Dynamic Preference for Flexibility*

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Abstract

We consider a decision maker who has preference for flexibility when faced with dynamic decision situations that involve intertemporal tradeoffs, such as those in consumption savings problems, and axiomatize three recursive representations of choice over infinite horizon consumption problems. The first features uncertain consumption utilities that evolve according to a subjective process that is iid, in the second utilities follow a subjective Markov process, and the third allows the uncertainty about utilities to depend on exogenous states of the world that follow a subjective Markov process. The parameters of the representations, which are the (evolution of) beliefs over consumption utilities, and the discount factor, are uniquely identified from behavior. We characterize a natural notion of greater preference for flexibility in terms of dilations of beliefs.

1. Introduction

Uncertainty about future consumption utilities, for instance future risk aversion, influences how economic agents make decisions. A decision maker (DM) who is uncertain about future consumption utilities prefers not to commit to a course of future action today, and therefore has preference for flexibility. Jones and Ostroy [1984] point out that preference for flexibility, of which preference for liquidity is a particular instance, is a pervasive theme in economics. They provide an extensive discussion relating many instances in macroeconomics and finance where DM's dynamic behavior exhibits preference for liquidity in particular, and preference for flexibility in general, as in Goldman [1974]. For example, DM might be willing to forfeit current consumption if this allows him to delay a decision about future consumption. While this intuition is inherently dynamic,

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standard models that accommodate preference for flexibility, most prominently Kreps [1979] and Dekel, Lipman and Rustichini [2001] (henceforth DLR[1]), are static in the sense that there is no intertemporal tradeoff (although Kreps [1979] suggests that an infinite horizon model of preference for flexibility is desirable). We provide foundations for dynamic models of preference for flexibility.

We follow Gul and Pesendorfer [2004], henceforth GP, and consider infinite horizon consumption problems (IHCPs) as the domain of choice. IHCPs are defined recursively as menus of lotteries over pairs of consumption and a new IHCP as a continuation problem.

For a simple example of an IHCP that features only degenerate lotteries, suppose DM has income y in every period and current wealth w. The price of consumption, p, is fixed. In each period DM can choose to consume an amount $k \in \left[0, \frac{w+y}{p}\right]$ at cost kp. This will leave him with wealth $w' = w + y - kp$ for the next period. Given this technology, we can rephrase the consumption problem he faces recursively as a collection of feasible consumption and continuation problem pairs,

$$x_w := \left\{(k, x_{w'}) : k \in [0, \tfrac{w+y}{p}],\ w' = w + y - kp\right\}$$

The domain of IHCPs is rich and future choice behavior can be complicated.[2] For example, future choices may depend directly on time or the consumption history. We rule out such dependencies in order to stay as close to the standard model as possible.[3] Given two IHCPs x and y, a standard decision makeris indifferent between x and $x \cup y$ whenever he prefers x to y. Following Kreps [1979], we refer to this property as ***strategic rationality***. Infinite Horizon Consumption Streams (IHCSs) can be defined as degenerate IHCPs that give DM, in each period, a lottery over instantaneous consumption and a consumption stream beginning in the next period (see Appendix C for a formal description of the set of IHCSs). If DM has in mind a ranking of all consumption streams and evaluates any IHCP according to the best stream contained in it, then his choice will satisfy strategic rationality. In contrast, a decision maker who values flexibility satisfies ***Monotonicity***, that is, he weakly prefers $x \cup y$ to x.

Strategic rationality is a natural point of departure for any analysis. If strategic rationality with respect to *all* IHCPs is violated in a particular situation, then it is desirable to understand the nature of this violation, by asking if DM is strategically rational with respect to some smaller class of IHCPs.

We propose three versions of this question. First, we suppose that DM is strategically rational when restricting next period's choice of a continuation problem. This is the content of our axiom *Continuation Strategic Rationality (CSR)*, representing the simplest deviation from full strategic rationality.

(1) A relevant corrigendum is Dekel, Lipman, Rustichini and Sarver [2007] (henceforth DLRS).

(2) We only model the initial choice of a consumption problem, but our representation suggests dynamically consistent future choice.

(3) Dependence of choice on the consumption history is central to the notion of habits. We are independently working on a model of habit formation on the domain of IHCPs.

Second, if CSR is violated, we further relax the requirement of strategic rationality and suppose that DM is strategically rational with respect to continuation problems, contingent on some event that will be observable by the modeller. In a setting without exogenous (ie, contractable) states of the world, the only candidate for such an event is past choice behavior, and in the simplest case the choice of current consumption (given a sufficiently rich set of consumption choices). This is our axiom *Choice Contingent CSR*.

Finally, in many applications there exists a collection of exogenous states of the world that may evolve over time, and that can be contracted upon. If DM is fully strategically rational contingent on the state at the time of choice, a state dependent version of the standard model can be recovered (see corollary 13 below). Suppose full strategic rationality is violated even contingent on the state. Then we can ask whether DM is strategically rational with respect to continuation problems, contingent on the state. Since continuation problems are not relevant for consumption until the next period, CSR should hold both, contingent on the state at the time of choice, and contingent on the state at the next period. This is our axiom *State Contingent CSR*.

For each of these relaxations of strategic rationality, we provide a representation of dynamic preference for flexibility that is the solution to a Bellman equation, and can therefore be analyzed using standard dynamic programming techniques. All three representations explain preference for flexibility via uncertain future consumption utilities. For example, DM might be uncertain about future risk aversion (the concavity of future consumption utilities), and, independently, about future intertemporal marginal rate of substitution (the scaling of utilities). The representations differ only in the process that governs the evolution of the uncertainty. Under CSR, this process is iid over time and the model, therefore, constitutes a minimal deviation from the standard model of dynamic choice. Choice Contingent CSR allows uncertainty to evolve according to a particular Markov process, providing a model that can accommodate persistent endogenous taste shocks. Finally, State Contingent CSR yields a model where the evolution of uncertainty about future consumption utilities is driven by the evolution of the (exogenous) state of the world, which in turn follows a subjective Markov process.

