TA session #4
ECON 342

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Problem 1. Consider the following binary choice panel data model:

$$
\begin{aligned}
y_{it}^* &= \alpha_i + x_{it}'\beta_0 + \varepsilon_{it} \\
y_{it} &= \mathcal{I}\left[y_{it}^* > 0\right]
\end{aligned}
$$

for $t = 1, 2$, and $i = 1, \ldots, N$. The econometrician observed y_{it}, x_{it} but not the 'individual specific effect' α_i, nor the disturbance term ε_{it}, which we will assume is independent and identically distributed across i and t. We assume that $ve)i$ is distributed independently of x_{i1}, x_{i2}, α_i and has a Logistic distribution,

$$\mathbb{P}\left(\varepsilon_{it} \leq x\right) = \frac{1}{1 + \exp[-x]} = \Lambda(x)$$

The parameter of interest is β_0 which we will try to estimate in this question.

(a) Evaluate,

$$\mathbb{P}\left(y_{i1} = 1 | y_{i1} + y_{i2} = 1, x_{i1}, x_{i2}, \alpha_i\right)$$

How does it depend on α_i?

(b) Evaluate,

$$\mathbb{P}\left(y_{i2} = 1 | y_{i1} + y_{i2} = 1, x_{i1}, x_{i2}, \alpha_i\right)$$

How does it depend on α_i?

(c) Propose a MLE for β_0, only using individuals where $(y_{i1} = 1, y_{i2} = 0)$ or $(y_{i1} = 0, y_{i2} = 1)$.

Solution. Consider an individual that switches states, i.e., $y_{i1} + y_{i2} = 1$. The probability of being at state 1 at $t = 1$ given she switched in $t = 2$ equals to,

$$\mathbb{P}\left(y_{i1} = 1 | y_{i1} + y_{i2} = 1, x_{i1}, x_{i2}, \alpha_i\right) = \frac{\mathbb{P}\left(y_{i1} = 1, y_{i1} + y_{i2} = 1 | x_{i1}, x_{i2}, \alpha_i\right)}{\mathbb{P}\left(y_{i1} + y_{i2} = 1 | x_{i1}, x_{i2}, \alpha_i\right)} \tag{1}$$

The denominator in (1) can be written as,

$$\begin{aligned}
\mathbb{P}\left(y_{i1}=1, y_{i1}+y_{i2}=1|x_{i1},x_{i2},\alpha_i\right) =& \mathbb{P}\left(y_{i1}=1, y_{i2}=0|x_{i1},x_{i2},\alpha_i\right)\\
=& \mathbb{P}\left(y^*_{i1}>0, y^*_{i2}\leq 0|x_{i1},x_{i2},\alpha_i\right)\\
=& \mathbb{P}\left(\alpha_i + x'_{i1}\beta_0+\varepsilon_{i1}>0, \alpha_i + x'_{i2}\beta_0+\varepsilon_{i2}\leq 0|x_{i1},x_{i2},\alpha_i\right)\\
=& \mathbb{P}\left(\varepsilon_{i1}>-\alpha_i - x'_{i1}\beta_0, \varepsilon_{i2}\leq -\alpha_i - x'_{i2}\beta_0|x_{i1},x_{i2},\alpha_i\right)\\
=& \mathbb{P}\left(\varepsilon_{i1}>-\alpha_i - x'_{i1}\beta_0|x_{i1},x_{i2},\alpha_i\right)\mathbb{P}\left(\varepsilon_{i2}\leq -\alpha_i - x'_{i2}\beta_0|x_{i1},x_{i2},\alpha_i\right)\\
=& \frac{\exp[\alpha_i + x'_{i1}\beta_0]}{1+\exp[\alpha_i + x'_{i1}\beta_0]}\times\frac{1}{1+\exp[\alpha_i + x'_{i2}\beta_0]}\\
=& \frac{\exp[\alpha_i + x'_{i1}\beta_0]}{\left(1+\exp[\alpha_i + x'_{i1}\beta_0]\right)\left(1+\exp[\alpha_i + x'_{i2}\beta_0]\right)}
\end{aligned}$$

