{n_{t+1}}$ as n_{t+1} goes to zero.

Our results imply that even if investors disagree only once for a small n_{t+1} and agree at all other trading periods, the perceived gains are finite. If investors disagree at all trading periods, when T goes to infinity but ρ^i remains the same, the perceived gains go to infinity. On the other hand, the gains from trading in the underlying stock will go to zero if n_t is arbitrarily small. Consequently, we can conclude that frequent trading cannot substitute for options trading, contrary to the results obtained in models with common interpretation of public signals [e.g., Brennan and Cao (1996)].

From expression (26), when the number of trading sessions is finite, the trading gains with options trading goes to zero if $(\rho^i - 1)$ goes to 0 or if differences of opinion are arbitrarily small. However, when the number of trading sessions, T, goes to infinity, if $(\rho^i - 1)$ approaches zero in the order of $1/T^\delta$, the trading gains may not be zero. Suppose that

$$\rho^i - 1 = \frac{\omega^i(T)}{T^\delta},$$

where $\omega^i(T)$ goes to a constant ω^i when T goes to infinity. Further suppose that as T goes to infinity, $n_{t+1}/[K_t K_{t+1}]$ goes to zero. As a result, x_j, as expressed in equation (27), converges to zero in the order of $(\rho^i - 1)$. Using equation (26), we have

$$\begin{aligned} \frac{\gamma_o^i(T)}{T^{1-2\delta}} &= \frac{\tau}{2T^{1-2\delta}}\sum_{j=0}^{T-1}[x_j^i - \ln(1 + x_j^i)] = \frac{\tau}{4T^{1-2\delta}}\sum_{j=0}^{T-1}[(x_j^i)^2 + O((x_j^i)^3)] \\ &= \frac{\tau}{4T^{1-2\delta}}\sum_{j=0}^{T-1}[(x_j^i)^2 + O((x_j^i)^3)] = \frac{\tau}{4}[(\omega^i)^2 + O(1/T^\delta)]. \end{aligned}$$

When T goes to infinity, we obtain

$$\lim_{T\to\infty} \frac{\gamma_o^i(T)}{T^{1-2\delta}} = \frac{\tau(\omega^i)^2}{4}.$$

It is clear that as trading becomes more frequent, that is, $T \to \infty$, even when $\rho^i - 1$ converges to zero, $\gamma_o^i(T)$ does not necessarily diminish to zero. We consider three scenarios: $\delta > 1/2$, $\delta < 1/2$, and $\delta = 1/2$. When $\delta < 1/2$, the perceived gains go to infinity. When $\delta = 1/2$, the perceived gains go to a finite value. Only when $\delta > 1/2$ would the perceived gains go to zero. This illustrates that even when the disagreement among investors is arbitrarily small, when trading becomes more frequent, the perceived gains can be finite or even go to infinity. Options are not redundant assets as long as the investors' trading gains are positive.

3.4 Expected Utility Gains under Alternative Views

We have shown that all investors believe that they are better off with both additional rounds of trading and the introduction of options from their own points of view. While investor j believes that he is better off with additional trading opportunities, some other investors may be worse off from investor j's point of view. In this subsection, we consider the effects of additional trading sessions and the introduction of options under alternative views. More specifically, let $W^i(T)$ be the wealth of investor i at time 1 in an economy with T trading sessions without options; let $W_o^i(T)$ be the wealth of investor i at time 1 in an economy with T trading sessions and options with all possible strike prices. When $T2 > T1$, we have shown that using i's belief, we always have $E_0^i[U(W^i(T2))] \geq E_0^i[U(W^i(T1))]$ and $E_0^i[U(W_o^i(T))] \geq E_0^i[U(W^i(T))]$. However, this may not be true from another investor's point of view, as illustrated in the following proposition.

Proposition 3 *The addition of more trading sessions and the introduction of options can make an investor worse off from another investor's point of view. That is, there may exist investors i and j such that $E_0^j[U(W^i(T2))] < E_0^j[U(W^i(T1))]$ and $E_0^j[U(W_o^i(T))] < E_0^j[U(W^i(T))]$, where $T2 > T1$.*

While all investors are better off with respect to more frequent trading and the introduction of options from their own points of view, they could be worse off under an alternative view. This may have interesting implications with respect to government regulations in emerging markets. From a government's point of view, if it believes that investors are overconfident or have exuberant beliefs, then increasing frictions or delaying the introduction of options may make investors better off.

4 Trading Volume in Stocks and Options

In this section we use the results obtained in Theorems 1-3 to analyze the investors' trading strategies and trading volume in stocks and options.

4.1 Trading Strategies and Price Dynamics

This subsection considers the trading strategies and price dynamics without options. To characterize investors' trading strategies we consider the change in investor i's demand for the stock between successive market sessions. Using equations (6) and (7), we have

$$D^i_{t+1} - D^i_t = \tau(\rho^i - 1)h(P_{t+1} - P_t). \tag{28}$$

It is clear that trading occurs as investors disagree on how to interpret public information as well as the stock price changes. Investors who have high confidence ($\rho^i > 1$) about a public signal put more weight on the signal and thus trade in the direction of the signal. If the public signal is very positive, the price will go up. But investors with high confidence still believe that the price has not fully incorporated the positive signal due to the presence of investors with low confidence. Hence, investors with high confidence believe that the stock price will go up even further and demand more shares of the stock. On the other hand, investors who have low confidence ($\rho^i < 1$) about a public signal put less weight on it. When the stock price goes up, they believe that the price is overreacting to the public signal due to the presence of investors with high confidence. Hence, investors with low confidence sell the stock. The following proposition characterizes the relations between trades and price changes from different investors' perspectives.

Proposition 4 *(i) The trades of investors with high confidence ($\rho^i > 1$) at time τ_t are positively correlated with the price change at time τ_t.*
(ii) The trades of investors with low confidence ($\rho^i < 1$) at time τ_t are negatively correlated with the price change at time τ_t.
(iii) Investors with the average confidence level ($\rho^i = 1$) do not trade.

This proposition states that investors with high confidence buy, whereas investors with low confidence sell as price goes up. It also implies that depending on their confidence levels, investors may have different expectations about future stock prices conditional on the current stock price. When the current stock price is low due to bad news, investors with low confidence believe that investors on average had overreacted to the bad news, so the stock price will rebound in the next period. On the other hand, investors with high confidence believe that the investors with low confidence had underreacted to the bad news, so the stock price will go down even further in the next period. The next proposition summarizes the price dynamics based on different investors' perspectives.

Proposition 5 *(i) For an investor with high confidence ($\rho^i > 1$), the price change at time τ_{t+1} is positively correlated with the price at time τ_t and is positively correlated with the price change at time τ_t, that is,*

$$\mathrm{Cov}^i(P_{t+1} - P_t, P_t) > 0, \tag{29}$$

$$\mathrm{Cov}^i(P_{t+1} - P_t, P_t - P_{t-1}) > 0. \tag{30}$$

(ii) For an investor with low confidence ($\rho^i < 1$), the price change at time τ_{t+1} is negatively correlated with the price at time τ_t and is negatively correlated with the price change at time τ_t, that is,

$$\mathrm{Cov}^i(P_{t+1} - P_t, P_t) < 0, \tag{31}$$

$$\mathrm{Cov}^i(P_{t+1} - P_t, P_t - P_{t-1}) < 0. \tag{32}$$

(iii) For the average investor ($\rho^i = 1$), the price change at time τ_{t+1} is uncorrelated with the price at time τ_t and is uncorrelated with the price change at time τ_t, that is,

$$\mathrm{Cov}^i(P_{t+1} - P_t, P_t) = 0, \tag{33}$$

$$\mathrm{Cov}^i(P_{t+1} - P_t, P_t - P_{t-1}) = 0. \tag{34}$$

We next apply Theorem 1 to study the equilibrium trading volume of the stock.

4.2 Trading Volume

Many empirical studies have examined the contemporaneous behavior of volume and absolute price changes and found a positive correlation between the two [e.g., Karpoff (1987)]. Since the dynamics of price volatility and trading volume can only be studied in a multiple trading session economy, this subsection presents additional results on the autocorrelation properties of trading volume as well as the relation between trading volume and the number of trading sessions between time 0 and time 1.

Let $\Delta P_t = P_t - P_{t-1}$ denote the price change at time τ_t, where $t = 1, \ldots, T-1$. Let trading volume at time τ_t, V_t, be defined as one half the sum of all purchases and sales, that is,

$$\begin{aligned} V_t &= \frac{1}{2}\int_0^1 |D_t^i - D_{t-1}^i| di = \frac{1}{2}\int_0^1 \tau K_t \left| \frac{K_t^i}{K_t} - \frac{n_t^i}{n_t} \right| |\Delta P_t| \, di \\ &= \frac{1}{2}\int_0^1 \tau h \left| \rho^i - 1 \right| |\Delta P_t| \, di. \end{aligned} \tag{35}$$

Note that there is no hedging demand for the stock in our model and that all trades are due to the differences of opinion about public signals. As a result, we obtain a simple result that the trading volume in each period is proportional to the product of the absolute price change and the dispersion of investors' public information precisions. In Proposition 2, we have shown that price changes are serially correlated if the true precision is different from the average precision. Since the correlation coefficient between the absolute values of two normally distributed variables x and y with correlation $r(x, y)$ and means of zero is given by

$$\text{Corr}(|x|, |y|) = \frac{2}{\pi - 2}\int_0^{r(x,y)} \arcsin t dt > 0,$$

the following lemma is immediate.[13]

Lemma 3 *(i) Trading volume and absolute price changes are positively correlated.*
(ii) Trading volume and absolute change of the precision weighted average forecast are positively correlated.
(iii) Trading volume is higher when the public signal is very informative (a high value of n_t).
(iv) Trading volume increases with the dispersion of beliefs among investors.
(v) For any investor whose precision is different from the average precision, the absolute price change and trading volume are positively serially correlated.

