TA session #2
ECON 342

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Problem 1. You plan to estimate following panel data model:

$$y_{it} = \mathbf{x}_{it}\beta + \varepsilon_{it}$$

with $\varepsilon_{it} = \alpha_i + u_{it}$. Re-write the model for all T periods as follows:

$$\mathbf{y}_i = \mathbf{X}_i\beta + \varepsilon_i$$

where:

$$\mathbf{y}_i = (y_{i1}, \ldots, y_{iT})'; \;\; \mathbf{u}_i = (u_{i1}, \ldots, u_{iT})'$$
$$\mathbf{X}_i = \begin{pmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \\ \vdots \\ \mathbf{x}_{iT} \end{pmatrix}; \;\; \varepsilon_i = \alpha_i \mathbf{i}_T + \mathbf{u}_i; \;\; \mathbb{E}\left(\varepsilon_i \varepsilon_i'\right) = \Omega$$

After long meditation you conclude that it is reasonable to assume that:

1. $\mathbb{E}\left(u_{it}|\mathbf{x}_i, \alpha_i\right) = 0$ and $\mathbb{E}\left(\alpha_i \mathbf{x}_i\right) = 0$
2. $\mathbb{E}\left(\mathbf{x}_i'\Omega^{-1}\mathbf{x}_i\right) = \mathbf{A}$ is nonsingular (has rank $K = K_1 + \cdots + K_M$)
3. (a) $\mathbb{E}\left(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, \alpha_i\right) = \sigma_u^2 I_T$, and (b) $\mathbb{E}\left(\alpha_i^2|\mathbf{x}_i\right) = \sigma_\alpha^2$

a) Obtain the asymptotic properties of your RE estimator.

b) Your colleague suggests that (1), (2) and (3a) hold but (3b) does might not. Are your estimates of β still valid?

Solution. Our assumptions imply:

$$\mathbb{E}\left(\alpha_i^2\right) = \mathbb{E}\left(\mathbb{E}\left(\alpha_i^2|\mathbf{x}_i\right)\right) = \sigma_\alpha^2 \tag{1}$$
$$\mathbb{E}\left(u_{it}^2\right) = \mathbb{E}\left(\mathbb{E}\left(u_{it}^2|\mathbf{x}_i, \alpha_i\right)\right) = \sigma_u^2 \tag{2}$$
$$\mathbb{E}\left(u_{it}u_{is}\right) = 0 \tag{3}$$

Thus, the diagonal elements of Ω contain:

$$\mathbb{E}\left(\varepsilon_{it}^2\right) = \mathbb{E}\left((\alpha_i + u_{it})^2\right) = \mathbb{E}\left(\alpha_i^2\right) + \mathbb{E}\left(u_{it}^2\right) + 2\mathbb{E}\left(\alpha_i u_{it}\right) = \sigma_u^2 + \sigma_\alpha^2$$

while the diagonal elements are:

$$\mathbb{E}\left(\varepsilon_{it}\varepsilon_{is}\right) = \mathbb{E}\left((\alpha_i + u_{it})(\alpha_i + u_{is})\right) = \sigma_\alpha^2$$

therefore,

$$\Omega = \begin{pmatrix} \sigma_u^2 + \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \ddots & 0 & \vdots \\ \vdots & & & \\ \sigma_\alpha^2 & \cdots & & \sigma_u^2 + \sigma_\alpha^2 \end{pmatrix} \tag{4}$$

Using,

$$\hat{\Omega} \xrightarrow{p} \Omega$$

we have:

$$\hat{\beta}_{RE} = \left[\frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\mathbf{x}_i\right]^{-1} \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\mathbf{y}_i \tag{5}$$

The limiting distribution of the RE estimator is obtained by writing:

$$\hat{\beta}_{RE} - \beta = \left[\frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\mathbf{x}_i\right]^{-1} \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\varepsilon_i$$
$$\sqrt{N}(\hat{\beta}_{RE} - \beta) = \left[\frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\mathbf{x}_i\right]^{-1} \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\varepsilon_i$$

Using the LLN and assumption (2) we have:

$$\left[\frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i'\hat{\Omega}^{-1}\mathbf{x}_i\right]^{-1} \xrightarrow{p} \mathbf{A}^{-1}$$

and the CLT along with (1)-(3) yields,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{x}_i'\hat{\Omega}^{-1}\varepsilon_i \xrightarrow{p} N(\mathbf{0},\mathbf{H})$$

with,

$$\mathbf{H}=\mathbb{E}\left(\mathbf{x}_i'\Omega^{-1}\varepsilon_i\varepsilon_i'\Omega^{-1}\mathbf{x}_i\right)=\mathbb{E}\left(\mathbf{x}_i'\Omega^{-1}\mathbb{E}\left(\varepsilon_i\varepsilon_i'|\mathbf{x}_i\right)\Omega^{-1}\mathbf{x}_i\right)=\mathbb{E}\left(\mathbf{x}_i'\Omega^{-1}\mathbf{x}_i\right)$$

which together imply:

$$\begin{aligned}\sqrt{N}(\hat{\beta}_{RE}-\beta) \xrightarrow{d} & N(\mathbf{0},\underbrace{\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1}}_{\mathrm{A}^{-1}})\\ =&N(\mathbf{0},\mathbf{A}^{-1})\end{aligned}$$

