For independent but not identically distributed r.v.s the Berry–Esseen result takes the form

$$\sup_z |F_n(z) - F(z)| \leq c \frac{\sum_1^n \mu_{3i}}{(\sum_1^n \sigma_i^2)^{3/2}}.$$

A disadvantage of the above approximation is that it does not provide guidance in choosing between asymptotically equivalent results. This leads to the concept of asymptotic expansions of the approximation errors. The idea behind an asymptotic expansion of the distribution function is that it can be expressed as

$$F_n(z) = F(z) + \frac{F_1(z)}{\sqrt{n}} + \frac{F_2(z)}{n} + \cdots,$$

where the first term is $F(z) \sim \mathcal{N}(0, 1)$ and $F_i(z)$ are some functions of z. Such expansions are known as an Edgeworth expansion or a Gram–Charlier type expansion. For details on these expansions, see Kendall and Stuart (1973), Phillips (1977), and Rothenberg (1984).

A.4 Order of Magnitudes (Small o and Large O)

Sometimes it is useful to have a measure of the order of magnitude of a particular sequence, say, $\{X_n\}$. The magnitude is determined by looking into the behavior of X_n for large n. The following definitions are useful in this context:

Definition 1: The sequence $\{X_n\}$ of real numbers is said to be at most of order n^k and is denoted by

$$X_n = O(n^k), \quad \text{if } \frac{X_n}{n^k} \to c$$

as $n \to \infty$ for some constant $c > 0$. Further if $\{X_n\}$ is a sequence of r.v.s then

$$X_n = O_p(n^k) \quad \text{or} \quad O_{a.s.}(n^k)$$

if, as $n \to \infty$,

$$\frac{X_n}{n^k} - c_n \to 0 \quad \text{in prob.} \quad \text{or} \quad \text{a.s.},$$

respectively, where c_n is a nonstochastic sequence. The $O_p(n^k)$, for example, represents at most of order n^k in probability.

Definition 2: The sequence $\{X_n\}$ of real numbers is said to be of smaller order than n^k and is denoted by

$$X_n = o(n^k), \quad \text{if } \frac{X_n}{n^k} \to 0$$

as $n \to \infty$. Further if $\{X_n\}$ is stochastic then

$$X_n = o_p(n^k) \quad \text{or} \quad o_{a.s.}(n^k)$$

if

$$\frac{X_n}{n^k} \to 0 \quad \text{in prob. or a.s.}$$

In the above definitions k can take any real value (positive or negative). As an example consider a nonstochastic sequence

$$\{X_n\} = \frac{1}{n+4}. \tag{A.57}$$

It is easy to verify that for $k = -1$ in definition 1,

$$\frac{X_n}{n^{-1}} = \frac{n}{n+4} \to 1 \tag{A.58}$$

as $n \to \infty$. Thus $X_n = O(n^{-1}) = O(1/n)$.

The sequence $X_n = 1/(n+4)$ is also $o(1) = o(n^0)$. This is because, for $k = 0$, using Definition 2, $X_n/n^0 = X_n = 1/n + 4 \to 0$ as $n \to \infty$.

As another example, note that $X_n = \sum_{i=1}^n z_i^2 = O(n)$, where z_i is nonstochastic, if $\sum_1^n z_i^2/n \to c$ as $n \to \infty$. However, if z_i is such that $\sum_{i=1}^n z_i^2/n^3 \to c$ as $n \to \infty$ then $\sum_1^n z_i^2 = O(n^3)$. For example, if $z_i = i$, then $\sum_1^n z_i^2 = n(n+1)(2n+1)/6 = O(n^3)$; the reader should verify that $n + 1 = O(n) = o(n^2)$; $e^{-n} = o(n^{-\lambda})$, $\lambda > 0$; $\log_e n = o(n^{\lambda})$, $\lambda > 0$; $30n^2 + 20n = O(n^2) = o(n^3)$.

Now consider a stochastic sequence $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, where $EX_i = \mu$ and $V(X_i) = \sigma^2$. Note that, for $k = -1/2$ in Definition 1,

$$\frac{\bar{X}_n}{n^{-1/2}} = n^{1/2}\bar{X}_n. \tag{A.59}$$

But $n^{1/2}\bar{X}_n - n^{1/2}\mu$ is $O_p(1)$ by using Chebychev's inequality. Thus $\bar{X}_n - \mu = O_p(n^{-1/2})$. It also follows that $\bar{X}_n = \mu + o_p(1)$.

The small o and capital O satisfy the following properties.

If $X_n = O(n^k)$ and $Y_n = O(n^m)$ then

$$\begin{aligned} &\text{(i)} && X_n Y_n = O(n^{k+m}), \\ &\text{(ii)} && X_n^r = O(n^{rk}), \\ &\text{(iii)} && X_n + Y_n = O(n^{k_0}), \quad k_0 = \max(k, m). \end{aligned} \tag{A.60}$$

The same results hold for small o in place of capital O.

Further, if $X_n = O(n^k)$ but $Y_n = o(n^m)$, then

$$\begin{aligned} &\text{(i)} && X_n + Y_n = O(n^k), \\ &\text{(ii)} && X_n Y_n = o(n^{k+m}). \end{aligned} \tag{A.61}$$