The first three implications are consistent with the empirical evidence summarized in Karpoff (1987). Implication (*iv*) implies that trading volume may be related to the dispersion among financial analysts' forecasts. Empirically, Frankel and Froot (1990) examine foreign exchange data and find a positive relation between volume and dispersion, and Ajinkya, Atiase, and Gift (1991) also obtain a positive relation between stock volume of trading and the dispersion

[13]Harris and Raviv (1993) derive results (*i*), (*ii*), and (*iii*) based on differences of opinion with risk-neutral investors. Brennan and Cao (1996) also obtain predictions (*i*) and (*v*) using a partially revealing rational expectations model with differentially informed investors.

in financial analysts' earnings forecasts, both of which seem to support our prediction (iv). Implication (v) may be tested using survey data of investors' beliefs about the stock price changes.

4.3 Open Interest and Trading Volume in Options

This subsection considers the open interest and trading volume in options. From equation (17), investors' open interest in options increases with n_t, the average precision of the signal. If we interpret n_t as the informativeness of a public signal, then investors' holdings in options increase with the informativeness of the signal. Let ΔD^i_{CZt} denote investor i's amount of trading for a call option with strike price Z, which, according to Theorem 2, is given by

$$\Delta D^i_{CZt} = \tau \left[\frac{K_t K_{t+1}}{n_{t+1}} - \frac{K_{t-1} K_t}{n_t} \right] (1 - \rho^i). \tag{36}$$

Let the trading volume in options, V_{CE}, be defined as half of the sum of the absolute trades. We have

$$V_{CZt} = \tau \left| \frac{K_t K_{t+1}}{n_{t+1}} - \frac{K_{t-1} K_t}{n_t} \right| \int_0^1 |1 - \rho^i| di = \tau \left| \frac{1}{\mathrm{Var}[\Delta P_t]} - \frac{1}{\mathrm{Var}[\Delta P_{t+1}]} \right| \int_0^1 |1 - \rho^i| di. \tag{37}$$

Because the average of ρ^i is one, $\int_0^1 |1 - \rho^i| di$ is a measure of investors' dispersion of beliefs. Equations (21) and (37) then lead to the following result regarding the open interest and trading volume in options.[14]

Proposition 6 *Investors' open interest in options increases with the informativeness of a public signal and investors' trading volume in options is higher when they have higher dispersion of beliefs.*

These results are intuitive. With a higher dispersion of beliefs, investors disagree on the volatility of the stock payoff more; equivalently, they disagree on the value of options more, the trading volume for options naturally increases. Keeping dispersion of beliefs the same, a more informative signal makes investors trade more aggressively on their differences and thus further increases open interest in options.

In practice, a proxy for the dispersion of beliefs is the dispersion among financial analysts' forecasts. To connect the trading volume in options with the dispersion among analysts' forecasts, recall that investor i's conditional expectation of the final payoff u at time t is given

[14] The results for put options can be obtained similarly.

by

$$\mu_t^i = \frac{h\bar{u} + \rho^i \sum_{j=0}^t n_j y_{ij}}{h + \rho^i \sum_{j=0}^t n_j}. \tag{38}$$

The difference between investor i's forecast and the consensus forecast is

$$\mu_t^i - \mu_t = \frac{\rho^i - 1}{K_t^i}(\mu_t - \bar{u}). \tag{39}$$

When ρ^i is close to one, as a first-order approximation, we have the dispersion of forecasts D_{ft} as

$$D_{ft} \equiv \int_0^1 |\mu_t^i - \mu_t| di \approx \left| \frac{\mu_t - \bar{u}}{K_t} \right| \int |\rho^i - 1| di. \tag{40}$$

Consequently, we have the following relation between the trading volume and the dispersion of forecasts:

$$V_{CZt} \approx \left| \frac{K_t \tau \left[\frac{K_t K_{t+1}}{n_{t+1}} - \frac{K_{t-1} K_t}{n_t} \right]}{4(\mu_t - \bar{u})} \right| D_{ft}, \tag{41}$$

which means that investors' trading volume in options is higher when there is a larger dispersion in investors' forecasts. This result is consistent in spirit with the empirical result of Buraschi and Jiltsov (2003), who find that open interest and trading volume in the index options market increase with their index of difference in beliefs.

When the public signal in trading session t is very informative or the average precision of the signal n_t is very large, open interest should be very high. It is likely that earnings announcements, mergers and acquisitions, or spinoffs represent informative public events and open interest should be higher around the dates of these public events. High open interest also suggests that trading volume in options may be high around these dates. Indeed, assuming that public information is very informative at time τ_t or n_t is so large such that $\text{Var}[\Delta P_t] - \text{Var}[\Delta P_{t+1}] > 0$, then the volume of trades in options increases in the average precision of public information in that session. Thus trading in options will also be more active around informative public announcements. These predictions may be tested using event studies to analyze the behavior of open interest and trading volume in options around informative public events.

4.4 Stock Trading Volume in the Presence of Options

The literature has provided some empirical evidence that the introduction of options tends to increase the trading volume of the underlying stock. See, for example, Skinner (1990). In this subsection, we examine the effects of options on the trading volume of the underlying stock.

Following the results of Theorem 2, we have

$$\begin{aligned} \Delta D_t^i &= \tau\left[(n_t^i y_t - K_t P_t + K_{t-1})P_{t-1} + (\rho^i - 1)\left(\frac{K_t^2}{n_{t+1}}P_t - \frac{K_{t-1}^2}{n_t}P_{t-1}\right)\right] \\ &= \tau(\rho^i - 1)\left[\frac{K_t K_{t+1}}{n_{t+1}}P_t - \frac{K_{t-1}K_t}{n_t}P_{t-1}\right]. \end{aligned} \tag{42}$$

The trading volume of the underlying stock is then given by

$$V_t \equiv \frac{1}{2}\int_0^1 |\Delta D_t^i| di = \tau\left|\frac{K_t K_{t+1}}{n_{t+1}}P_t - \frac{K_{t-1}K_t}{n_t}P_{t-1}\right| \int |(\rho^i - 1)| di. \tag{43}$$

It is clear that the trading volume can be positive even if the stock price remains the same, that is, $\Delta P_t = 0$. The reason is that in the presence of options, investors trade in the stock to hedge options. Even if the stock price remains unchanged, there may still be a need to hedge options because option prices may change due to differences of opinion about public signals. This result extends the result of Harris and Raviv (1993), who obtain that in the absence of options, trading takes place only when the stock price changes. It also complements the result of Kandel and Pearson (1995), thus offering an alternative explanation for their empirical finding that trading may occur even if the stock price remains the same. Our result also indicates that trading volume is related not only to the current price change but also to the past price changes. In addition, the coefficient on lagged price change should be larger for stocks with options than for stocks without options.

To offer sharper empirical predictions, we next assume that the volatility of the stock price change is stationary across trading sessions for the average investor, that is, $\mathrm{Var}[\Delta P_t] = \mathrm{Var}[\Delta P_{t+1}]$ for all t. This serves as a sufficient condition for the result regarding the expected trading volume of the underlying stock given in the following proposition.

Proposition 7 *When* $\mathrm{Var}[\Delta P_t] = \mathrm{Var}[\Delta P_{t+1}]$*, the introduction of options increases the expected trading volume of the underlying stock. Moreover, the expected trading volume is more sensitive both to the price changes of the stock and to the dispersion of forecasts among investors.*

The results of this proposition are due to investors' hedging demands for options. For example, with options, a change in the stock price affects the properties of the options associated with the stock, which requires more hedging for options. As a result, the expected trading volume of the stock is more sensitive to stock price changes in the presence of options. To test the implications of Proposition 7, one may take two approaches. The first approach is to conduct an event study to analyze the amount of trading volume before and after the introduction of options. The second approach is to perform a cross-sectional study to compare trading volume and its sensitivity to price changes and the dispersion of forecasts among investors between stocks with options and those without options.

5 Multiple Stocks

We have shown that the trading volume is related to the price change in a single-stock model. Empirical studies have shown that the trading volume of a stock is related to the price changes of not only that stock but also those with related payoffs [Huberman and Regev (2001)]. We next consider a multi-stock dynamic model and examine the relationship between trading volume and price changes. For tractability, we omit options in this model.

The payoffs of the M stocks are realized at time 1, and are represented by an $M \times 1$ normally distributed random vector $\tilde{U}$ with mean $\bar{U}$ and precision matrix H. Each investor $i, i \in [0, 1]$, is endowed at time 0 with risky assets denoted by the vector X^i; investors are characterized by negative exponential utility functions as defined earlier. The vector of the aggregate per capita supply of the risky assets is X.