When (3b) does not hold we have

$$\mathbb{E}\left(\alpha_i^2|\mathbf{x}_i\right)=f(\mathbf{x}_i),$$

therefore,

$$\begin{aligned}\mathbb{E}\left(\varepsilon_{it}^2\right)=&\sigma_u^2+f(\mathbf{x}_i)\\ \mathbb{E}\left(\varepsilon_{it}\varepsilon_{is}\right)=&f(\mathbf{x}_i)\end{aligned}$$

In this case,

$$\mathbb{E}\left(\varepsilon_i\varepsilon_i'|\mathbf{x}_i\right)=\begin{pmatrix}\sigma_u^2+f(\mathbf{x}_i) & f(\mathbf{x}_i) & \cdots & f(\mathbf{x}_i)\\ f(\mathbf{x}_i) & \ddots & 0 & \vdots\\ \vdots & & & \\ f(\mathbf{x}_i) & \cdots & & \sigma_u^2+f(\mathbf{x}_i)\end{pmatrix}$$

which does not have the form of 4. The usual RE estimator 5 is still consistent but the variance-covariance matrix is no longer valid. In this case, the CLT along with (1)-(3a) yields,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{x}_i'\hat{\Omega}^{-1}\varepsilon_i \xrightarrow{p} N(\mathbf{0},\mathbf{H})$$

where,

$$\mathbf{H}=\mathbb{E}\left(\mathbf{x}_i'\Omega^{-1}\varepsilon_i\varepsilon_i'\Omega^{-1}\mathbf{x}_i\right)$$

Then,

$$\sqrt{N}(\hat{\beta}_{RE}-\beta) \xrightarrow{d} N(\mathbf{0},\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1})$$

$\Diamond$

Problem 2. Consider the model from the previous problem. After performing a Hausman test you realize that you should use a fixed effect estimator.

a) You decide to estimate your model using the difference estimator to eliminate the unobserved effect α_i. Under what conditions is your estimator consistent?

b) What is the variance-covariance matrix of your estimator?

Solution. Taking differences we have:

$$
\begin{array}{ll}
y_{i1} = \mathbf{x}_{i1}\beta + \alpha_i + u_{i1} & \cdot \\
y_{i2} = \mathbf{x}_{i2}\beta + \alpha_i + u_{i2} & y_{i2} - y_{i1} = (\mathbf{x}_{i2} - \mathbf{x}_{i1})\beta + (u_{i2} - u_{i1}) \\
\vdots & \Rightarrow \vdots \\
y_{i,T-1} = \mathbf{x}_{i,T-1}\beta + \alpha_i + u_{i,T-1} & y_{i,T-1} - y_{i,T-2} = (\mathbf{x}_{i,T-1} - \mathbf{x}_{i,T-2})\beta + (u_{i,T-1} - u_{i,T-2}) \\
y_{iT} = \mathbf{x}_{iT}\beta + \alpha_i + u_{iT} & y_{iT} - y_{i,T-1} = (\mathbf{x}_{iT} - \mathbf{x}_{i,T-1})\beta + (u_{iT} - u_{i,T-1})
\end{array}
$$

then the model 'looses' $t = 1$ and for $t \geq 2$ we have:

$$\Delta y_{it} = \Delta \mathbf{x}_{it}\beta + \Delta u_{it}$$

Stacking all $T - 1$ equations we can re-write the system of equations as:

$$\Delta \mathbf{y}_i = \Delta \mathbf{x}_i \beta + \Delta \mathbf{u}_i$$

The pooled OLS estimator from this model will be consistent if:

$$\mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbf{u}_i\right) = \mathbf{0}$$