All the parameters in our representations are uniquely identified, which makes it possible to compare behavior in terms of those parameters. We characterize a notion of *greater preference for flexibility* in terms of a dilation (which is the multi-dimensional generalisation of a mean preserving spread) of beliefs over consumption utilities. Based on this characterization, Krishna and Sadowski [2012b] analyze how prices in a particular Lucas tree economy react to a dilation of the beliefs of a representative agent who has dynamic preference for flexibility.

Identification is also relevant for model based forecasting, which is a central reason for the use of formal models. It involves estimating a relevant parameter in one context in order to forecast outcomes in another. Model based forecasting, thus, requires that (i) the relevant parameter is uniquely identified and (ii) the modeler is willing to assume that the parameter is meaningful outside the context of the observed data. In our models beliefs over consumption utilities are uniquely identified. Those beliefs might, for example, forecast future choice frequencies of alternatives from continuation problems, as discussed in Sadowski [2011].[4]

The remainder of the paper is structured as follows. Section 1.1 gives a preview of our results while section 1.2 reviews related literature. Section 2 lays out the basic framework as well as assumptions that are maintained throughout most of the paper. Based on CSR, section 3 provides the foundations for a recursive representation where DM's utilities follow a subjective iid process. Section 4 considers the case where DM's preferences satisfy Choice Contingent CSR, and derives a representation where consumption utilities follow a particular Markov process. Section 5 considers an environment with objective states and preferences that satisfy State Contingent CSR, and derives a representation where beliefs about consumption utilities depend on the exogenous state, and states follow a subjective Markov process. A characterization of *greater preference for flexibility* can be found in section 6. All proofs, as well as a general representation of preference for flexibility with an infinite prize space, are established in the appendices.

1.1. Preview of Results

Choice from an infinite horizon consumption problem (IHCP), $x \in Z$, determines a lottery, p, over consumption in the present period, $k \in K$, and a continuation problem, $z \in Z$, which is a new infinite horizon consumption problem starting next period. We explicitly model choice between consumption problems from an ex-ante perspective, before consumption begins. We consider *continuous* and *non-trivial* preferences over such problems.

The behavioral content of preference for flexibility is *Monotonicity*, the central assumption in Kreps [1979], which states that more options are always weakly preferred. In DLR, menus contain lotteries over some finite prize space. The set of all continuation problems, in contrast, is not finite. Assuming a form of *Independence*, we provide a representation result after DLR for infinite prize spaces.[5]

We make the following simplifying assumptions. Preferences over IHCPs are *separable* with respect to present consumption and continuation problems. We assume that DM has *stationary* preferences, that is, choice does not directly depend on time, and is *indifferent to the timing* of the resolution of objective uncertainty.

(4) An important question in practice is how much data is needed to estimate the model. This question is not specific to our model. For a discussion in the context of consumer theory, see Mas-Colell [1978], who shows that data that grows (in a regular way) provides increasingly tight bounds on the set of utility representations that can rationalize the data.

(5) See appendix B.1. In the case of a finite prize space, the collection of future utilities that are relevant (in the sense of DLR) is essentially unique. This uniqueness may fail when the prize space is infinite. Our other axioms put additional structure on preferences, so that we obtain representations with a meaningful state space.

CSR, together with the axioms above, is the behavioral content of a representation of Preference for Flexibility with Constant Beliefs (PFC),

$$V(x) = \int_{\mathcal{U}} \max_{p\in x} \Big[\int_{K\times Z} \big[u(k) + \delta V(z)\big] \, \mathrm{d}p(k, z)\Big] \, \mathrm{d}\mu(u)$$

where $\delta \in (0, 1)$ is a (uniquely determined) discount factor, $\mathcal{U}$ is the space of consumption utilities (ie, the space of von Neumann-Morgenstern (vN-M) utilities over the set of consumption prizes K that are identified up to an additive constant), and μ is a probability measure on $\mathcal{U}$, which is unique up to scaling (see definition 3 below). Unique identification is possible because continuation problems are valued by δV, which is independent of the realised consumption utility $u \in \mathcal{U}$, thereby providing a numeraire.[6] The PFC representation provides a dynamic theory of preference for flexibility: It takes the recursive form of a Bellman equation and features uncertain consumption utilities in a setting where choice of current consumption can affect what is available in the future. The evolution of utilities is iid in the sense that current utilities do not affect the distribution of future utilities.

Our second model relaxes CSR to become *Choice Contingent CSR*. This axiom states, roughly, that DM is strategically rational with respect to continuation problems, contingent on choosing a particular consumption alternative from a sufficiently large menu. We simplify the problem by posing an axiom that limits the number of relevant consumption rankings to be *finite*. Weakening CSR reduces the power of stationarity. To compensate for this, preference for flexibility is assumed to be *persistent*, in the sense that continuation alternatives that add value from the ex ante perspective continue to do so contingent on any particular consumption choice.