Immediately prior to trading session t, a vector of *public* signals is released. The public signals are represented by the $M \times 1$ vector $\tilde{Y}_t$, where

$$\tilde{Y}_t = \tilde{U} + \tilde{\eta}_t.$$

Investors have differential interpretation about the public signals. For each investor $\tilde{\eta}_t$ is normally distributed with mean zero and precision matrix N_t^i.[15] The average precision matrix for the public signal is $N_t \equiv \int_i N_t^i di$.

Let P_t denote the vector of equilibrium risky asset prices, D_t^i the vector of investor's demands for the risky assets, and F_t the public information set including the prices P_t, all at trading session t. The following theorem describes the asset prices and the investors' asset demands at each market session in a sequential equilibrium.

Theorem 4 *There exists a sequential equilibrium. (i) The vectors of risky asset prices, P_t, and*

[15] We assume that $N_{0i} = O$ and $N_{Ti}^{-1} = O$, where O denotes the zero matrix. This assumption is consistent with the earlier assumption that there is no public information at time 0 and that all risky asset returns are realized at session T.

investor i's asset demands, D_t^i, are given by

$$P_t = K_t^{-1}[K_t \mu_t - \tau X], \tag{44}$$

$$D_t^i = \tau K_t^i [\mu_{ti} - P_t], \tag{45}$$

where

$$\mu_t^i \equiv \mathrm{E}_t^i[U] = K_{ti}^{-1}(H\bar{U} + \sum_{j=0}^{t} N_j \tilde{Y}_j), \quad \mu_t \equiv K_t^{-1} \int_0^1 K_t^i \mu_t^i di,$$

$$K_t^i \equiv [\mathrm{V}ar_t^i[\tilde{U}]]^{-1} = H + \sum_{j=1}^{t} N_j^i, \quad K_t \equiv \int_0^1 K_{ti} di = H + \sum_{j=0}^{t} N_j.$$

(ii) The optimal trading strategy of investor i, ΔD_t^i, and the trading volume of the stocks, V_t, are given by

$$\Delta D_t^i \equiv D_t^i - D_{t-1}^i = \tau (K_t - K_t^i)[P_t - P_{t-1}], \tag{46}$$

$$V_t = \int_i |\Delta D_t^i| di = \int_i |\tau (K_t - K_t^i)[P_t - P_{t-1}]| di. \tag{47}$$

Again, the equilibrium stock prices are determined by the average investor whose belief is equal to the average of the beliefs of all investors. In other words, this average investor serves as the representative agent in our economy with heterogenous beliefs.

Notice that in equilibrium, the stock price change at time $t+1$ is given by

$$\Delta P_{t+1} \equiv P_{t+1} - P_t = K_{t+1}^{-1} N_{t+1} (Y_{t+1} - P_t). \tag{48}$$

Equation (47) shows that the trading volume of a stock depends not only on the price change of that stock but also on the price changes of other related stocks. For example, if stocks 1 and 2 are correlated, then the corresponding terms, $K_t(1,2)$ and $K_t^i(1,2)$, in the K_t and K_t^i matrices are nonzero. As a result, the price changes of stocks 1 and 2, $\Delta P_{1(t+1)}$ and $\Delta P_{2(t+1)}$, contribute to the trading volume of these stocks as given in equation (47). Even if $\Delta P_{1(t+1)}$ is zero, the trading volume of stock 1 may not be zero.

Following equation (48), the price change is a linear function of the shocks in the signals $(\tilde{Y}_{t+1} - P_t)$. Even when the shocks of signals are not zero, the prices of some stocks may not change. However, the trading volume of a stock can still be positive even when the price change of that stock is zero. Notice that the relationship between an investor's optimal trading strategy or the trading volume of a stock and the absolute price change in the multiple stock model can be quite complicated. Specifically, the trading volume of a stock is related to the sum of the price changes of the related stocks, weighted by the absolute value of the precision difference. It shows that even if there are no differences of opinion or no signals about a stock's payoff, there may still be trading in that stock due to differences of opinion about the payoffs of other related stocks. These results may be used to explain the empirical findings of Kandel and Pearson (1995) and Huberman and Regev (2001).

We next show that the Capital Asset Pricing Model (CAPM) holds for the average investor a. The belief of the average investor a is represented by the expected payoff μ_t and precision matrix K_t. Using equation (48), we obtain that the expected dollar return of investor a is given by

$$\mathrm{E}_t^a \Delta P_{t+1} = K_{t+1}^{-1} N_{t+1} K_t^{-1} X = \mathrm{Cov}_t^a[\Delta P_{t+1}, \Delta P_{M(t+1)}] X,$$

where $\Delta P_{M(t+1)}$ denotes the return of the market portfolio defined as $P_{Mt} = P_t^T X$, with the superscript "T" representing the transpose of a vector. This calculation indicates that from the average investor's point of view, the CAPM holds, yielding

$$\mathrm{E}_t^a[\Delta P_{k(t+1)}] = \frac{\mathrm{Cov}_t^a[\Delta P_{k(t+1)}, \Delta P_{M(t+1)}]}{\mathrm{Var}_t^a[\Delta P_{M(t+1)}]} \mathrm{E}_t^a[\Delta P_{M(t+1)}] \equiv \beta_{kt} \mathrm{E}_t^a[\Delta P_{M(t+1)}], \ k = 1, 2, \cdots, N, \tag{49}$$

where k is the index for the number of stocks. Recall that the risk free rate is assumed to be zero throughout this paper. In our model, the average investor holds the market portfolio and the market portfolio is on his efficient frontier at all times. As a result, the average investor does not trade dynamically and the CAPM holds each period from his point of view. On the contrary, the investors whose precision matrices are different from that of the average investor trade dynamically in this model. The trading of an investor is proportional to the product of the difference of the precision matrix between that investor and that of the average investor, and the vector of the price changes. Trading is more active for investors holding views away from the consensus. Our result suggests that if the real data reflects the true view of the stock payoff, then the CAPM may not be confirmed by the data unless the average belief happens to be the correct one.

6 Conclusion

In this paper we develop a model of trading based on differences of opinion regarding public signals. It is shown that the trading volume in the stock is proportional to the dispersion of precisions among investors, which is shown to be approximately proportional to the dispersion of investors' forecasts. In the absence of options, trading may take place even when there are no differences of opinion about future public signals, as long as there is a disagreement in the past. As the number of trading sessions goes to infinity, investors' perceived gains increase and converge to a finite limit, but the Pareto optimal allocation cannot be achieved. If differences of opinion approach zero in every trading session, then the trading gains converge to zero.

When options are added to the economy, the Pareto optimal allocation can be restored. This can be accomplished by adding either a continuum of call and put options or a single option with quadratic payoffs of the stock. All assets are priced as if there existed a representative agent whose belief is equal to the average of all investors' beliefs.[16] More specifically, all option contracts in our model are priced in accordance with the risk-neutral pricing principle of Black and Scholes (1973). With differences of opinion, the volatility implied by the option price may not be the same as the realized volatility of the stock price, unless the average agent's belief happens to be the correct one. We demonstrate that when the implied volatility is greater than the realized volatility, the stock returns exhibit positive autocorrelation or momentum and when the implied volatility is lower than the realized volatility, the stock returns exhibit negative autocorrelation. We further demonstrate that both the implied volatility and the realized volatility have explanatory power for the expected stock returns.

We show that even if the differences of opinion are arbitrarily small, the gains of trading options may remain positive whereas those of trading stocks converge to zero in the limit of continuous trading. In other words, options are not redundant assets. We also show that even if a public signal is arbitrarily noisy, the perceived gains of trading options remain finite whereas those of trading stocks are zero. These results highlight the importance of incorporating options in models with differences of opinion. In addition, this study offers some unique empirical implications regarding the trading volume. It is shown that the open interest and trading volumes in options are higher both around the dates of public announcements and for stocks that have higher differences of opinion. It is also shown that in the presence of options, the trading volume of the underlying stock is not only higher but also more responsive to the stock price changes.

[16] Trading volumes, however, reflect the underlying differences of opinion among investors.

In a multiple stock model, we demonstrate that there may be trading in a stock even if its price remains unchanged and that the trading volume of a stock is related to the investors' differences of opinion about that stock's payoff as well as to those about the payoffs of other correlated stocks. Assets are priced according to a representative agent whose belief is equal to the average of all investors' beliefs. The expected asset returns based on this investor's belief are governed by the CAPM.

For tractability, we have adopted some simplifying assumptions. For example, because differences of opinion may affect investors' consumption decisions, a generalization would be to consider an intertemporal consumption model with multiple consumption dates. To focus on trading based only on differences of opinion, we have ignored the effects of private information acquisition. It would be of interest to develop a general model that incorporates both differences of opinion and asymmetric information in the presence of options. Finally, in the presence of both risk-averse investors and options, the model becomes intractable with short-sale constraints on stocks and position limits on options. Taking portfolio constraints on stocks and options into account may provide additional insight on the dynamics of trading volume and price process and how they relate to the dispersion of beliefs among investors. We leave them for future research.

Appendix: Proofs

Proof of Theorems 1 & 4: We here present the proof of Theorem 4, which includes Theorem 1 as a special case in which the number of stocks is one.

We prove this theorem using mathematical induction. It is straightforward to show that the price function stated in the theorem clears the market; then we only need to show that the investors' demands are optimal.

