TA Session #9

ECON 341: Econometrics I

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November 21, 2008

Problem 1

Model:

$$y_i = \mathbf{x}_i'\beta + \gamma z_i + \varepsilon_i$$

$$V(e_i|\mathbf{x}_i, z_i) = \sigma^2 \qquad E(\mathbf{x}'z) = \mathbf{0}$$

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Procedure 1 $\hat{\beta}$ which is obtained from the estimation of y on $\mathbf{x}$ and z.

Procedure 2 $\tilde{\beta}$ that is obtained from the regression of y on x.

Consistency of $\hat{\beta}$

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and obtain:

$$M_z\mathbf{y} = M_z(\mathbf{X}\beta + \gamma\mathbf{z} + \varepsilon)$$

What do you obtain?

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then:

$$\hat{\beta} = (\mathbf{X}'M_z'M_z\mathbf{X})^{-1}\mathbf{X}'M_z'M_z\mathbf{y} = (\mathbf{X}'M_z\mathbf{X})^{-1}\mathbf{X}'M_z\mathbf{y}$$

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The sampling error equals:

$$\begin{aligned}
\hat{\beta} &= (\mathbf{X}' M_z \mathbf{X})^{-1} \mathbf{X}' M_z \mathbf{y} \\
&= (\mathbf{X}' M_z \mathbf{X})^{-1} \mathbf{X}' M_z (M_z \mathbf{X} \beta + M_z \varepsilon) \\
\hat{\beta} - \beta &= (\mathbf{X}' M_z \mathbf{X})^{-1} \mathbf{X}' M_z \varepsilon
\end{aligned}$$

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In terms of sums:

$$\begin{aligned} \mathbf{X}' M_z \varepsilon = \mathbf{X}'[\mathbf{I} - \mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}]\varepsilon &= \mathbf{X}'\varepsilon - \mathbf{X}'\mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'\varepsilon \\ &= \sum_{i=1}^{n} \mathbf{x}_i \varepsilon_i - \left(\sum_{i=1}^{n} \mathbf{x}_i \mathbf{z}_i \right) \left(\sum_{i=1}^{n} \mathbf{z}_i^2 \right)^{-1} \left(\sum_{i=1}^{n} \mathbf{z}_i \varepsilon_i \right) \end{aligned}$$

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In terms of sums:

$$\begin{aligned}\mathbf{X}' M_z \varepsilon &= \mathbf{X}'[\mathbf{I} - \mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}]\varepsilon = \mathbf{X}'\varepsilon - \mathbf{X}'\mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'\varepsilon \\ &= \sum_{i=1}^{n} \mathbf{x}_i \varepsilon_i - \left(\sum_{i=1}^{n} \mathbf{x}_i \mathbf{z}_i\right)\left(\sum_{i=1}^{n} \mathbf{z}_i^2\right)^{-1}\left(\sum_{i=1}^{n} \mathbf{z}_i \varepsilon_i\right)\end{aligned}$$

$$\begin{aligned}\mathbf{X}' M_z \mathbf{X} &= \mathbf{X}'[\mathbf{I} - \mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}]\mathbf{X} = \mathbf{X}'\mathbf{X} - \mathbf{X}'\mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'\mathbf{X} \\ &= \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' - \left(\sum_{i=1}^{n} \mathbf{x}_i \mathbf{z}_i\right)\left(\sum_{i=1}^{n} \mathbf{z}_i^2\right)^{-1}\left(\sum_{i=1}^{n} \mathbf{z}_i \mathbf{x}_i'\right)\end{aligned}$$

Consistency of $\hat{\beta}$

$$\hat{\beta} - \beta = \left[\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' + \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{z}_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i^2 \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \mathbf{x}_i' \right) \right]^{-1}$$

$$\times \left[\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \varepsilon_i + \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{z}_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i^2 \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \varepsilon_i \right) \right]$$

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$\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' \xrightarrow{p}$

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Consistency of $\hat{\beta}$

$$\hat{\beta}-\beta=\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'+\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{x}_i'\right)\right]^{-1}$$

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$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' \xrightarrow{p} E\left(\mathbf{x}_i\mathbf{x}_i'\right)$

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$\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' \xrightarrow{p} E\left(\mathbf{x}_i \mathbf{x}_i'\right)$ $\qquad$ $\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i^2 \xrightarrow{p}$

$\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{z}_i \xrightarrow{p} E\left(\mathbf{x}_i \mathbf{z}_i\right) = \mathbf{0}$ $\qquad$ $\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \varepsilon_i \xrightarrow{p}$