We call the corresponding representation a representation of Preference for Flexibility with Ranking Persistent Utilities (PFR). We illustrate the representation using an example. Consider $\mathcal{U}_M := \{u_1, \lambda u_1, u_2\}$ and a Markov process on $\mathcal{U}_M$ with transition probabilities $M(u, u')$, for $u, u' \in \mathcal{U}_M$, where $M(\lambda u_1, u) = M(u_1, u)$ for all $u \in \mathcal{U}_M$. This corresponds to the preference satisfying Consumption Contingent CSR, as the choice of consumption lotteries can, at best, identify the ordinal ranking of those lotteries, and u_1 and λu_1 represent the same vN-M ranking. Further, $0 < M(u, u_2) < 1$ for all $u \in \mathcal{U}_M$, so that all rankings are recurrent. It is easy to see that such a Markov process has a unique invariant measure μ_0. Then, $V(x, \mu_0) := \sum_{u\in\mathcal{U}_M} \mu_0(u)V(x, u)$ is a PFR representation, where

$$V(x, u) = \sum_{u'\in\mathcal{U}_M} \max_{p\in x} \Big[\int_{K\times Z} \big[u'(k) + \delta V(z, u')\big] \, \mathrm{d}p(k, x)\Big] \, M(u, u')$$

(6) Note that in DLR, the subjective space of utilities is *twice* normalized, so that vN-M utility functions on K are identified up to additive constant as well as up to a scaling factor. Thus, if $u \in \mathcal{U}$ is a vN-M utility function on K, the distinction between u and $2u$ is not behaviorally meaningful in the DLR setting.

The PFR representation is unique up to a scaling of the transition probabilities M. Notice that the PFR representation does not feature a numeraire, as the continuation utility $V(z, u)$ is ranking dependent. The standard indeterminacy of state dependent expected utility suggests that one might be able to arbitrarily rescale utilities, $u(k)+\delta V(z, u)$, if the weights of the transition probabilities are adjusted accordingly. However, V is determined implicitly, and our proof demonstrates that rescaling a particular consumption utility, u, implies the same rescaling for $V(z, u)$ if, and only if, all other consumption utilities are rescaled by the same factor. Thus, the mechanism that allows for identification of beliefs in the absence of a numeraire is the self-referential nature of the recursive representation.

Our third model assumes an exogenous state space, S. Choice is over State Contingent IHCPs (SIHCPs), which are acts $f \in H$, that determine for every state $s \in S$ a continuation problem. A continuation problem is a set of lotteries, where each lottery yields consumption and another state contingent IHCP, which conditions on the state at the next period. We adapt our maintained assumptions (section 2) to this domain. In addition, we assume *State Contingent CSR*, which requires that strategic rationality with respect to continuation problemsholds (a) contingent on the state at the time the DM chooses an SIHCP for the following period, and (b) contingent on the state in the following period, at which time the DM will face a particular continuation problem.

This results in a representation of Preference for Flexibility with Exogenous States (PFX). It features state contingent measures μ_s on $\mathcal{U}$, and a subjective Markov process on the set of states, S, with transition probabilities $\Pi(s, s')$. The process is fully connected, ie, $\Pi(s, s') > 0$ for all $s, s' \in S$, and therefore has a unique invariant distribution π_0. Then, $V(\cdot, \pi_0) := \sum_{s \in S} \pi_0(s) V(\cdot, s)$ represents $\succsim$, where

$$V(f, s) = \sum_{s' \in S} \Pi(s, s') \max_{p \in f(s)} \left[\int_{K \times H} \left[u(k) + \delta V(g, s') \right] \mathrm{d}p(k, g) \right] \mathrm{d}\mu_s(u)$$

The PFX representation is unique up to a common scaling of the measures $(\mu_s)_{s \in S}$. As in the PFR representation, continuation utility is state dependent, but identification is nevertheless possible because of the recursive nature of the representation, which ensures that the collection of continuation utilities $V(g, s')$ are jointly identified up to a common scaling.

Consider two decision makers DM and DM*. We say that DM* has a *greater preference for flexibility* than DM if DM* prefers a non-degenerate IHCP over an infinite horizon consumption stream (IHCS), which does not offer any flexibility at all, whenever DM does. Our characterization is as follows: Suppose DM and DM* have PFC representations (μ, δ) and (μ^*, δ^*) respectively, and suppose neither is indifferent between all IHCSs. Then, DM* has a greater preference for flexibility than DM if, and only if, μ^* is a dilation of μ, and $\delta = \delta^*$. If the supports of μ and μ^* are one dimensional, the condition amounts to saying that μ^* is a mean preserving spread of μ.

1.2. Related Literature

Our work builds on standard axiomatic models of preference for flexibility. The innovation that makes it possible to think about preference for flexibility, introduced by Kreps [1979], is the investigation of choice over *menus* of consumption outcomes. The second seminal paper in this literature, DLR, modifies the domain to consider menus of lotteries over outcomes as objects of choice. This facilitates the interpretation of the subjective states as tastes, where a taste is simply a vN-M ranking over consumption outcomes, a distinct advantage over Kreps since the space of all (twice normalized) vN-M functions is a well behaved and analytically tractable set. While menu choice can capture DM's attitude towards the future, implied choice in these models is actually static, in the sense that there is no opportunity for any intertemporal tradeoff. We provide a model of preference for flexibility due to uncertain consumption utilities in the tradition of these earlier papers, where the implied choice is dynamic.

We adopt the recursive domain of Infinite Horizon Consumption Problems (IHCPs), first analyzed by GP, who show that this recursive domain is well defined. GP provide a dynamic model of consumption with temptation preferences. We allow, instead, for uncertain utilities on the domain of IHCPs. In principle, this requires an infinite dimensional subjective state space, which complicates the analysis. Higashi, Hyogo and Takeoka [2009], henceforth HHT, use the same domain and also consider preference for flexibility. They provide a stationary representation, where the preference for flexibility stems exclusively from a random discount factor. This elegantly avoids the issue of the dimensionality of the state space, but the model cannot capture uncertainty about future consumption utilities, for example, uncertain risk aversion, and does not accommodate a dynamic evolution of uncertainty. Takeoka [2006] considers choice between menus of menus of lotteries, and derives a three period version of DLR. The model cannot capture intertemporal tradeoffs, and consequently the representation is only jointly identified, as in DLR. Rustichini [2002] is also interested in preference for flexibility in intertemporal problems, modeled as lotteries over sets of infinite consumption paths. While an additive representation can be obtained, the domain precludes a recursive value function and does not allow unique identification of the representation.