In the last trading session, since there is only one trading session left, it is well known that investor i's trading strategy is as described in the theorem:

$$D^i_{T-1} = \tau K^i_{T-1}[\mu^i_{T-1} - P_{T-1}]. \tag{50}$$

Given this equilibrium trading strategy, investor i's expected utility in the last trading session is then given by

$$\mathrm{E}^i_{T-1}[U^i] = -\exp\left[-\frac{1}{\tau}W^i_{T-1} - \frac{1}{2}(b^i_{T-1})^T K^i_{T-1} b^i_{T-1}\right], \tag{51}$$

where the superscript "T" denotes the transpose of a matrix and where $b^i_{T-1} \equiv \mu^i_{T-1} - P_{T-1}$ denotes investor i's excess return at session $(T-1)$. The first term in the exponential comes from the investor's wealth, and the second term represents the certainty equivalent gains in the expected utility from trading in the last session.

Suppose that at trading session $(t+1)$, investor i's demand and expected utility are given by

$$D^i_{t+1} = \tau K^i_{t+1}[\mu^i_{t+1} - P_{t+1}] \tag{52}$$

and

$$\mathrm{E}^i_{t+1}[U^i] \propto -\exp\left[-\frac{1}{\tau}W^i_{t+1} - \frac{1}{2}(b^i_{t+1})^T K^i_{t+1} b^i_{t+1}\right], \tag{53}$$

respectively. If at session t, investor i's optimal trading strategy and expected utility are equal to

$$D^i_t = \tau K^i_t[\mu^i_t - P_t] \quad \text{and} \quad \mathrm{E}^i_t[U^i] \propto -\exp\left[-\frac{1}{\tau}W^i_t - \frac{1}{2}(b^i_t)^T K^i_t b^i_t\right], \tag{54}$$

then the proposed equilibrium holds by induction.

At trading session t, let a_t denote the price change, b^i_t denote the expected excess return for investor i, and W^i_t denote the wealth of investor i, that is,

$$a_t = P_t - P_{t-1},\ b^i_t = \mu^i_t - P_t,\ W^i_t = W^i_{t-1} + (D^i_t)^T(P_t - P_{t-1}).$$

To determine the optimal strategy of investor i at session t, we need to calculate the expected utility given any strategy D_t^i at session t, that is, $\mathrm{E}_t^i[U^i]$. This can be determined in two steps: $\mathrm{E}_t^i[U^i] = \mathrm{E}_t^i[\mathrm{E}_t^i[U^i|a_{t+1}]] = \mathrm{E}_t^i[\mathrm{E}_t^i[\mathrm{E}_t^i[U^i|a_{t+1}, b_{t+1}]|a_{t+1}]]$.

From equation (6), we have

$$K_{t+1}P_{t+1} - K_t P_t = n_{t+1}y_{t+1},$$

which implies that

$$K_{t+1}a_{t+1} = K_{t+1}(P_{t+1} - P_t) = n_{t+1}(y_{t+1} - P_t).$$

Thus we can rewrite a_{t+1} as $a_{t+1} = A_{t+1}(u - P_t + \eta_{t+1})$, where A_{t+1} is defined by $A_{t+1} = K_{t+1}^{-1}(K_{t+1} - K_t)$.

Define:

$$\Lambda_t^i \equiv (\mathrm{Var}_t^i[u|a_{t+1}])^{-1}, \quad \mu_{b_{t+1}^i} \equiv \mathrm{E}_t^i[b_{t+1}^i|a_{t+1}] = B_t^i a_{t+1} + C_t^i b_t^i,$$

$$\mu_{a_{t+1}}^i \equiv \mathrm{E}_t^i[a_{t+1}] = A_{t+1}b_t^i, \quad B_t^i \equiv [I - C_t^i]A_{t+1}^{-1} - I, \quad C_t^i \equiv (\Lambda_t^i)^{-1}K_t^i,$$

$$\Gamma_t^i \equiv (\mathrm{Var}_t^i[b_{t+1}^i|a_{t+1}])^{-1} = [\mathrm{Var}_t^i[u|a_{t+1}] - \mathrm{Var}_t^i[u|a_{t+1}, b_{t+1}^i]^{-1} = [(\Lambda_t^i)^{-1} - (K_t^i)^{-1}]]^{-1},$$

$$\Psi_t^i \equiv (\mathrm{Var}_t^i[a_{t+1}])^{-1} = ((A_{t+1})^{-1})^T[(K_t^i)^{-1} + (\Lambda_t^i - K_t^i)^{-1}]^{-1}(A_{t+1})^{-1} = (B_t^i + I)K_t^i A_{t+1}^{-1}.$$

We now calculate the expected utility conditional on a_{t+1}; dropping irrelevant terms, this is given by

$$\begin{aligned}
\mathrm{E}_t^i[U^i|a_{t+1}] \quad &\propto \quad -\int_R \exp\left[-\frac{1}{\tau}(D_t^i)^T a_{t+1} - \frac{1}{2}(b_{t+1}^i)^T K_{t+1}^i b_{t+1}^i - \frac{1}{2}(b_{1+1}^i - \mu_{b_{t+1}^i})^T \Gamma_t^i (b_{1+1}^i - \mu_{b_{t+1}^i})\right] db_{t+1}^i \\
&\propto \quad -\exp\left[-\frac{1}{\tau}(D_t^i)^T a_{t+1} - \frac{1}{2}\mu_{b_{t+1}^i}^T \Gamma_t^i \mu_{b_{t+1}^i} + \frac{1}{2}(\mu_{b_{t+1}^i}\Gamma_t^i)^T (K_{t+1}^i + \Gamma_t^i)^{-1}(\mu_{b_{t+1}^i}\Gamma_t^i)\right] \\
&= \quad -\exp\left[-\frac{1}{\tau}(D_t^i)^T a_{t+1} - \frac{1}{2}\mu_{b_{t+1}^i}^T [(K_{t+1}^i)^{-1} + (\Gamma_t^i)^{-1}]^{-1}\mu_{b_{t+1}^i}\right] \\
&= \quad -\exp\left[-\frac{1}{\tau}(D_t^i)^T a_{t+1} - \frac{1}{2}\mu_{b_{t+1}^i}^T \Lambda_t^i \mu_{b_{t+1}^i}\right] \\
&= \quad -\exp\left[-\frac{1}{\tau}(D_t^i)^T a_{t+1} - \frac{1}{2}(B_t^i a_{t+1} + C_t^i b_t^i)^T \Lambda_t^i (B_t^i a_{t+1} + C_t^i b_t^i)\right]. \qquad (55)
\end{aligned}$$

Taking the expectation with respect to a_{t+1}, we get

$$\begin{aligned} \mathrm{E}_t^i[U^i] &\propto -\int_R \exp\left[-\frac{1}{\tau}(D_t^i)^T a_{t+1} - \frac{1}{2}(B_t^i a_{t+1} + C_t^i b_t^i)^T \Lambda_t^i (B_t^i a_{t+1} + C_t^i b_t^i)\right] \\ &\quad \times \exp\left[-\frac{1}{2}(a_{t+1} - \mu_{a_{t+1}}^i)^T \Psi_t^i (a_{t+1} - \mu_{a_{t+1}}^i)\right] da_{t+1} \\ &\propto -\sqrt{\frac{1}{G_t^i}} \exp\left[\frac{1}{2}(F_t^i)^T G_t^i (F_t^i) - \frac{1}{2}(b_t^i)^T H_t^i b_t^i\right], \end{aligned} \tag{56}$$

where

$$F_t^i = D_t^i/\tau + B_t^i \Lambda_t^i C_t^i b_t^i - \Psi_t^i \mu_{a_{t+1}}^i, \tag{57}$$

$$G_t^i = [(B_t^i)^T \Lambda_t^i B_t^i + \Psi_t^i]^{-1}, \tag{58}$$

$$H_t^i = (C_t^i)^T \Lambda_t^i C_t^i + A_{t+1}^T \Psi_t^i A_{t+1} = C_t^i K_t^i + (B_t^i + I) A_{t+1} K_t^i = [C_t^i + I - C_t^i] K_t^i = K_t^i. \tag{59}$$

Because Λ_t^i and Ψ_t^i are positive definite, G_t^i is positive definite.

The first-order condition simplifies to:

$$F_t^i = 0, \tag{60}$$

which implies that investor i's optimal demand for the tth trading session is given by

$$D_t^i = \tau[\Psi_t^i \mu_{a_{t+1}}^i - B_t^i \Lambda_t^i C_t^i b_t^i] = \tau[B_t^i + 1 - B_t^i] K_t^i b_t^i = \tau K_t^i [\mu_t^i - P_t]. \tag{61}$$

The optimal demand in equation (61) has the same form as in equation (52) except for the time subscript. Thus the optimal demand at session t is unaffected by the existence of future trading sessions. Substituting equations (57), (58), and (59) into equation (56), we have

$$\mathrm{E}_t^i[U^i] \propto -\exp\left[-\frac{1}{\tau} W_t^i - \frac{1}{2}(b_t^i)^T K_t^i b_t^i\right]. \tag{62}$$

Q.E.D.