while the denominator is equal to,

$$\begin{aligned}
\mathbb{P}\left(y_{i1}+y_{i2}=1|x_{i1},x_{i2},\alpha_i\right) =& \mathbb{P}\left(y_{i1}=1, y_{i2}=0|x_{i1},x_{i2},\alpha_i\right)+\mathbb{P}\left(y_{i1}=0, y_{i2}=1|x_{i1},x_{i2},\alpha_i\right)\\
=& \frac{\exp[\alpha_i + x'_{i1}\beta_0]}{1+\exp[\alpha_i + x'_{i1}\beta_0]}\times\frac{1}{1+\exp[\alpha_i + x'_{i2}\beta_0]}\\
&+\frac{1}{1+\exp[\alpha_i + x'_{i1}\beta_0]}\times\frac{\exp[\alpha_i + x'_{i2}\beta_0]}{1+\exp[\alpha_i + x'_{i2}\beta_0]}\\
=& \frac{\exp[\alpha_i + x'_{i1}\beta_0]+\exp[\alpha_i + x'_{i2}\beta_0]}{\left(1+\exp[\alpha_i + x'_{i1}\beta_0]\right)\left(1+\exp[\alpha_i + x'_{i2}\beta_0]\right)}
\end{aligned}$$

Replacing these two expressions back into (1) we have:

$$\begin{aligned}
\mathbb{P}\left(y_{i1}=1|y_{i1}+y_{i2}=1,x_{i1},x_{i2},\alpha_i\right) =& \frac{\exp[\alpha_i + x'_{i1}\beta_0]}{\exp[\alpha_i + x'_{i1}\beta_0]+\exp[\alpha_i + x'_{i2}\beta_0]}\\
=& \frac{\cancel{\exp[\alpha_i]}\exp[x'_{i1}\beta_0]}{\cancel{\exp[\alpha_i]}(\exp[x'_{i1}\beta_0]+\exp[x'_{i2}\beta_0])}\\
=& \frac{\exp[x'_{i1}\beta_0]}{\exp[x'_{i1}\beta_0]+\exp[x'_{i2}\beta_0]}\\
=& \frac{\exp[-(x_{i2}-x_{i1})'\beta_0]}{1+\exp[-(x_{i2}-x_{i1})'\beta_0]} = 1-\Lambda[(x_{i2}-x_{i1})'\beta_0]
\end{aligned}$$

which does not depend on α_i. Similarly we can show that,

$$\mathbb{P}\left(y_{i2}=1|y_{i1}+y_{i2}=1,x_{i1},x_{i2},\alpha_i\right) = \frac{1}{1+\exp[-(x_{i2}-x_{i1})'\beta_0]} = \Lambda[(x_{i2}-x_{i1})'\beta_0]$$

Therefore, the conditional likelihood of observation i is equal to,

$$\ell_i(\beta) = \mathcal{I}[y_{i1}=1, y_{i2}=0]\times\Big(1-\Lambda[(x_{i2}-x_{i1})'\beta_0]\Big)+\mathcal{I}[y_{i1}=0, y_{i2}=1]\times\Lambda[(x_{i2}-x_{i1})'\beta_0]$$

then, the MLE of β is equal to,

$$\hat{\beta} = \arg\max_{\beta} \sum_{i=1}^{N} \mathcal{I}[y_{i1} + y_{i1} = 1]\ell_i(\beta)$$

Interestingly, this problem is nothing more that a cross-sectional logit regression and analogous to differencing in the linear case with $T = 2$! ♢

Problem 2. Consider the Logit model with $\mathbb{P}(y_i = 1|x_i) = \Lambda(\beta_0 + \beta_1 x_i)$, where the scalar random variable x_i is binary, taking values of 0 and 1.

(a) Show that this model can be written as a linear probability model $\mathbb{P}(y_i = 1|x_i) = \gamma_0 + \gamma_1 x_i$ and derive γ_0 and γ_1 as functions of β_0 and β_1

(b) Show that the MLE of γ_0 and γ_1 is equal to the least squares estimate of y_i on x_i with intercept.