Static models of preference for flexibility have difficulty uniquely identifying parameters from behavior, as beliefs and utilities can not be distinguished. DLR suggest the introduction of a numeraire good (with state independent evaluation) to their model, so as to identify beliefs uniquely in the same sense that they are uniquely identified in Anscombe and Aumann [1963].[7] The source of identification in the PFC representation is similar, where the numeraire arises naturally in the form of continuation problems. Sadowski [2011] considers a situation where observable states of the world contain some information that is 'relevant' for future preferences and shows that beliefs over subjective states can then be identified. Instead of choice between menus, Gul and

(7) Schenone [2010] formalizes this argument.

Pesendorefer [2006] study random choice from menus. Observed choice frequencies naturally correspond to a unique measure over utilities, but the scaling of utilities remains arbitrary in their model. Ahn and Sarver [2011] simultaneously model choice between menus and random choice from menus. They achieve full identification by requiring the beliefs in the representation of choice between menus to correspond to frequencies of choice from menus. Dillenberger and Sadowski [2012] and Takeoka [2007] fully identify representations of preference for flexibility which features uncertain future beliefs, rather than uncertain future tastes.

That a decision maker can be uncertain about future preferences in a dynamic setting is noted by Koopmans [1964], who distinguishes between *once-and-for-all* planning, where the agent selects an action for all possible future contingencies, and *piecemeal* planning where, in each period, the agent chooses an action for the current period, while simultaneously narrowing the set of alternatives for the future. Jones and Ostroy [1984] consider a non-axiomatic (but dynamic) model of choice where an agent prefers flexibility due to uncertainty about future utilities. As mentioned in the introduction, they discuss many instances in macroeconomics and finance where such preferences occur. They also discuss a notion of *greater variability of beliefs* which roughly corresponds to our notion of a dilation of beliefs. A special case of the PFC representation is used in empirical work by Hendel and Nevo [2006], who study the problem of a decision maker who maintains an inventory.

A growing literature argues that variations in preferences over time, and in particular in risk-aversion, are central in explaining various market behaviors. Indeed, estimated risk aversion has been suggested as a useful index of market sentiment[8] and there is evidence that the largest component of changes of the equity risk premium is variation in risk aversion, rather than the quantity of risk.[9] More generally, variation of risk aversion over time has been considered in representative agent settings to improve our understanding of asset pricing phenomena. For instance, Campbell and Cochrane [1999] identify variations in risk premia that correlate with the fundamentals of the economy as a crucial aspect of many dynamic asset pricing phenomena. Our PFX representation, where the evolution of utilities (and in particular risk aversion) depends on the state of the world, can generate such correlation.[10] Bekaert et al [2010] show that allowing stochastic risk aversion that is not driven by, or perfectly correlated with, the fundamentals of the economy can explain a wide range of asset pricing phenomena, as well as fitting important features of bond and stock markets simultaneously. In our PFC and PFR representations utilities evolve independent of the state of the world. As mentioned above, an immediate application of our model is provided in Krishna and Sadowski [2012b], who investigate a Lucas tree economy with an investment stage and a representative agent whose preferences

(8) See, for example, Bollerslev et al [2011].

(9) See Smith and Whitelaw [2009].

(10) In Campbell and Cochrane [1999] the correlation is generated indirectly, via habit forming consumption choices.

have a PFC representation. They show that greater variation in risk aversion results in greater price volatility, and also in underinvestment, since investing in the productive asset reduces liquidity.

2. A Model of Dynamic Preference for Flexibility

In this section we describe the environment and provide a collection of maintained assumptions.

2.1. Environment

For a compact metric space Y, let $\mathscr{P}(Y)$ denote the space of probability measures endowed with the topology of weak convergence, so that $\mathscr{P}(Y)$ is compact and metrizable. Let $\mathscr{F}(Y)$ denote the space of closed subsets of a compact metric space Y, endowed with the Hausdorff metric, which makes $\mathscr{F}(Y)$ a compact metric space.

Let K be a finite set of consumption prizes with typical member k. We follow GP in defining an infinite horizon consumption problem (IHCP) as a collection of lotteries that yield a prize in the present period and a new infinite horizon problem starting in the next period. Let Z be the collection of all IHCPs.[11] GP show that Z is a compact metric space, and that each $z \in Z$ can be identified with a compact set of probability measures over $K \times Z$. In particular, it can be shown that Z is linearly homeomorphic to the space of all closed subsets of $\mathscr{P}(K \times Z)$. We shall denote this linear homeomorphism as $Z \simeq \mathscr{F}\big(\mathscr{P}(K \times Z)\big)$. Typical elements $x, y, z \in Z$ are interpreted as menus of lotteries over consumption and continuation problems, while $p, q \in \mathscr{P}(K \times Z)$ are typical lotteries, with p_k and p_z denoting the marginal distributions of p on K and Z.

We explicitly model choice between consumption problems from an ex-ante perspective, before consumption begins. That is, we analyze a binary relation $\succsim \subset Z \times Z$, which we refer to as a *preference*. We let $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succsim$. The recursive domain of IHCPs is rich; for instance, it can accommodate temporal lotteries as in Kreps and Porteus [1978]. It is also amenable to analysis by stochastic dynamic programming. Its construction follows the *descriptive approach* of Kreps and Porteus [1978] in that it more closely describes how economic agents act and, as mentioned above, embodies what Koopmans [1964] refers to as *piecemeal planning*: instead of choosing a consumption stream that determines consumption for all time, at each instant the decision maker chooses immediate consumption as well as a set of alternatives for the future.