Proof of Theorem 2: It is easy to check that the proposed equilibrium prices clear the proposed equilibrium demands. We only need to show that the proposed equilibrium demands are optimal given the prices. In the last trading session $(T-1)$, investor i's wealth in the next period is given by

$$\begin{aligned} W_T^i &= W_{T-1}^i + D_{T-1}^i(u - P_{T-1}) + \int_{-\infty}^0 D_{PZ(T-1)}^i[(Z-u)^+ - P_{PZ(T-1)}]dZ \\ &\quad + \int_0^\infty D_{CZ(T-1)}^i[(u-Z)^+ - P_{CZ(T-1)}]dZ. \end{aligned}$$

Given the conjectured demands and prices for the stock and options, we have

$$W_T^i = W_{T-1}^i + \tau(\mu_{t-1}^i - P_t)(u - P_{T-1}) + \tau\left[K_{T-1} - K_{T-1}^i\right]\left[(u-P_{T-1})^2 - \frac{1}{K_{T-1}}\right].$$

It can be shown that in this proposed equilibrium, investors achieve the Pareto optimal allocation and the Euler conditions for the stock and options are satisfied, that is,

$$\mathrm{E}_{T-1}^i[(u - P_{T-1})U'(W_T^i)] = 0,$$

$$\mathrm{E}_{T-1}^i[((Z-u)^+ - P_{PZ(T-1)})U'(W_T^i)] = 0,$$

$$\mathrm{E}_{T-1}^i[((u-Z)^+ - P_{CZ(T-1)})U'(W_T^i)] = 0.$$

Thus the proposed equilibrium holds in the last period.

Suppose that the proposed equilibrium holds for period t. If we can show that the proposed demands and prices also constitute an equilibrium at trading session $(t-1)$, then the demands and prices form an equilibrium for all t. Let $U^i(W_t^i, P_t)$ denote the expected utility that investor i achieves conditional on his wealth and belief in trading session t. As in the proof of Theorem 3, it can be shown that

$$U^i(W_t^i, P_t) \propto -\exp\left[-\frac{W_t^i}{\tau} - \frac{K_t^i}{2}(\mu_t^i - P_t)^2\right].$$

To prove that the proposed demands are optimal, we need to show that the following Euler conditions hold:

$$\mathrm{E}_{t-1}^i[(P_t - P_{t-1})U'(W_t^i, P_t)] = 0,$$

$$\mathrm{E}_{t-1}^i[(P_{PZt} - P_{PZ(t-1)})U'(W_t^i, P_t)] = 0,$$

$$\mathrm{E}_{t-1}^i[(P_{CZt} - P_{CZ(t-1)})U'(W_t^i, P_t)] = 0.$$

Next, we show that the marginal rates of substitution are equalized across all investors. In trading session t, investors observe a public signal y_t. Let f^i denote investor i's probability density of y_t. Consider the marginal rate of substitution M^i_{tkl} between two realizations of y_t: y_{tk} and y_{tl}, we have

$$M^i_{tkl} = \frac{f^i(y_{tk})U'(W^i_t(y_{tk}), P_t(y_{tk}))}{f^i(y_{tl})U'(W^i_t(y_{tl}), P_t(y_{tl}))}. \tag{63}$$

We now determine the probability weighted marginal utility $f^i(y_{tk})U'(W^i_t(y_{tk}), P_t(y_{tk}))$. Dropping terms unrelated to y_{tk}, we have

$$\begin{aligned}
& f^i(y_{tk})U'(W^i_t(y_{tk}), P_t(y_{tk})) \\
\Rightarrow\ & \exp\left\{-\frac{W^i_{t-1}}{\tau} - \frac{D^i_{t-1}[P_t(y_{tk}) - P_{t-1}]}{\tau} - \frac{K^i_t}{2}[\mu^i_t(y_{tk}) - P_t(y_{tk})]^2\right. \\
& \left. -\frac{1-\rho^i}{2}\left(\frac{K_{t-1}K_t}{n_t} - h\right)[P_t(y_{tk}) - P_{t-1}]^2 - \frac{n^i_t K^i_{t-1}}{2K^i_t}(y_{tk} - \mu^i_{t-1})^2\right\} \\
=\ & \exp\left\{-K^i_{t-1}(\mu^i_{t-1} - P_{t-1})\frac{n_t}{K_t}(y_{tk} - P_{t-1})\right. \\
& - \frac{1}{2K^i_t}\left[\left(n^i_t - \frac{K^i_t n_t}{K_t}\right)(y_{tk} - P_{t-1}) + K^i_{t-1}(\mu^i_{t-1} - P_{t-1})\right]^2 \\
& \left. -\frac{1-\rho^i}{2}\left(\frac{K_{t-1}K_t}{n_t} - h\right)\frac{n^2_t}{K^2_t}(y_{tk} - P_{t-1})^2 - \frac{n^i_t K^i_{t-1}}{2K^i_t}(y_{tk} - \mu^i_{t-1})^2\right\} \\
\Rightarrow\ & \exp\left\{-\left[\frac{K^i_{t-1}n_t}{K_t} + \frac{K^i_{t-1}}{K^i_t}\left(n^i_t - \frac{n_t K^i_t}{K_t}\right) - \frac{K^i_{t-1}n^i_t}{K^i_t}\right]\right. \\
& \times(\mu^i_{t-1} - P_{t-1})(y_{tk} - P_{t-1}) - \frac{1}{2}\left[\frac{K^i_{t-1}n^i_t}{K^i_t} + \frac{1}{K^i_t}\left(n^i_t - \frac{n_t K^i_t}{K_t}\right)^2\right. \\
& \left.\left. +(1-\rho^i)\left(\frac{K_{t-1}K_t}{n_t} - h\right)\frac{n^2_t}{K^2_t}\right](y_{tk} - P_{t-1})^2\right\} \\
=\ & \exp\left[\frac{K_{t-1}K_t}{n_t}(y_{tk} - P_{t-1})^2\right],
\end{aligned} \tag{64}$$

where " $\Rightarrow$ " means that a multiplier (a proportional factor) that is unrelated to y_{tk} has been omitted. Consequently, the marginal rates of substitution are unrelated to i and are equal for all investors.

Because $f^i(y_{tk})U'(W_t^i(y_{tk}), P_t(y_{tk}))$ is independent of investor i's information, the Euler equations for all investors differ by only a multiplier. As a result, if the Euler conditions for one investor (e.g., the average investor) are satisfied, then the Euler equations for all other investors will be satisfied. Consequently, we only need to show that the Euler equations hold for the average investor who does not trade in the market.

Let a denote the average investor with $\rho^a = 1$. Following Brennan (1979), we have

$$P_t U'(W_t^a, P_t) = \mathrm{E}_t^a[P_T U'(W_T^a, P_T)], \quad U'(W_t^a, P_t) = \mathrm{E}_t^a[U'(W_T^a, P_T)],$$

which imply that

$$\begin{aligned}
\mathrm{E}_{t-1}^a[(P_t - P_{t-1})U'(W_t^a, P_t)] &= \mathrm{E}_{t-1}^a[P_t U'(W_t^a, P_t) - P_{t-1}U'(W_t^a, P_t)] \\
&= \mathrm{E}_{t-1}^a[P_T U'(W_T^a, P_T) - P_{t-1}U'(W_{t-1}^a, P_{t-1})] = 0.
\end{aligned}$$

Similarly, it can be shown that

$$\mathrm{E}_{t-1}^a[(P_{PZt} - P_{PZ(t-1)})U'(W_t^a, P_t)] = 0,$$

$$\mathrm{E}_{t-1}^a[(P_{CZt} - P_{CZ(t-1)})U'(W_t^a, P_t)] = 0.$$

Thus, the Euler equations hold for investor a at session $(t-1)$, which further implies that the Euler equations hold for all investors. The proposed demands are optimal for all investors at session $(t-1)$. By mathematical induction, the proposed equilibrium demands are optimal in all periods.

Q.E.D.

Proof of Theorem 3: We prove this theorem using mathematical induction. It is straightforward to show that the price function stated in the theorem clears the market; then we only need to show that the investors' demands are optimal.

In the last trading session, substituting investor i's terminal wealth into the utility function, using the conjectured equilibrium price of the option, and taking expectations, we can write investor i's portfolio problem as

$$\begin{aligned}
\max_{D^i_{Q(T-1)}, D^i_{T-1}} \mathrm{E}_{T-1}^i[U^i] &= -\sqrt{\frac{1}{1 + \frac{2D^i_{Q(T-1)}}{\tau K^i_{T-1}}}} \exp\left\{ \frac{\left[\frac{D^i_{T-1}}{\tau} - K^i_{T-1}(\mu^i_{T-1} - P_{T-1})\right]^2}{K^i_{T-1} + \frac{2D^i_{Q(T-1)}}{\tau}} \right\} \\
&\quad \times \exp\left\{ \frac{D^i_{Q(T-1)} P_{Q(T-1)}}{\tau} - \frac{K^i_{T-1}}{2}(\mu^i_{T-1} - P_{T-1})^2 \right\},
\end{aligned}$$

where $D^i_{Q(T-1)}$ and D^i_{T-1} are the number of units of the quadratic option and the stock purchased by investor i, respectively. The optimal solutions are then given by

$$D^i_{T-1} = \tau K^i_{T-1}(\mu^i_{T-1} - P_{T-1}),$$

$$D^i_{Q(T-1)} = \frac{1}{2}\tau\left(\frac{1}{P_{Q(T-1)}} - K^i_{T-1}\right).$$

Given the equilibrium trading strategies in the last trading session, investor i's expected utility in the last trading session is:

$$\mathrm{E}^i_{T-1}[U^i] \propto -\exp\left[-\frac{W^i_{T-1}}{\tau} - \frac{1}{2}(b^i_{T-1})^2 K^i_{T-1}\right]. \tag{65}$$

The first term in the exponential comes from the investor's wealth, the second term represents the gains from trading in the last session, and $b^i_{T-1} \equiv \mu^i_{T-1} - P_{T-1}$.