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$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' \xrightarrow{p} E\left(\mathbf{x}_i\mathbf{x}_i'\right) \qquad \frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2 \xrightarrow{p} E\left(\mathbf{z}_i^2\right)$$

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Replacing we have

$$\hat{\beta} \xrightarrow{p} \beta$$

Asymptotic distribution of $\hat{\beta}$

$$\sqrt{n}(\hat{\beta}-\beta) = \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{z}_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^2\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{x}_i'\right)\right]^{-1}$$

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$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \varepsilon_i \right) \xrightarrow{d}$$

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$$\sqrt{n}\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\mathbf{x}_i\varepsilon_i\right) \xrightarrow{d} \quad N\left(\mathbf{0}, \sigma^2 E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right)$$

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$$\sqrt{n}\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\mathbf{x}_i\varepsilon_i\right) \xrightarrow{d} \quad N\left(\mathbf{0},\sigma^2 E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right)$$

Then:

$$\sqrt{n}(\hat{\beta}-\beta) \xrightarrow{d} \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} N\left(\mathbf{0},\sigma^2 E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right) = N\left(\mathbf{0},\sigma^2\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1}\right)$$

Consistency of $\tilde{\beta}$

$$\mathbf{y} = \mathbf{X}\beta + v$$

with $v = \gamma \mathbf{z} + \varepsilon$.

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$$\tilde{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'y$$

Thus, the sampling error is:

$$\tilde{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + v) \Rightarrow \tilde{\beta} - \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'v$$

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Thus, the sampling error is:

$$\tilde{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + v) \Rightarrow \tilde{\beta} - \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'v$$

Is the estimator consistent?

$$\tilde{\beta} - \beta = (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i')^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i v_i$$

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$$\mathbf{y} = \mathbf{X}\beta + v$$

with $v = \gamma \mathbf{z} + \varepsilon$.

$$\tilde{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'y$$

Thus, the sampling error is:

$$\tilde{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + v) \Rightarrow \tilde{\beta} - \beta = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'v$$

Is the estimator consistent?

$$\tilde{\beta} - \beta = (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i')^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i v_i$$

Yes! Because:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' \xrightarrow{p} E\left(\mathbf{x}_i\mathbf{x}_i'\right)$$

$$\sum_{i=1}^{n}\mathbf{x}_i v_i \xrightarrow{p} E\left(\mathbf{x}_i v_i\right) = E\left(\gamma\mathbf{x}_i z_i + \mathbf{x}_i\varepsilon_i\right) = \mathbf{0}$$

Asymptotic distribution of $\tilde{\beta}$

$$\sqrt{n}(\tilde{\beta} - \beta) = (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i')^{-1}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i v_i\right)$$

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$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i v_i \right) \xrightarrow{d} N\left(\mathbf{0}, E\left(v_i^2 \mathbf{x}_i \mathbf{x}_i'\right)\right)$$

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We have:

$$E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right) = E\left((\gamma^2 z_i^2 + \varepsilon_i^2 + 2\gamma z_i\varepsilon_i)\mathbf{x}_i\mathbf{x}_i'\right)$$

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Using the law of iterated expectations:

$$\begin{aligned} E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right) &= E\left(E\left(\gamma^2 z_i^2\mathbf{x}_i\mathbf{x}_i + \varepsilon_i^2\mathbf{x}_i\mathbf{x}_i + 2\gamma z_i\varepsilon_i\mathbf{x}_i\mathbf{x}_i'|x_i\right)\right) \\ &= E\left(\gamma^2 E\left(z_i^2|x_i\right)\mathbf{x}_i\mathbf{x}_i' + E\left(\varepsilon_i^2|x_i\right)\mathbf{x}_i\mathbf{x}_i' + 2\gamma E\left(z_i\varepsilon_i|x_i\right)\mathbf{x}_i\mathbf{x}_i'\right) \\ &= \gamma^2 E\left(E\left(z_i^2|x_i\right)\mathbf{x}_i\mathbf{x}_i'\right) + \sigma^2 E\left(\mathbf{x}_i\mathbf{x}_i'\right) \end{aligned}$$

Asymptotic distribution of $\tilde{\beta}$

$$\begin{aligned}\sqrt{n}(\tilde{\beta} - \beta) &\xrightarrow{d} \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} N\left(\mathbf{0}, E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right)\right) \\ &\xrightarrow{d} N\left(\mathbf{0}, \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right)\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1}\right)\end{aligned}$$

Asymptotic distribution of $\tilde{\beta}$

$$\begin{aligned}\sqrt{n}(\tilde{\beta}-\beta) &\xrightarrow{d} \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} N\left(\mathbf{0}, E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right)\right) \\ &\xrightarrow{d} N\left(\mathbf{0}, \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right)\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1}\right)\end{aligned}$$

Then,

$$\begin{aligned}&\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(v_i^2\mathbf{x}_i\mathbf{x}_i'\right)\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} \\ =&\gamma^2\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(E\left(z_i^2|x_i\right)\mathbf{x}_i\mathbf{x}_i'\right)\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1}+\sigma^2\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1}\end{aligned}$$

Is $\tilde{\beta}$ more efficient?