We will also consider the space of menus of consumption lotteries, $\mathscr{F}\big(\mathscr{P}(K)\big)$, with typical members being a, b, c. By the recursive nature of Z, continuation problems are members of Z. Let A, B, C denote typical elements of the collection of menus of

(11) See GP for the recursive construction of Z. We provide a sketch in appendix C.

continuation lotteries, $\mathcal{F}\big(\mathcal{P}(Z)\big)$. To ease notational burden, we will often write $\mathcal{F}$ for $\mathcal{F}\big(\mathcal{P}(K \times Z)\big)$, $\mathcal{F}_K$ for $\mathcal{F}\big(\mathcal{P}(K)\big)$, and $\mathcal{F}_Z$ for $\mathcal{F}\big(\mathcal{P}(Z)\big)$. When there is no risk of confusion, we identify prizes and continuation problems with degenerate lotteries and lotteries with singleton menus. For example, we denote the lottery over continuation problems that yields z with certainty by z, and the lottery that yields current consumption k and continuation problem x with certainty by (k, x).

2.2. Maintained Assumptions

Our first two axioms on $\succsim$ collect standard requirements.

Axiom 1 (Non-triviality)**.** $\succsim$ is non-trivial, in the sense that there exist $x, y \in Z$ such that $x \succ y$.

Axiom 2 (Continuous Order)**.** $\succsim$ satisfies the following:

(a) $\succsim$ is complete and transitive.

(b) $\succsim$ is continuous, in the sense that $\{y : y \succsim x\}$ and $\{y : x \succsim y\}$ are closed.

We take the convex sum of sets to be the Minkowski sum, namely $\lambda x + (1-\lambda)y := \{\lambda p + (1-\lambda)q : p \in x, q \in y\}$ whenever $\lambda \in [0, 1]$. Notice that if $x, y \in \mathcal{F}$, then $\lambda x + (1-\lambda)y$ is also closed, and hence is in $\mathcal{F}$. The following axiom is von Neumann-Morgenstern's Independence axiom.

Axiom 3 (Independence)**.** $x \succ y$ implies $\lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z$ for all $\lambda \in (0, 1]$ and $z \in Z$.[12]

A standard decision maker is one whose preferences are ***strategically rational*** in the sense that $x \succsim y$ implies $x \cup y \sim x$. A standard decision maker who satisfies Axiom 2 chooses as if he evaluates each set by its best element. There exists, then, a continuous function $w : \mathcal{P}(K \times Z) \to \mathbb{R}$ that is linear, such that the functional $x \mapsto \max_{p \in x} w(p)$ represents $\succsim$.[13]

Instead of a standard DM, we are interested in a DM who values flexibility.

Axiom 4 (Monotonicity)**.** $x \cup y \succsim x$ for all $x, y \in Z$.

This is the central axiom in Kreps [1979]. It says that additional alternatives are always weakly beneficial.

(12) A lottery $p \in \mathcal{P}(K \times Z)$ is a *singleton* menu. A weaker version of Independence is *Singleton Independence*, which says that Independence holds for all singleton menus. GP show that Singleton Independence along with Stationarity (Axiom 6) and Indifference to Timing (Axiom 7) imply Independence (Axiom 3) assumed here.

(13) See footnote 5 of Gul and Pesendorfer [2004] for a formal argument.

Theorem 5 in appendix B provides a first representation result for preferences that satisfy Monotonicity (Axiom 4). It establishes that Axioms 1–4 are necessary and sufficient to afford $\succsim$ a ***finitely additive EU representation***. In particular, there exists a subjective state space, $\mathfrak{U}_{K\times Z}$, which is a collection of all the (twice-normalized) vN-M utility functions on $K \times Z$, along with the Borel algebra $\mathscr{A}_{\mathfrak{U}_{K\times Z}}$, and a charge μ on $\mathfrak{U}_{K\times Z}$ that induces the preference functional

$$V(x) := \int_{\mathfrak{U}_{K\times Z}} \max_{p\in x} \mathrm{u}(p)\ \mathrm{d}\mu(\mathrm{u})$$

The state space $\mathfrak{U}_{K\times Z}$ consists of all continuous functions on $K \times Z$ that are identified up to positive affine transformation. In particular, all the utility functions in $\mathfrak{U}_{K\times Z}$ (i) have the same utility for some $x^* \in Z$, and (ii) have the same (supremum) norm. The first requirement corresponds to normalizing the constant term to 0, and the second requirement amounts to normalizing the scaling factor to 1.[14]

We remark that even though K is finite, Z is infinite and infinite dimensional. This implies we cannot appeal to the additive EU representation theorem of DLR. The representation above is an infinite dimensional version of their representation. There, the state space and the charge over the subjective state space are jointly identified, in the sense that, given the normalization of the state space, the charge is unique. In contrast, the finitely additive EU representation that we obtain in Theorem 5 is not jointly identified. The following axioms leverage the recursive nature of the domain to impose more structure on the vN-M utility functions $\mathrm{u} \in \mathfrak{U}_{K\times Z}$.

Each probability measure p over $K \times Z$ induces marginal distributions p_k and p_z over K and Z respectively. The next axiom says that DM does not care about correlations between outcomes in K and Z, but only cares about the marginal distributions induced by the lotteries in the menu. In particular, if two lotteries induce the same marginal distributions over K and Z, then DM does not value the flexibility of having both lotteries available for choice.

Axiom 5 (Separability)**.** If, for $p, q \in \mathscr{P}(K \times Z)$, the marginal distributions satisfy $p_k = q_k$ and $p_z = q_z$, then $\{p, q\} \sim \{p\}$.