Suppose that at trading session $(t+1)$, we have

$$D^i_{t+1} = \tau K^i_{t+1}[\mu^i_{t+1} - P_{t+1}], \tag{66}$$

$$\begin{aligned} D^i_{Q(t+1)} &= \frac{\tau}{2}\left[(1-\rho^i)\frac{K^2_{t+1}}{n_{t+2}} + \frac{1}{P_{Q(t+1)} - P^2_{t+1} - 1/K_{t+2}} - K^i_{t+1}\right] \\ &= \frac{\tau}{2}(1-\rho^i)\left[\frac{K_t K_{t+1}}{n_{t+1}} - h\right], \end{aligned} \tag{67}$$

$$\mathrm{E}^i_{t+1}[U^i] \propto -\exp\left[-\frac{W^i_{t+1}}{\tau} - \frac{1}{2}(b^i_{t+1})^2 K^i_{t+1}\right]. \tag{68}$$

If at session t, we have

$$D^i_t = \tau K^i_t[\mu^i_t - P_t], \tag{69}$$

$$D^i_{Qt} = \frac{\tau}{2}\left[(1-\rho^i)\frac{K^2_t}{n_{t+1}} + \frac{1}{P_{Qt} - P^2_t - 1/K_{t+1}} - K^i_t\right], \tag{70}$$

$$\mathrm{E}^i_t[U^i] \propto -\exp\left[-\frac{W^i_t}{\tau} - \frac{1}{2}(b^i_t)^2 K^i_t\right], \tag{71}$$

then the proposed equilibrium holds by induction.

We now calculate the expected utility conditional on a_{t+1}; dropping irrelevant terms, this is given by:

$$\begin{aligned}
\mathrm{E}_t^i[U^i|a_{t+1}] &\propto -\int_R \exp\left[-\frac{D_t^i a_{t+1} + D_{Qt}^i(P_t^2 + 2P_t a_{t+1} + a_{t+1}^2 + 1/K_{t+1} - P_{Qt})}{\tau}\right] \\
&\quad \times \exp\left[-\frac{(b_{t+1}^i)^2 K_{t+1}^i}{2} - \frac{(b_{1+1}^i - \mu_{b_{t+1}^i})^2 \Gamma_t^i}{2}\right] db_{t+1}^i \\
&= -\exp\left[-\frac{D_t^i a_{t+1} + D_{Qt}^i(P_t^2 + 2P_t a_{t+1} + a_{t+1}^2 + 1/K_{t+1} - P_{Qt})}{\tau}\right] \\
&\quad \times \exp\left[-\frac{(B_t^i a_{t+1} + C_t^i b_t^i)^2 \Lambda_t^i}{2}\right]. \qquad (72)
\end{aligned}$$

Taking the expectation with respect to a_{t+1}, we get

$$\begin{aligned}
\mathrm{E}_t^i[U^i] &\propto -\int_R \exp\left[-\frac{D_t^i a_{t+1} + D_{Qt}^i(P_t^2 + 2P_t a_{t+1} + a_{t+1}^2 + 1/K_{t+1} - P_{Qt})}{\tau}\right] \\
&\quad \times \exp\left[-\frac{(B_t^i a_{t+1} + C_t^i b_t^i)^2 \Lambda_t^i}{2} - \frac{(a_{t+1} - \mu_{a_{t+1}}^i)^2 \Psi_t^i}{2}\right] da_{t+1} \\
&\propto -\frac{1}{\sqrt{G_{tQ}^i}} \exp\left[-\frac{D_{Qt}^i(P_t^2 + 1/K_{t+1} - P_{Qt})}{\tau} + \frac{(F_{tQ}^i)^2 G_{tQ}^i}{2} - \frac{(b_t^i)^2 H_{tQ}^i}{2}\right], \qquad (73)
\end{aligned}$$

where

$$F_{tQ}^i = (D_t^i + 2D_{Qt}^i P_t)/\tau + B_t^i \Lambda_t^i C_t^i b_t^i - \Psi_t^i \mu_{a_{t+1}}^i, \qquad (74)$$

$$G_{tQ}^i = [(B_t^i)^2 \Lambda_t^i + \Psi_t^i + 2D_{Qt}^i/\tau]^{-1}, \qquad (75)$$

$$H_{tQ}^i = (C_t^i)^2 \Lambda_t^i + A_{t+1}^2 \Psi_t^i = C_t^i K_t^i + (B_t^i + I)A_{t+1} K_t^i = [C_t^i + I - C_t^i]' K_t^i = K_t^i. \qquad (76)$$

Since Λ_t^i and Ψ_t^i are positive, G_t^i is positive.

The first-order conditions simplify to:

$$F_{tQ}^i = 0, \qquad (77)$$

$$2G_{tQ}^i + P_t^2 + \frac{1}{K_{t+1}} - P_{Qt} = 0, \qquad (78)$$

which imply that investor i's optimal demands in the tth trading session are given by

$$D_t^i = \tau K_t^i[\mu_t^i - P_t] - 2D_{Qt}^i P_t, \tag{79}$$

$$D_{Qt}^i = \frac{1}{2}\tau[(1-\rho^i)\frac{K_t^2}{n_{t+1}} + \frac{1}{P_{Qt} - P_t^2 - 1/K_{t+1}} - K_t^i]. \tag{80}$$

These are indeed the proposed equilibrium strategies. Substituting equations (74), (75), and (76) into equation (73), we have

$$\mathrm{E}_t^i[U^i] \propto -\exp\left[-\frac{W_t^i}{\tau} - \frac{(b_t^i)^2 K_t^i}{2}\right]. \tag{81}$$

Q.E.D.

Proof of Proposition 1: Assume that $K_{t-1}^i = K_{t-1}$, $n_t^i \neq n_t$, and $n_j^i = n_j$ for $j > t$. From investor i's demand function, we get

$$\Delta D_t^i = \tau\left(\frac{n_t^i}{n_t} - 1\right) K_{t-1}\Delta P_t \neq 0$$

when $\Delta P_t \neq 0$. In addition, because there is agreement on the public signals after session t, we have

$$\Delta D_j^i = \tau(K_j - K_j^i)\Delta P_j \neq 0, \quad \text{for} \quad \Delta P_j \neq 0.$$

Q.E.D.

Proof of Proposition 2: Notice that the option price P_{Q_t} is independent of ρ^i. Hence, even if $\rho^i = 1$ for all investors, the option prices would remain the same. Because all state contingent claims can be synthesized using options, all option prices can be determined according to the average investor and thus all assets can be priced using the same principle.

Proof of Proposition 3: We prove this proposition by examples. Consider the case in which $T1 = 1$ and $T2 = 2$. When $T1 = 1$, investor i's expected utility is

$$\mathrm{E}U^i = -\exp\left[\frac{-\bar{u}x^i}{\tau} + \frac{(x^i)^2}{2\tau^2 h} - \frac{(x^i - x)^2}{2\tau^2 h}\right].$$

When investors can trade with an additional trading session, his wealth at time 1 is

$$W_1^i = xP_0 + D_0^i(P_1 - P_0) + D_1^i(u - P_1).$$

Straightforward calculation shows that investor i's wealth at time 1 is:

$$W_1^i = xu + \tau \frac{(n_1^i - n_1)}{h + n_1} \left[h(y - \mu) + \frac{x}{\tau} \right] \left(u - \frac{h\mu + n_1 y}{h + n_1} + \frac{x}{\tau K_1} \right).$$

Consider the view from investor j whose $n_1^j = n_1$. From j's point of view, the optimal strategy for investor i should be $\hat{D}_1^i = \hat{D}_1^j = x$. That is, investor i is better off not trading in the second session from investor j's point of view. However, since investor i will trade in the second trading session, from investor j's point of view, investor i is worse off when the second trading session is added.

Next, we consider the case with options. For ease of exposition, consider the case in which there are two trading sessions, 0 and 1 and options are only available in trading session 1. In the absence of options, investor i's wealth at time 1 is given by

$$W_1^i = xP_1 + D_1^i(u - P_1) = x\mu + (D_1^i - x)(u - P_1),$$

where

$$P_1 = \frac{h\bar{\mu} + n_1 y}{h + n_1} - \frac{x}{(h + n_1)\tau},$$

$$D_1^i = \tau(h\bar{\mu} + n_1^i y - (h + n_1^i)P_1,$$

$$\begin{aligned} D_1^i - x &= \tau \left\{ h\bar{\mu} + n_1^i y - \frac{h + n_1^i)(h\bar{\mu} + n_1 y)}{h + n_1} + \frac{(h + n_1^i)x}{h + n_1)\tau} \right\} - x \\ &= \frac{\tau(n_1^i - n_1)}{h + n_1} \left[h(y - \bar{\mu}) + \frac{x}{\tau} \right]. \end{aligned}$$

W_1^i is then given by

$$W_1^i = x\mu + \frac{\tau(n_1^i - n_1)}{h + n_1} \left[h(y - \bar{\mu}) + x\tau \right] \left[u - \frac{h\bar{\mu} + n_1 y}{h + n_1} + \frac{x}{(h + n_1)\tau} \right],$$

or we write

$$-\frac{W_1^i}{\tau} \equiv -C - B^T\omega - \frac{1}{2}\omega^T A\omega,$$

where $\omega^T = (u, \eta_1)$.

