# TA Session #9

## ECON 341: Econometrics I

J. Marcelo Ochoa

November 20, 2008

## Problem 1

Model:

$$y_i = \mathbf{x}_i'\beta + \gamma z_i + \varepsilon_i$$

$$V(e_i|\mathbf{x}_i, z_i) = \sigma^2 \qquad E(\mathbf{x}'z) = \mathbf{0}$$

**Procedure 1** $\hat{\beta}$ which is obtained from the estimation of $y$ on $\mathbf{x}$ and $z$.

**Procedure 2** $\tilde{\beta}$ that is obtained from the regression of $y$ on $x$.

## Consistency of $\hat{\beta}$

How do we obtain $\hat{\beta}$? Use the annihilator matrix

## Consistency of $\hat{\beta}$

How do we show consistency?

The sampling error equals:

$$\hat{\beta} = (\mathbf{X}' M_z \mathbf{X})^{-1} \mathbf{X}' M_z \mathbf{y}$$

In terms of sums:

## Consistency of $\hat{\beta}$

$$\hat{\beta} - \beta = \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{x}_i'\right)\right]^{-1}$$
$$\times\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\varepsilon_i + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\varepsilon_i\right)\right]$$

$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' \xrightarrow{p}$

$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i \xrightarrow{p}$

$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\varepsilon_i \xrightarrow{p}$

$\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2 \xrightarrow{p}$

$\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\varepsilon_i \xrightarrow{p}$

Replacing we have

## Asymptotic distribution of $\hat{\beta}$

$$\sqrt{n}(\hat{\beta} - \beta) = \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{x}_i'\right)\right]^{-1}$$
$$\times\left[\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\varepsilon_i\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\varepsilon_i\right)\right]$$

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\varepsilon_i\right) \xrightarrow{d}$$

Then:

$\sqrt{n}(\hat{\beta}-\beta) \xrightarrow{d}$

## Consistency of $\tilde{\beta}$

$$\mathbf{y} = \mathbf{X}\beta + v$$

with $v = \gamma \mathbf{z} + \varepsilon$.

$$\tilde{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'y$$

Thus, the sampling error is:

Is the estimator consistent?

$$\tilde{\beta} - \beta = (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i')^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i v_i$$

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' \xrightarrow{p}$$

$$\sum_{i=1}^{n}\mathbf{x}_i v_i \xrightarrow{p}$$

## Asymptotic distribution of $\tilde{\beta}$

$$\sqrt{n}(\tilde{\beta} - \beta) = (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i')^{-1}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i v_i\right)$$

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i v_i\right) \xrightarrow{d}$$

We have:

$$E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right) =$$

Using the law of iterated expectations:

$$E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right) =$$
$$=$$
$$=$$

# Asymptotic distribution of $\tilde{\beta}$

$$\sqrt{n}(\tilde{\beta} - \beta) \xrightarrow{d}$$
$$\xrightarrow{d}$$

Then,

$$[E(\mathbf{x}_i \mathbf{x}_i')]^{-1} E(v_i^2 \mathbf{x}_i \mathbf{x}_i') [E(\mathbf{x}_i \mathbf{x}_i')]^{-1}$$
$$=$$

# Is $\tilde{\beta}$ more efficient?

We need to show if it has a smaller variance,

$$V(\tilde{\beta}) - V(\hat{\beta})$$

is positive definite.

$$V(\tilde{\beta}) - V(\hat{\beta}) =$$

# Problem 2

We observe $\mathbf{w}_i = (y_i, x_i)$ are iid, and we want an estimate of

$$\mu = E(y_i)$$

We have two moment conditions:

$$E\left(\mathbf{g}(\mathbf{w}_i, \mu)\right) = E\begin{pmatrix} y_i - \mu \\ x_i \end{pmatrix} = \mathbf{0}$$

but one parameter, so the model is overidentified.
The GMM estimator of $\mu$, solves:

$$\min_{\mu} n\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{g}(\mathbf{w}_i, \mu)\right)' \mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{g}(\mathbf{w}_i, \mu)\right) = n\bar{\mathbf{g}}_n(\mu)'\mathbf{W}\bar{\mathbf{g}}_n(\mu)$$

# What is the optimal choice of W?

Let,

$$E\left(\mathbf{g}(\mathbf{w}_i, \mu)\mathbf{g}(\mathbf{w}_i, \mu)'\right) =$$

$$=$$

Then,

$$\mathbf{W} =$$

# Efficient GMM estimator

$$J_n(\mu) =$$

$$=$$

$$= \frac{n}{\hat{\sigma}_x\hat{\sigma}_y - (\hat{\sigma}_{xy})^2}[\hat{\sigma}_x^2(\bar{y}-\mu)^2 + \hat{\sigma}_y^2\bar{x}^2 - 2\hat{\sigma}_{xy}(\bar{y}-\mu)\bar{x}]$$

First-order condition:

$$\frac{dJ_n(\hat{\mu})}{d\mu} =$$

Then:

$$\hat{\mu} =$$

# Problem 3

The model,

$$y_i = \mathbf{x}_i'\beta + \varepsilon_i$$

More moment-conditions:

$$E(\mathbf{g}(\mathbf{w}_i,\beta)) = E\begin{pmatrix} \mathbf{x}_i\varepsilon_i \\ \mathbf{q}_i\varepsilon_i \end{pmatrix} = \mathbf{0}$$

than parameters.
We use GMM:

$$J_n(\beta) = n\bar{\mathbf{g}}_n(\beta)'\mathbf{W}\bar{\mathbf{g}}_n(\beta)$$

## What is the optimal choice of W?

$$\mathbf{W} = $$

$$E\left(\mathbf{g}(\mathbf{w}_i,\mu)\mathbf{g}(\mathbf{w}_i,\mu)'\right) = $$

$$=$$

$$\hat{S} =$$

with,

$$\hat{\sigma}^2 =$$

## Efficient GMM estimator

$$\begin{aligned} J_n(\beta) =& n\bar{\mathbf{g}}_n(\mu)'\hat{\mathbf{W}}\bar{\mathbf{g}}_n(\mu) \\ =& n\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i(y_i - \mathbf{x}_i'\beta)\right]'\hat{S}^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i(y_i - \mathbf{x}_i'\beta)\right] \\ =& n\left[\mathbf{Z}'\mathbf{y} - \mathbf{Z}'\mathbf{X}\beta\right]'\hat{S}^{-1}\left[\mathbf{Z}'\mathbf{y} - \mathbf{Z}'\mathbf{X}\beta\right] \\ =& n\Big[\mathbf{y}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y} + \beta'\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\beta + \mathbf{y}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\beta \\ & + \beta'\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y}\Big] \end{aligned}$$

with $\mathbf{Z} = (\mathbf{X}\ \mathbf{Q})$

$$\hat{\beta}_{GMM} = \left[\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\right]^{-1}\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y}$$