Our version of Separability is stronger than the version of separability assumed by GP. Variants of the axiom also appear in HHT and Schenone [2010]. It is instructive to consider what Separability rules out. Let $k_1, k_2 \in K$ be prizes, and let $x_1, x_2 \in Z$ be IHCPs, such that k_2 is more valuable than k_1 (which is to say, $(k_2, x) \succ (k_1, x)$ for all $x \in Z$), and suppose $x_2 \succ x_1$. Let $p, q \in \mathscr{P}(K \times Z)$ be such that p is uniform over (k_2, x_1) and (k_1, x_2), while q is uniform over (k_1, x_1) and (k_2, x_2). Notice that by

(14) There exists a unique element $x^* \in Z$ such that $x^* \simeq (p_k^*, x^*) \in \mathscr{F}$, where x^* consists of the uniform lottery over K, namely $p_k^* \in \mathscr{P}(K)$, in each period. Analogous to the definition in DLRS, we formally have $\mathfrak{U}_{K\times Z} := \{\mathrm{u} \in C(K \times Z) : \|\mathrm{u}\|_\infty = 1,\ \sum_{k\in K} \mathrm{u}(k, x^*) = 0\}$, where $C(K \times Z)$ is the Banach space of continuous functions on $K \times Z$ with the supremum norm.

construction, $p_k = q_k$ and $p_z = q_z$. Nevertheless, p hedges between the present and the future, while q does not, and so it does not seem unreasonable for DM to rank $\{p\} \succ \{q\}$. Separability rules this out by insisting that only the respective marginals matter. We hasten to add that virtually all dynamic models used in applications satisfy separability.

Versions of the next two axioms appear in GP, who provide a more detailed discussion. We are interested in stationary preferences, where the ranking of continuation problems does not depend on time. The recursive nature of the domain allows us to capture this notion via the following axiom, which says that if $x \succsim y$, then x is also better than y as a continuation problem after consumption of k. Recall that (k, x) denotes the degenerate lottery that gives $(k, x) \in K \times Z$ with probability one.

Axiom 6 (Stationarity)**.** $\{(k, x)\} \succsim \{(k, y)\}$ if, and only if, $x \succsim y$.

As mentioned above, the domain of IHCPs is rich enough to describe temporal lotteries. While attitudes towards the timing of the resolution of uncertainty are interesting, we abstract from a preference for a particular pattern of resolution by imposing the following axiom.

Axiom 7 (Singleton Indifference to Timing)**.** $\{\lambda(k, x) + (1 - \lambda)(k, y)\} \sim \{(k, \lambda x + (1 - \lambda)y)\}$ for all $\lambda \in [0, 1]$.

The axiom states that DM is indifferent between (i) receiving lottery $\lambda(k, x) + (1 - \lambda)(k, y)$, which yields consumption k and determines whether the continuation problem will be x or y, (early resolution) and (ii) receiving with certainty consumption k and the continuation menu $\lambda x + (1 - \lambda)y$ (late resolution).

3. Constant Beliefs

This section addresses the case where DM is strategically rational with respect to continuation problems.

3.1. Continuation Strategic Rationality (CSR)

Axiom 8 (CSR)**.** $\{(k, x)\} \succsim \{(k, y)\}$ implies $\{(k, x)\} \sim \{(k, x), (k, y)\}$.

In all the menus compared here, next period's consumption is fixed at k. The axiom says that there is no preference for flexibility with respect to continuation problems, given k. The axiom does not imply that DM is certain about his consumption utility for periods after next period. It only implies that he does not expect this uncertainty to be resolved prior to next period's choice. In particular, the axiom is silent on how DM values the option of retaining the alternatives from *both* x and y for choice two periods ahead. In fact, if $x \cup y \succ x$, which is consistent with Monotonicity (Axiom 4), then Stationarity (Axiom 6) implies $\{(k, x \cup y)\} \succ \{(k, x)\}$. We interpret this ranking, which is not precluded by Axiom 8, as the manifestation of DM's uncertainty about his consumption utility, two periods ahead.

3.2. Preference for Flexibility with Constant Beliefs

Axiom 8 suggests that preference for flexibility arises entirely due to uncertainty about next period's value for consumption (lotteries over K), and that there is no uncertainty about how continuation problems should be valued. In line with this intuition, the subjective state space relevant for our first representation is the set of all vN-M utility functions over K that are identified up to an additive constant, $\mathcal{U} := \{u \in \mathbb{R}^K : \sum u_i = 0\}$, endowed with the Borel sigma algebra. Subjective states $u \in \mathcal{U}$ are naturally interpreted as consumption utilities, and the two terms are viewed as synonyms. Similarly, in what follows, all probability measures on subjective state spaces are interpreted as subjective beliefs, and the two terms are used interchangeably. To ensure that expected consumption utility under a measure μ is well defined, the measure μ must be ***nice***, ie, must satisfy $\mu u := \int_{\mathcal{U}} u \, \mathrm{d}\mu(u) \in \mathcal{U}$, which is equivalent to requiring that the expected utility from every prize $k \in K$ is finite.

Definition 1. Let μ be a nice probability measure on (the Borel sigma-algebra of) $\mathcal{U}$, and $\delta \in (0,1)$. We say that $\succsim$ has a representation of ***Preference for Flexibility with Constant Beliefs*** (PFC), (μ, δ), if there exists a continuous function $V : Z \to \mathbb{R}$ that satisfies

$$V(x) = \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta V(p_z) \right] \mathrm{d}\mu(u) \tag{3.1}$$

and represents $\succsim$.