It can be shown that based on investor j's belief, the expected utility of investor i is equal to $-\frac{\sqrt{h n_1^j}}{\sqrt{det|A|}} \exp(-C + 1/2B^T A^{-1} B)$, where $C = -\frac{(n_1^i - n_1)}{h n_1} \frac{x^2}{\tau^2 (h + n_1)}$, the two elements in the B vector are given by $B_1 = \frac{x}{\tau} + \frac{(n_1^i - n_1)}{h+n_1}\left[(-h\bar{\mu} + x/\tau)(\frac{h}{h+n_1} + h(-\frac{h\bar{\mu}}{h+n_1} + \frac{x}{(h+n_1)\tau})\right]$ and $B_2 = \frac{n_1^i - n_1}{h+n_1}\left[(-h\bar{\mu} + x/\tau)(\frac{-n_1}{h+n_1}) + h(-\frac{h\bar{\mu}}{h+n_1} + \frac{x}{h+n_1}\tau)\right]$, and the four elements of the A matrix are given by $A_{11} = h + \frac{h^2}{h+n_1}\frac{2(n_1^i - n_1)}{h+n_1}$, $A_{12} = A_{21} = \frac{n_1^i - n_1}{h+n_1}\left[-\frac{hn_1}{h+n_1} + \frac{h^2}{h+n_1}\right]$, and $A_{22} = n_1^j + \frac{2(n_1^i - n_1)}{h+n_1}\left(-\frac{hn_1}{h+n_1}\right)$.

In the presence of options, the wealth of investor i in period 1 is given by

$$W_1^i = xP_0 + D_0^i(P_1 - P_0) + D_1^i(u - P_1) + D_{Q0}^i(P_{Q1} - P_{Q0}) + D_{Q1}^i(u^2 - P_{Q0}).$$

Similarly, we can calculate the expected utility of investor i based on the belief of investor j, yielding

$$-\frac{\sqrt{h}n_1^j}{\sqrt{det|A^o|}}\exp(-C^o + 1/2B^{oT}(A^o)^{-1}B^o).$$

Here C^o, B^o, and A^o are given by

$$C^o = C - (n_1^i - n_1)\frac{x^2}{(h+n_1)^2\tau^2},$$

$$B_1^o = B_1 - (n_1^i - n_1)\frac{h}{h+n_1}\left(-\frac{h\bar{\mu}}{h+n_1} + \frac{x}{(h+n_1)\tau}\right),$$

$$B_2^o = B_2 - (n_1^i - n_1)\frac{n_1}{h+n_1}\left(\frac{h\bar{\mu}}{h+n_1} - \frac{x}{(h+n_1)\tau}\right),$$

$$A_{11}^0 = A_{11} - (n_1^i - n_1)\frac{h^2}{(h+n_1)^2}, \quad A_{12} = A_{21} = A_{12} - (n_1^i - n_1)\frac{h}{h+n_1}\left(-\frac{n_1}{h+n_1}\right),$$

$$A_{22}^o = A_{22} - (n_1^i - n_1)\left(\frac{n_1}{h+n_1}\right)^2 = n_1^j - \left(\frac{n_1^i - n_1}{h+n_1}\right)\left(\frac{2hn_1 + n_1^2}{h+n_1}\right).$$

We need to show that the expected utility of investor i can be less with options trading from investor j's point of view. Notice that when $n_1^i > n_1$, as $n_1^j \to \left(\frac{n_1^i - n_1}{h+n_1}\right)\left(\frac{2hn_1 + n_1^2}{h+n_1}\right)$, $A_{22}^o \to 0$ while $A_{22} \to (n_1^i - n_1)\frac{n_1^2}{(h+n_1)^2} > 0$. This means that as $n_1^j \to \left(\frac{n_1^i - n_1}{h+n_1}\right)\left(\frac{2hn_1 + n_1^2}{h+n_1}\right)$, investor i's expected utility through trading in options from investor j's point of view goes to negative infinity,[17]

[17]Note that $\sqrt{det|A^o|}$ goes to zero in this case.

whereas investor i's expected utility through trading in stocks from investor i's point of view goes to a finite value. Thus, there exists parameter values that from investor j's point of view, investor i is worse off trading in options.
Q.E.D

Proof of Proposition 4: This proposition follows directly from equation (28).

Proof of Proposition 5: From equation (6), we have

$$\begin{aligned}\mathrm{Cov}^i(P_{t+1}-P_t,P_t) &= \frac{n_{t+1}}{K_{t+1}}\mathrm{Cov}^i(u-P_t,P_t)\\ &= \frac{n_{t+1}}{K_{t+1}}\mathrm{Cov}^i(u-\rho^iP_t+(\rho^i-1)P_t,P_t)\\ &= \frac{n_{t+1}}{K_{t+1}}(\rho^i-1)\mathrm{Var}^i(P_t), \end{aligned} \tag{82}$$

where $(u-\rho^iP_t)$ is independent of P_t. Similarly, we have

$$\begin{aligned}\mathrm{Cov}^i(P_{t+1}-P_t,P_t-P_{t-1}) &= \frac{n_tn_{t+1}}{K_tK_{t+1}}\mathrm{Cov}^i(u-P_t,u+\eta_t-P_{t-1})\\ &= \frac{n_tn_{t+1}}{K_tK_{t+1}}\mathrm{Cov}^i(u-\rho^iP_t+(\rho^i-1)P_t,P_t)\\ &= \frac{hn_tn_{t+1}}{K_{t-1}K_t^2K_{t+1}}\frac{\rho^i-1}{\rho^i}. \end{aligned} \tag{83}$$

Q.E.D.

Proof of Proposition 6: The proposition follows directly from equations (21) and (37).

Proof of Proposition 7: Let $\Delta D_t^i(n)$ and $\Delta D_t^i(o)$ denote the trading volume in session t for investor i in the economy without and with options, respectively. With the assumption of stationary consensus price change volatility, we have

$$\Delta D_t^i(n)=\tau(\rho_i-1)h\Delta P_t,$$

$$\Delta D_t^i(o)=\tau(\rho_i-1)\frac{K_tK_{t+1}}{n_{t+1}}\Delta P_t.$$

Because $K_{t+1}\geq n_{t+1}$ and $K_t\geq h$, we have that $|\Delta D_t^i(o)|\geq|\Delta D_t^i(n)|$ and that $\Delta D_t^i(o)$ is more sensitive to ΔP_t than $\Delta D_t^i(n)$ is.
Q.E.D.

Proof of Lemma 1: We provide the proof for the economy with two trading sessions. The proof for the general case is similar and is thus omitted.

Investor i's wealth at time 1 is given by

$$W_1^i = W_0^i + D_0^i(u - P_0) + (D_{1i} - D_0^i)(u - P_1).$$

Notice that

$$P_1 - P_0 = \frac{n_1}{K_1}(u + \eta_1 - P_0).$$

The wealth function, ignoring some irrelevant terms, is then given by

$$W_1^i = \frac{n_2}{K_2}\left[(u+\eta_1 - P_0)\left(u - P_0 - \frac{n_1}{K-1}(u+\eta_1 - P_0)\right)\right]\left(K_0^i - \frac{n_{1i}}{n_1}K_0\right).$$

Taking the expectation of investor i's utility function with respect to u and η_1, we have

$$\mathrm{E}U^i = -\sqrt{\frac{K_0^i n_{1i}}{\det|\Omega|}}\exp\left[\frac{-\bar{u}x^i}{\tau} + \frac{(x^i)^2}{2\tau^2 h} - \frac{(x^i - x)^2}{2\tau^2 h}\right] = \sqrt{\frac{K_0^i n_{1i}}{\det|\Omega|}}\mathrm{E}U^i(1),$$

where

$$\Omega = \begin{Bmatrix} \frac{1}{2}K_{0i} + \frac{k_0 n_1}{k_1^2}(\frac{n_{1i}}{n_1}K_1 - K_{1i}), & \frac{n_1}{K_1}(\frac{K_0 - n_1}{K_1})(\frac{n_{1i}}{n_1}K_1 - K_{1i}) \\ \frac{n_1}{K_1}(\frac{K_0 - n_1}{K_1})(\frac{n_{1i}}{n_1}K_1 - K_{1i}), & \frac{1}{2}n_{1i} + \frac{n_1^2}{K_1^2}(\frac{n_{1i}}{n_1}K_1 - K_{1i}) \end{Bmatrix}.$$

Further algebra gives the results in the lemma for the two trading session case.
Q.E.D.

