TA session #3
ECON 342

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Problem 1. We will use MATLAB to solve the following problem.

1. Simulate data for the following model:

$$y_i = I[\beta_1 + \beta_2 x_i + \epsilon_i > 0]$$

where $\alpha_0 = \beta_0 = 1, x_i \sim N(0,1), \epsilon_i \sim N(0,1)$, and ϵ_i, x_i are independent. Generate 100 i.i.d observations.

2. Estimate α_0, β_0 using NLS and a Logit cdf to model the conditional expectation of y.

Solution. **The model**

Consider n individuals, $i = 1, \dots, n$ where each is faces with a binary decision or choice $j = 0, 1$. If we let

$$p_i = \mathbb{P}\left(y_i = 1 | \mathbf{x}_i\right)$$

i.e., the probability that Y_i is one, and $1 - p_i$ the probability that Y_i is 0, then

$$Y_i \sim \text{ Bernoulli } (p_i)$$

therefore

$$\mathbb{E}\left(Y_i | \mathbf{x}_i\right) = p_i$$

and

$$V\left(Y_i | \mathbf{x}_i\right) = p_i(1 - p_i).$$

We can represent the probability that $y_i = 1$ as,

$$p_i = \mathbb{P}\left(\mathbf{x}_i\beta + \varepsilon_i > 0\right) = \mathbb{P}\left(\varepsilon_i > -\mathbf{x}_i\beta\right)$$

where $\mathbf{x}_i = (1, x_i)$ and $\beta = (\beta_1, \beta_2)'$.

Specifying a functional form for the CDF of ε_i say $F(\varepsilon_i)$ we have,

$$p_i = 1 - F(-\mathbf{x}_i\beta)$$

with a PDF symmetric around zero we can re-write this results as,

$$p_i = F(\mathbf{x}_i\beta)$$

In our case we will assume that,

$$F(\alpha) = \Lambda(\alpha) = \frac{\exp[\alpha]}{1 + \exp[\alpha]}$$

i.e., a logistic cdf. You can show that $\Lambda(\cdot)$ is symmetric and has first- and second-derivatives equal to,

$$\Lambda'(\alpha) = \Lambda(\alpha)\Big(1 - \Lambda(\alpha)\Big)$$
$$\Lambda''(\alpha) = \Lambda'(\alpha)\Big(1 - 2\Lambda(\alpha)\Big)$$

which are useful at the time of taking derivatives of this cdf. [1]

Optimization problem and Newton's method

In this problem we will obtain our estimates solving the following optimization problem:

$$\min_{\beta} \sum_{i=1}^{N} \Big(y_i - p_i(\beta)\Big)^2$$

where,

$$p_i(\beta) = \Lambda(\mathbf{x}_i\beta)$$

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[1] For a standard normal cdf, $\Phi(\alpha)$, we have:

$$\phi'(\alpha) = \alpha\phi(\alpha)$$

Let $f(\beta)$ be our objective function. To solve this problem we will use the following algorithm:

Initialization Choose a guess β_0 and stopping parameters δ and $\epsilon > 0$

Step 1: Compute the gradient $\nabla f(\beta^m)$ and hessian $H(\beta^m)$

Step 2: Compute,

$$\beta^{m+1} = \beta^m - [H(\beta^m)]^{-1} \nabla f(\beta^m)$$

Step 3: If

$$\frac{\|\beta^m - \beta^{m+1}\|}{1 + \|\beta^m\|} < \epsilon$$

go to step 4; else go to step 1.

Step 4: If

$$\frac{\|\nabla f(\beta^{m+1})\|}{1 + \|f(\beta^m)\|} < \delta$$

STOP and report sucess; else STOP and report convergence to a nonoptimal point.

In our problem,

$$f(\beta) = \sum_{i=1}^{N} \Big(y_i - \Lambda(\mathbf{x}_i\beta)\Big)^2$$

Then,

$$\nabla f(\beta) = \sum_{i=1}^{N} -2 * \Big(y_i - \Lambda(\mathbf{x}_i\beta)\Big)\Lambda'(\mathbf{x}_i\beta)\mathbf{x}_i'$$

$$H(\beta) = \sum_{i=1}^{N} -2\Big[\Lambda''(\mathbf{x}_i\beta)\Big(y_i - \Lambda(\mathbf{x}_i\beta)\Big) - \Lambda'(\mathbf{x}_i\beta)^2\Big]\mathbf{x}_i'\mathbf{x}_i$$