Note that,

$$\Delta \mathbf{x}_i' \Delta \mathbf{u}_i = (\Delta \mathbf{x}_{i2}', \Delta \mathbf{x}_{i3}', \cdots, \Delta \mathbf{x}_{iT}) \times \begin{pmatrix} \Delta u_{i2} \\ \Delta u_{i3} \\ \vdots \\ \Delta u_{iT} \end{pmatrix}$$

which implies that we need:

$$\mathbb{E}\left(\Delta \mathbf{x}_{it}' \Delta u_{it}\right) = \mathbb{E}\left((\mathbf{x}_{it} - \mathbf{x}_{i,t-1})(u_{it} - u_{it-1})\right) 0; \text{ for } t = 2, \ldots, T$$

which will fail if u_{it} is correlated with $\mathbf{x}_{it}$ or $\mathbf{x}_{i,t-1}$, and u_{it-1} with $\mathbf{x}_{it}$ which is the same as saying u_{it} is correlated with $\mathbf{x}_{i,t+1}$. For consistency of our estimator we can set our first assumption as follows:

$$\mathbb{E}(u_{it} | \mathbf{x}_{t-1}, \mathbf{x}_t, \mathbf{x}_{t+1}) = 0, \text{ for } t = 2, \ldots, T$$

A stronger assumption would be to assume strong exogeneity,

$$\mathbb{E}\left(\Delta u_{it}|\Delta \mathbf{x}_2, \cdots, \Delta \mathbf{x}_T\right) = 0, \text{ for } t = 2, \ldots, T$$

With this assumption at hand we have:

$$\beta = \left[\mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbf{x}_i\right)\right]^{-1} \Delta \mathbf{x}_i' \Delta \mathbf{y}_i$$

For $\left[\mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbf{x}_i\right)\right]^{-1} = \left[\sum_{t=2}^{T} \mathbb{E}\left(\Delta \mathbf{x}_{it}' \Delta \mathbf{x}_{it}\right)\right]^{-1}$ to be well defined we need that it is invertible, thus we cannot have a non-time varying variable in $\mathbf{x}_i$. To obtain the FE estimator we write:

$$\hat{\beta}_{FE} = \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T} \Delta \mathbf{x}_{it}' \Delta \mathbf{x}_{it}\right]^{-1} \frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T} \Delta \mathbf{x}_{it}' \Delta y_{it}$$

To obtain the asymptotic variance-covariance matrix of this estimator we need to make some assumptions about u_{it}. We will keep it simple (for now) and assume that the error term is serially uncorrelated and conditionally homoskedastic. As usual, re-write the estimator as:

$$\sqrt{N}(\hat{\beta}_{FE} - \beta) = \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=2}^{T} \Delta \mathbf{x}_{it}' \Delta \mathbf{x}_{it}\right]^{-1} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} \Delta \mathbf{x}_{it}' \Delta \mathbf{u}_{it}$$

Using the CLT we know,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} \Delta \mathbf{x}_{it}' \Delta \mathbf{u}_{it} \xrightarrow{p} N(\mathbf{0}, \mathbf{H})$$

with,

$$\mathbf{H} = \mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbf{u}_i \Delta \mathbf{u}_i' \Delta \mathbf{x}_i\right) \tag{6}$$

Under our assumptions we can develop this further. Notice that

$$\Omega = \mathbb{E}\left(\mathbf{u}_i \mathbf{u}_i'\right)$$

has diagonal elements equal to:

$$\mathbb{E}\left(\Delta u_{it}^2\right) = \mathbb{E}\left(u_{it}^2 + u_{i,t-1}^2 - 2u_{it}u_{i,t-1}\right) = 2\sigma_u^2$$

and non-diagonal elements equal to,

$$\mathbb{E}\left(\Delta u_{it} \Delta u_{i,t-j}\right) = \begin{cases} \mathbb{E}\left(-u_{i,t-1}^2\right) = -\sigma_u^2, & \text{when} j = 1; \\ 0, & \text{when} j > 1 \end{cases}$$

Then,

$$\Omega = \sigma_u^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & & \\ 0 & \ddots & \ddots & & \\ & & & & -1 \\ 0 & & & -1 & 2 \end{pmatrix} = \sigma_u^2 \Sigma$$

which implies that 6 can be written as:

$$\mathbf{H} = \mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbb{E}\left(\mathbf{u}_i \Delta \mathbf{u}_i' | \mathbf{x}_i\right) \Delta \mathbf{x}_i\right) = \sigma_u^2 \mathbb{E}\left(\Delta \mathbf{x}_i' \Sigma \Delta \mathbf{x}_i\right)$$

Finally, we conclude that:

$$\sqrt{N}(\hat{\beta}_{FE} - \beta) \xrightarrow{d} N(\mathbf{0}, \mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1})$$

where,

$$\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1} = \sigma_u^2 \left[\mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbf{x}_i\right)\right]^{-1} \mathbb{E}\left(\Delta \mathbf{x}_i' \Sigma \Delta \mathbf{x}_i\right) \left[\mathbb{E}\left(\Delta \mathbf{x}_i' \Delta \mathbf{x}_i\right)\right]^{-1}$$

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