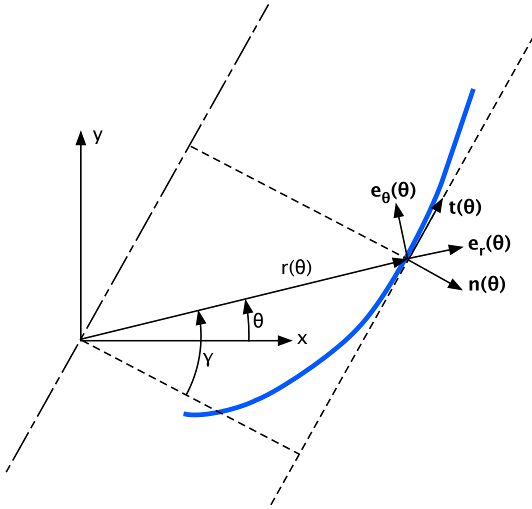
Derivation of the Cam Shape Equation for Constant Cam Angle:



In polar coordinates the cam shape is given by the vector location versus angle:

$$\mathbf{r}(\theta) = r(\theta)\mathbf{e}_r(\theta) \qquad \text{where } \mathbf{e}_r(\theta) = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

A tangent vector (not a unit vecrtor) to the cam curve is:

$$\mathbf{t}(\theta) = r'(\theta)\mathbf{e}_r(\theta) + r(\theta)\mathbf{e}_\theta(\theta) \qquad \text{where } \mathbf{e}_\theta(\theta) = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$$

The objective is to find the cam shape that holds the angle γ constant as θ varies.

Since the dot product is the product of the magnitudes times the cosine of the angle, note that:

$$\mathbf{r}(\theta)\bullet\mathbf{t}(\theta) = r(\theta)[r'(\theta)^2 + r(\theta)^2]^{1/2}\cos\left(\frac{\pi}{2} - \gamma\right) = \left[\frac{1}{4}\left(\frac{d}{d\theta}[r(\theta)^2]\right)^2 + r(\theta)^4\right]^{1/2}\sin\gamma$$

But also using the vector forms directly:

$$\mathbf{r}(\theta)\bullet\mathbf{t}(\theta) = r(\theta)r'(\theta) = \frac{1}{2}\frac{d}{d\theta}[r(\theta)^2]$$

Equating these two expressions:

$$\frac{1}{2}\frac{d}{d\theta}[r(\theta)^2] = \left[\frac{1}{4}\left(\frac{d}{d\theta}[r(\theta)^2]\right)^2 + r(\theta)^4\right]^{1/2}\sin\gamma$$

Now solve for $r(\theta)$ assuming γ is held constant.

Squaring both sides gives:

$$\left[\frac{1}{4}\left(\frac{d}{d\theta}[r(\theta)^2]\right)^2 + r(\theta)^4\right]\sin^2\gamma = \frac{1}{4}\left(\frac{d}{d\theta}[r(\theta)^2]\right)^2$$

After regrouping terms:

$$\frac{(1-\sin^2\gamma)}{4}\left(\frac{d}{d\theta}[r(\theta)^2]\right)^2 = [r(\theta)^2]^2\sin^2\gamma$$

Using a trigonometric identities and taking the square root of both sides:

$$\frac{1}{2}\frac{d}{d\theta}[r(\theta)^2] = \pm[r(\theta)^2]\tan\gamma$$

Carrying out the differentiation and canceling the extra factor of $r(\theta)$ gives two first-order linear governing equations, one for each sign choice:

$$\frac{d}{d\theta}r(\theta) \mp r(\theta)\tan\gamma = 0$$

The solutions of the two equations are:

$$r_1(\theta) = C_1 e^{(\tan\gamma)\theta} \qquad \text{and} \qquad r_2(\theta) = C_2 e^{-(\tan\gamma)\theta}$$

Choosing the solution with the increasing exponential (assuming positive angle γ), gives a very simple result:

$$r_1(\theta) = C_1 e^{(\tan\gamma)\theta}$$

which is also the logarithmic spiral form:

$$\ln[r_1(\theta)] = C_1(\tan\gamma)\theta$$

When written in exponential form, the appearance of $\tan\gamma$ in the exponent is interesting. Also note that typically $\tan\gamma << 1$. Reinterpreting the constant in front of the exponential, the expression can also be written as

$$r_1(\theta) = \widehat{C}_1 e^{(\tan\gamma)(\theta - \theta_{ref})}$$

For physical understanding, and a much easier derivation of the governing equation, note the figure below. Because the tangent of the cam must have a fixed angle gamma to the radius vector to keep the same cam angle as a function of gap, namely:

$$\tan\gamma = \frac{dr}{rd\theta}$$

which is really just a re-arrangement of the differential equation already derived.