In the representation above, V is linear on Z, which simply means $V\big(\lambda x + (1-\lambda)y\big) = \lambda V(x) + (1-\lambda)V(y)$ for all $x, y \in Z$, $\lambda \in [0,1]$. We abuse notation and let $u(p_k) = \sum_{k' \in K} p_k(k') u(k')$ denote the extension of u by linearity. Similarly, $V(p_z)$ denotes the linear extension (by continuity) of V from Z to $\mathcal{P}(Z)$, that is, $V(p_z) = \int_Z V(x) \, \mathrm{d}p_z(x)$.

The PFC representation suggests that future utilities over K are uncertain and beliefs follow a subjective process that is iid over time, described by the measure μ. The uncertainty about u implies that there is uncertainty about the tradeoff between instantaneous and future payoffs, that is, the relative magnitude of u as compared to δV.

Proposition 2. Each PFC representation (μ, δ) induces a unique continuous function $V \in C(Z)$ that satisfies equation (3.1) above.

The proposition is a corollary of proposition 12 below. In order to identify equivalent PFC representations of $\succsim$, we need the following definition.

Definition 3. Two probability measures μ and μ' on $\mathcal{U}$ are ***identical up to scaling,*** if there is $\lambda > 0$ such that $\mu(D) = \mu'(\lambda D)$ for all measurable $D \subset \mathcal{U}$, where $\lambda D := \{\lambda u : u \in D\}$.

Given a PFC representation (μ, δ) of $\succsim$, scaling μ corresponds to a scaling of the corresponding value function V. Intuitively, for $\lambda = \frac{1}{2}$, all the relevant subjective utilities are twice as large in (μ', δ) as in (μ, δ), and hence the induced preference functional V' is also twice as large as V.

Theorem 1. *The binary relation $\succsim$ satisfies Axioms 1–8 if, and only if, it has a PFC representation, (μ, δ). Moreover, μ is unique up to scaling, and δ is unique.*

The theorem is a corollary of Theorem 3 in section 5 below. We provide some intuition for a direct proof in section 3.3.

Theorem 1 accommodates the representation in Hendel and Nevo [2006], where an agent faces random taste shocks that are iid. They consider an environment where, in each period, the agent chooses an amount of a good for instantaneous consumption and an amount to add to his inventory. The budget set in each period depends on the realization of prices, which follow a Markov process, and on the inventory. Thus, the choice of inventory constitutes a choice between lotteries over budget sets for next period. This is a version of the IHCP described in the Introduction, except that the prices here are random, and hence the lotteries over continuation problems are not degenerate.

The uniqueness statement in Theorem 1 is the strongest one could hope for.[15] Let us compare this to the identification obtained in the static model of DLR, where there are no continuation problems and preferences are defined over $\mathcal{F}_K$. In DLR, the subjective state space is the collection of all possible consumption *rankings* — formally, their state space is $\mathfrak{U}_K := \left\{r \in \mathcal{U} : \sum_i r_i^2 = 1\right\}$ — and preferences are represented by $V(a) = \int_{\mathfrak{U}_K} \max_{p_k \in a} r(p_k)\ \mathrm{d}\mu(r)$, where μ is a probability measure on $\mathfrak{U}_K$. This leaves two types of non-uniqueness.

First, there are always multiple representations on the larger subjective state space, $\mathcal{U} = \mathfrak{U}_K \times \mathbb{R}_+$, where multiple tastes of different intensity correspond to the same ranking of consumption outcomes. Two measures on $\mathcal{U}$ correspond to the same preference on $\mathcal{F}_K$ if, and only if, they induce the same marginal on $\mathfrak{U}_K$. Consider, for example, the measure μ that has support $\{r_1, r_2\} \subset \mathfrak{U}_K$, and induces the preference functional $V(a) = \sum_{i=1}^{2} \max_{p_k \in a} r_i(p_k)\mu(r_i)$. Now, consider the measure $\tilde{\mu}$ that has support on $\{u_1, u_2, u_3\} \subset \mathcal{U}$, and where $u_1 := \frac{1}{2} r_1$ and $u_2 := \frac{3}{2} r_1$ correspond to the same ranking of consumption lotteries as r_1, and $u_3 := r_2$. Moreover, $\tilde{\mu}(u_1) = \tilde{\mu}(u_2) = \frac{1}{2}\mu(r_1)$, and $\tilde{\mu}(u_3) = \mu(r_2)$. Clearly, $\tilde{V}(a) = \sum_{i=1}^{3} \max_{p_k \in a} u_i(p_k)\tilde{\mu}(u_i) = V(a)$.

Second, beliefs over the subjective state space and intensities of tastes cannot be disentangled. This is the standard indeterminacy of state dependent expected util-

(15) The value function V is at best unique up to positive affine transformations. As noted above, scaling the measure μ corresponds to a scaling of V. Adding constants to the vN-M utilities $u(\cdot)$ is also behaviorally meaningless. In the PFC representation, these constants are simply omitted.

ity, which is also best seen through an example. Suppose the measure μ has support $\{r_1, r_2\} \subset \mathfrak{U}_K$, and both states are equiprobable. Thus, the induced preference functional is $V(a) = \sum_{i=1}^{2} \frac{1}{2} \max_{p_k \in a} r_i(p_k)$. Now consider a different measure $\tilde{\mu}$ on $\{u_1, u_2\}$, where the utilities $u_i \in \mathcal{U}$ are given by $u_i := \left[\frac{1}{2}/\tilde{\mu}(u_i)\right] r_i$. Clearly $\tilde{V}(a) = \sum_{i=1}^{2} \max_{p_k \in a} u_i(p_k)\tilde{\mu}(u_i) = V(a)$.