Proof of Lemma 2: From the proof of Theorem 3 and equations (56) and (73), the added value of options trading in period t is given by

$$\frac{\tau}{2}\ln\left(\frac{G_{tQ}^i}{G_t^i}\right) - D_{Qt}^i\frac{n_{t+1}}{K_t K_{t+1}},$$

which reduces to

$$\frac{\tau}{2}\left[x_t^i - \ln(1 + x_t^i)\right],$$

where

$$x_t^i = \frac{n_{(t+1)i}K_t}{n_{t+1}K_{t+1}} + \frac{n_{t+1}K_{ti}}{K_t K_{t+1}} - 1 = (\rho^i - 1)\left(1 - \frac{hn_{t+1}}{K_t K_{t+1}}\right).$$

Q.E.D.

Proof of Lemma 3: Parts (i), (iv), and (v) follow immediately from investors' trades, price changes, and Proposition 2. Let $\Delta\mu_t \equiv \mu_t - \mu_{t-1}$, Part (ii) follows because

$$\Delta P_t = \Delta u_t + \frac{x n_t}{\tau K_{t-1} K_t}.$$

Part (iii) follows because $\mathrm{E}^i[\Delta P_t]$ and $\mathrm{Var}^i[\Delta P_t]$ are both increasing in n_t and because $V(\mu, \sigma)$ is increasing in μ and σ for positive μ and σ.

References

Admati, A., and P. Pfleiderer, 1988, A theory of intraday patterns: Volume and price variability, *Review of Financial Studies*, 1, 3-40.

Ajinkya, B. B., R. K. Atiase, and M. J. Gift, 1991, Volume of trading and the dispersion in financial analysts' earnings forecasts, *Accounting Review*, 66, 389-401.

Back, K., 1993, Asymmetric information and options, *Review of Financial Studies*, 3, 1-24.

Basak, S., 2000, A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk, *Journal of Economic Dynamic and Control*, 24, 63-95.

Bessembinder, H., K. Chan, and P. J. Seguin, 1996, An empirical examination of information, differences of opinion, and trading activity, *Journal of Financial Economics*, 40, 105-134.

Biais, B., and P. Bossaerts, 1998, Asset prices and trading volume in a beauty contest, *Review of Economic Studies*, 65, 307-340.

Biais, B., P. Bossaerts, and C. Spatt, 2003, Equilibrium asset pricing under heterogeneous information, working paper, Toulouse, Caltech, and Carnegie Mellon.

Biais, B., and P. Hillion, 1994, Insider and liquidity trading in stock and options markets, *Review of Financial Studies*, 7, 743-780.

Black, F., and M. S. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, 637-654.

Brav, A., and J. B. Heaton, 2002, Competing theories of financial anomalies, *Review of Financial Studies*, 15, 575-606.

Breeden, D., and R. Litzenberger, 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business*, 51, 621-651.

Brennan, M. J., 1979, The pricing of contingent claims in discrete time models, *Journal of Finance*, 34, 53-68.

Brennan, M. J., and H. H. Cao, 1996, Information, trade, and option securities, *Review of Financial Studies*, 9, 163-208.

Brennan, M. J., and A. Kraus, 1978, Necessary conditions for aggregation in security markets, *Journal of Financial and Quantitative Analysis*, 40, 7-18.

Brown, D. P., and R. H. Jennings, 1989, On technical analysis, *Review of Financial Studies*, 2, 527-551.

Buraschi, A., and A. Jilstov, 2003, Option volume and differences in beliefs, working paper, LBS.

Cao, H. H., 1999, The effect of option stocks on information acquisition and price behavior in a rational expectations equilibrium, *Review of Financial Studies*, 12, 131-163.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam, 1998, Investor psychology and securities market under- and overreaction, *Journal of Finance*, 53, 1839-1885.

David, A., 2003, Heterogeneous beliefs, trading risk, and the equity premium, working paper, Washington University in St. Louis.

DeMarzo, P., and C. Skiadas, 1998, Aggregation, determinacy, and information efficiency for a class of economies with asymmetric information, *Journal of Economic Theory*, 80, 123-152.

DeTemple, J., and Murthy, 1994, Intertemporal asset pricing with heterogenous information, *Journal of Economic Theory*, 62, 294-320.

DeTemple, J., and L. Selden, 1991, A general equilibrium analysis of option and stock market interactions, *International Economic Review*, 32, 279-302.

Duffie, D., N. Garleanu, and L. Pederson, 2002, Securities lending, shorting, and pricing, *Journal of Financial Economics*, 66, 307-339.

Easley, D., M. O'Hara, and P. S. Srinivas, 1998, Option volume and stock prices: Evidence on where informed traders trade, *Journal of Finance*, 53, 431-465.

Foster, F. D., and S. Viswanathan, 1993, The effect of public information and competition on trading volume and price volatility, *Review of Financial Studies*, 6, 23-56.

Frankel, J. F., and K. Froot, 1990, Chartists, fundamentalists and trading in the foreign exchange market, *American Economic Review*, 80, 181-185.

Grundy, B. D. and M. McNichols, 1989, Trade and the revelation of information through prices and direct disclosure, *Review of Financial Studies*, 2, 495-526.

Harris, M., and A. Raviv, 1993, Differences of opinion make a horse race, *Review of Financial Studies*, 6, 473-506.

Harrison, J. M., and D. M. Kreps, 1978, Speculative investor behavior in a stock market with heterogenous expectations, *Quarterly Journal of Economics*, 93, 323-336.

He, H., and J. Wang, 1995, Differential information and dynamic behavior of stock trading volume, *Review of Financial Studies*, 8, 919-972.

Holden, C. W., and A. Subrahmanyam, 1992, Long-lived private information and imperfect competition, *Journal of Finance* 47, 247-270.

Hong, H., and J. C. Stein, 2003, Differences of opinion, short-sales constraints, and market crashes, *Review of Financial Studies*, 16, 487-525.

Huberman, G., and T. Regev, 2001, Contagious speculation and a cure for cancer: A nonevent that made stock price soar, *Journal of Finance*, 56, 387-396

Jarrow, R., 1980, Heterogenous expectations, restrictions on short sales and equilibrium asset prices, *Journal of Finance*, 35, 1105-1114.

Kandel, E., and N. D. Pearson, 1995, Differential interpretation of public signals and trade in speculative markets, *Journal of Political Economy*, 103, 831-872.

Karpoff, J. M., 1987, The relation between price changes and trading volume: A survey, *Journal of Financial and Quantitative Analysis*, 22, 109-126.

Kim, O., and R. E. Verrcchia, 1991, Trading volume and price reactions to public announcements, *Journal of Accounting Research*, 29, 302-321.

Kraus, A., and M. Smith, 1996, Heterogenous beliefs and the effects of replicatable options on asset prices, *Review of Financial Studies*, 9, 723-756.

Kyle, A. S., 1985, Continuous auctions and insider trading, *Econometrica*, 53, 1315-1335.

Kyle, A. S., and T. Lin, 2002, Continuous trading with heterogeneous beliefs and no noise trading, working paper, Duke University and University of Hong Kong.

Leland, H. E., 1980, Who should buy portfolio insurance, *Journal of Finance*, 35, 581-594.

Liu, J., and J. Pan, 2003, Dynamic derivative strategies, *Journal of Financial Economics*, forthcoming.

Milgrom, P., and N. Stokey, 1982, Information, trade, and common knowledge, *Journal of Economic Theory*, 26, 17-27.

Morris, S., 1996, Speculative investor behavior and learning, *Quarterly Journal of Economics*, 111, 1111-1133.

Odean, T., 1998, Volume, volatility, price and profit when all traders are above average, *Journal of Finance*, 53, 1887-1934.

Pan, J., and A. M. Poteshman, 2003, The information in option volume for stock prices, working paper, MIT and UIUC.

Pfleiderer, P., 1984, The volume of trade and the variability of prices: A framework for analysis in noisy rational expectations equilibria, working paper, Stanford University.

Qu, S., L. Starks, and H. Yan, 2003, Risk, dispersion of analyst forecasts and stock returns, working paper, UT Austin.

Rubinstein, M., 1974, An aggregation theorem for securities markets, *Journal of Financial Economics*, 1, 225-244.

Scheinkman, J., and W. Xiong, 2003, Overconfidence, short-sale constraints, and bubbles, *Journal of Political Economy*, forthcoming.

Shalen, C. T., 1993, Volume, volatility, and the dispersion of beliefs, *Review of Financial Studies*, 6, 405-434.

Sharpe, W., 1964, Capital stock prices: A theory of capital market equilibrium under conditions of risk, *Journal of Finance*, 19, 425-442.

Skinner, D. J., 1990, Options markets and the information content of accounting earnings releases, *Journal of Accounting and Economics*, 13, 191-211.

Spiegel, M., and S. Subrahmanyam, 1992, Informed speculation and hedging in a noncompetitive securities market, *Review of Financial Studies*, 5, 307-330.

Swan, E., 2000, *Building the Global Market: A 4000 Year History of Derivatives*, Kluwer Law International.

Varian, H., 1989, Differences of opinion in financial markets," in C. Stone (Ed.), *Financial Risk: Theory, Evidence and Implications*, Kluwer Academic Publications.

Viswanathan, S., 2001, Strategic trading, heterogeneous beliefs, and short constraints, working paper, Duke University.

Wang, J., 1994, A model of competitive stock trading volume, *Journal of Political Economy*, 102, 127-168.

Zapatero, F., 1998, Effects of financial innovations on market volatility when beliefs are heterogenous, *Journal of Economic Dynamic and Control*, 22, 597-626.