We need to show if it has a smaller variance,

$$V(\tilde{\beta}) - V(\hat{\beta})$$

is positive definite.

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$$V(\tilde{\beta}) - V(\hat{\beta}) = \gamma^2 \left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1} E\left(E\left(z_i^2|x_i\right)\mathbf{x}_i\mathbf{x}_i'\right)\left[E\left(\mathbf{x}_i\mathbf{x}_i'\right)\right]^{-1}$$

Yes! this matrix is positive definite.

Problem 2

We observe $\mathbf{w}_i = (y_i, x_i)$ are iid, and we want an estimate of

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but one parameter, so the model is overidentified.
The GMM estimator of μ, solves:

$$\min_{\mu} n\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{g}(\mathbf{w}_i, \mu)\right)' \mathbf{W}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{g}(\mathbf{w}_i, \mu)\right) = n\bar{\mathbf{g}}_n(\mu)'\mathbf{W}\bar{\mathbf{g}}_n(\mu)$$

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Let,

$$E\left(\mathbf{g}(\mathbf{w}_i, \mu)\mathbf{g}(\mathbf{w}_i, \mu)'\right) = E\left(\begin{pmatrix} y_i - \mu \\ x_i \end{pmatrix} (y_i - \mu \; x_i)\right)$$

$$= \begin{pmatrix} E\left(y_i - \mu\right)^2 & E\left((y_i - \mu)x_i\right) \\ E\left((y_i - \mu)x_i\right) & E\left(x_i^2\right) \end{pmatrix}$$

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Then,

$$\mathbf{W} = \begin{pmatrix} \sigma_y^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_x^2 \end{pmatrix}^{-1} = \frac{1}{\sigma_x\sigma_y - (\sigma_{xy})^2}\begin{pmatrix} \sigma_x^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_y^2 \end{pmatrix}$$

But, do you see a problem with **W**?

Efficient GMM estimator

$$
\begin{aligned}
J_n(\mu) &= n\bar{\mathbf{g}}_n(\mu)'\hat{\mathbf{W}}\bar{\mathbf{g}}_n(\mu) \\
&= \frac{n}{\hat{\sigma}_x\hat{\sigma}_y - (\hat{\sigma}_{xy})^2}(\bar{y} - \mu\ \bar{x}) \begin{pmatrix} \hat{\sigma}_x^2 & -\hat{\sigma}_{xy} \\ -\hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{pmatrix} \begin{pmatrix} \bar{y} - \mu \\ \bar{x} \end{pmatrix} \\
&= \frac{n}{\hat{\sigma}_x\hat{\sigma}_y - (\hat{\sigma}_{xy})^2}[\hat{\sigma}_x^2(\bar{y} - \mu)^2 + \hat{\sigma}_y^2\bar{x}^2 - 2\hat{\sigma}_{xy}(\bar{y} - \mu)\bar{x}]
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\end{aligned}
$$

First-order condition:

$$
\frac{dJ_n(\hat{\mu})}{d\mu} = \frac{n}{\hat{\sigma}_x\hat{\sigma}_y - (\hat{\sigma}_{xy})^2}[-2\hat{\sigma}_x^2(\bar{y} - \hat{\mu}) + 2\hat{\sigma}_{xy}\bar{x}] = 0
$$

Efficient GMM estimator

$$\begin{aligned} J_n(\mu) &= n\bar{\mathbf{g}}_n(\mu)'\hat{\mathbf{W}}\bar{\mathbf{g}}_n(\mu) \\ &= \frac{n}{\hat{\sigma}_x\hat{\sigma}_y - (\hat{\sigma}_{xy})^2}(\bar{y} - \mu\ \bar{x}) \begin{pmatrix} \hat{\sigma}_x^2 & -\hat{\sigma}_{xy} \\ -\hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{pmatrix} \begin{pmatrix} \bar{y} - \mu \\ \bar{x} \end{pmatrix} \\ &= \frac{n}{\hat{\sigma}_x\hat{\sigma}_y - (\hat{\sigma}_{xy})^2}[\hat{\sigma}_x^2(\bar{y} - \mu)^2 + \hat{\sigma}_y^2\bar{x}^2 - 2\hat{\sigma}_{xy}(\bar{y} - \mu)\bar{x}] \end{aligned}$$

First-order condition:

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Then:

$$\hat{\mu} = \bar{y} - \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x^2}\bar{x}$$

Problem 3

The model,

$$y_i = \mathbf{x}_i' \beta + \varepsilon_i$$

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More moment-conditions:

$$E\left(\mathbf{g}(\mathbf{w}_i, \beta)\right) = E\begin{pmatrix} \mathbf{x}_i\varepsilon_i \\ \mathbf{q}_i\varepsilon_i \end{pmatrix} = \mathbf{0}$$

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than parameters.
We use GMM:

$$J_n(\beta) = n\bar{\mathbf{g}}_n(\beta)'\mathbf{W}\bar{\mathbf{g}}_n(\beta)$$

What is the optimal choice of W?

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$$\begin{aligned} E\left(\mathbf{g}(\mathbf{w}_i, \mu)\mathbf{g}(\mathbf{w}_i, \mu)'\right) &= E\left(\begin{pmatrix} \mathbf{x}_i\varepsilon_i \\ \mathbf{q}_i\varepsilon_i \end{pmatrix}(\mathbf{x}_i'\varepsilon_i \; \mathbf{q}_i'\varepsilon_i)\right) \\ &= \sigma^2 \begin{pmatrix} E\left(\mathbf{x}_i\mathbf{x}_i'\right) & E\left(\mathbf{q}_i\mathbf{q}_i'\right) \\ E\left(\mathbf{q}_i\mathbf{x}_i'\right) & E\left(\mathbf{q}_i\mathbf{q}_i'\right) \end{pmatrix} \end{aligned}$$

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$$\hat{S} = \hat{\sigma}^2 \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i' & \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i\mathbf{q}_i' \\ \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i\mathbf{x}_i' & \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i\mathbf{q}_i' \end{pmatrix}$$

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$$\mathbf{W} = \left[E\left(\mathbf{g}(\mathbf{w}_i, \mu)\mathbf{g}(\mathbf{w}_i, \mu)'\right)\right]^{-1} = S^{-1}$$

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$$= \sigma^2 \begin{pmatrix} E\left(\mathbf{x}_i\mathbf{x}_i'\right) & E\left(\mathbf{q}_i\mathbf{q}_i'\right) \\ E\left(\mathbf{q}_i\mathbf{x}_i'\right) & E\left(\mathbf{q}_i\mathbf{q}_i'\right) \end{pmatrix}$$

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with,

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \hat{e}_i^2 = \frac{1}{n}\sum_{i=1}^n (y_i - \mathbf{x}_i'\hat{\beta}_{OLS})$$

Efficient GMM estimator

$$
\begin{aligned}
J_n(\beta) =& n\bar{\mathbf{g}}_n(\mu)'\hat{\mathbf{W}}\bar{\mathbf{g}}_n(\mu) \\
=& n\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i(y_i - \mathbf{x}_i'\beta)\right]' \hat{S}^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i(y_i - \mathbf{x}_i'\beta)\right] \\
=& n\left[\mathbf{Z}'\mathbf{y} - \mathbf{Z}'\mathbf{X}\beta\right]' \hat{S}^{-1}\left[\mathbf{Z}'\mathbf{y} - \mathbf{Z}'\mathbf{X}\beta\right] \\
=& n\Big[\mathbf{y}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y} + \beta'\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\beta + \mathbf{y}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\beta \\
& + \beta'\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y}\Big]
\end{aligned}
$$

with $\mathbf{Z} = (\mathbf{X}\ \mathbf{Q})$

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=& n\left[\mathbf{Z}'\mathbf{y} - \mathbf{Z}'\mathbf{X}\beta\right]'\hat{S}^{-1}\left[\mathbf{Z}'\mathbf{y} - \mathbf{Z}'\mathbf{X}\beta\right] \\
=& n\Big[\mathbf{y}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y} + \beta'\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\beta + \mathbf{y}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\beta \\
& + \beta'\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y}\Big]
\end{aligned}
$$

with $\mathbf{Z} = (\mathbf{X}\ \mathbf{Q})$

$$
\hat{\beta}_{GMM} = \left[\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{X}\right]^{-1}\mathbf{X}'\mathbf{Z}\hat{S}^{-1}\mathbf{Z}'\mathbf{y}
$$