The examples illustrate that in DLR beliefs and utilities are only *jointly* identified, in the sense that an arbitrary choice of one representation in $\mathcal{U}$ for each vNM ranking is crucial to determining beliefs. In the PFC representation, by contrast, subjective beliefs are uniquely identified (up to scaling) on the state space $\mathcal{U}$. Roughly, this is because continuation problems are valued equally in every subjective state, $u \in \mathcal{U}$, which provides a natural numeraire.[16]

Since μ is unique up to scaling, the PFC representation permits the distinction between (stochastic) risk aversion and (stochastic) intertemporal marginal rate of substitution (IMRS). Suppose, for example, that the support of μ is $\{u_1, \lambda u_1, u_2, \lambda u_2\}$ for some $\lambda > 0$. Risk aversion is the same for tastes u_i and λu_i for $i = 1, 2$, but the implied IMRS will be different. Section 1.2 provides references that argue that varying risk aversion is central in explaining various asset pricing phenomena. For a textbook account that emphasizes the role of the IMRS in asset pricing models, see Skiadas [2009, Chapter 6].

In HHT, there is no uncertainty about the ranking of consumption lotteries, only uncertainty about the discount factor. This is a special case of our model where the support of μ is a subset of $\{u\} \times \mathbb{R}_+ \subset \mathcal{U}$ for some $u \in \mathcal{U}$, with a typical state being (u, λ). The utility u represents the ranking, while it is only the *intensity* $\lambda \in \mathbb{R}_+$ of the consumption taste that is uncertain. In that case the interpretation of δ/λ as a random discount factor is natural. In our general setting, the state space $\mathcal{U}$ can be written as $\mathfrak{U}_K \times \mathbb{R}_+$, so that any $u(\cdot)$ can be written as $\lambda r(\cdot)$ where $r \in \mathfrak{U}_K$ and $\lambda \in \mathbb{R}_+$. Thus, any u could be interpreted as consisting of a consumption ranking r and an intensity $\lambda > 0$. Note that in the PFC representation, the measure μ is generally *not* a product measure over $\mathfrak{U}_K \times \mathbb{R}_+$. Therefore, the interpretation that DM is uncertain about his consumption ranking and, independently, about his discount factor, δ/λ, is not possible. Instead, we find it natural to interpret $u(\cdot)$ as the (random) consumption utility, which specifies how much DM values one lottery over another in terms of the continuation value, about which there is no uncertainty.

3.3. Proof Intuition for the PFC Representation

Theorem 5 (appendix B.1) provides a finitely additive EU representation. Proposition 28 in the appendix shows that any finitely additive EU representation that satisfies Separability

(16) As noted in section 1.2, DLR suggest identification of beliefs in a static model of preference for flexibility on the smaller state space $\mathfrak{U}_K$ by introducing an artificial numeraire, like money. In the dynamic model the numeraire naturally appears in the form of continuation problems.

(Axiom 5) must take the form

$$V(x) = \int_{\mathfrak{U}_K \times [0,1] \times \mathfrak{U}_Z} \max_{p \in x} \left[\lambda r(p_k) + (1-\lambda) v(p_z) \right] \mathrm{d}\mu'(r, \lambda, v)$$

where $\mathfrak{U}_Z$ is a space of twice-normalized vN-M functions over Z. Recall that preferences over menus are strategically rational if, for all menus x and y, it is the case that $x \succsim y$ implies $x \sim x \cup y$. We show in appendix B.2 that any preference relation over menus that is represented by integration against a charge is strategically rational if, and only if, the carrier[17] of the charge is a singleton. Hence, CSR (Axiom 8) implies that we can drop $\mathfrak{U}_Z$ from the description of the state space. Next, lemma 30 shows that since the carrier of the charge μ' is compact, μ' can be extended to the (Borel) sigma-algebra of $\mathfrak{U}_K \times [0, 1]$ as a (countably additive) probability measure. The preference functional then takes the form

$$V(x) = \int_{\mathfrak{U}_K \times [0,1]} \max_{p \in x} \left[\lambda r(p_k) + (1-\lambda) v(p_z) \right] \mathrm{d}\mu'(r, \lambda)$$

Now consider the restricted collection of pairs of consumption and continuation problems $B := K \times \{z_\circ, z^\circ\}$, where $z_\circ = \arg\min_{z \in Z} v(z)$ and $z^\circ = \arg\max_{z \in Z} v(z)$, and notice that $B \subset K \times Z$. Let $\iota : \mathcal{F}\left(\mathcal{P}(B)\right) \to \mathcal{F}\left(\mathcal{P}(K \times Z)\right)$ be the natural inclusion map identifying closed subsets of $\mathcal{P}(B)$ with closed subsets of $\mathcal{P}(K \times Z)$. Define the restricted preference functional on $\mathcal{F}\left(\mathcal{P}(B)\right)$ as $W(x) = V(\iota x)$ for all $x \in \mathcal{F}\left(\mathcal{P}(B)\right)$. Since B is finite, we may invoke the additive EU representation theorem of DLR, which says that for W, and therefore for V, beliefs and utilities are jointly identified. That is, they are identified up to the two types of nonuniqueness discussed after Theorem 1.

Our strong uniqueness result in contrast shows that for the preference functional V, the state space is unique, and the measure μ' is unique up to a common rescaling of its support on [0, 1]. Before sketching the main idea, note the role played by the finite set $\{z_\circ, z^\circ\}$. It is crucial that v not be constant on this set since if v were constant, the support of the measure μ' in the representation of W would simply become $\mathfrak{U}_K$. Moreover, it suffices to consider such a two-element subset of Z since there is no preference for flexibility in continuation problems. Put differently, our choice of $\{z_\circ, z^\circ\}$ allows us to be cognizant of the tradeoff between consumption from K and consumption from Z.

To see the intuition for the uniqueness of μ up to scaling, use the transformation $\frac{\lambda}{1-\lambda} r \mapsto u \in \mathcal{U}$ (and transform the measure μ' on $\mathfrak{U}_K \times [0, 1]$ appropriately to become a measure μ on $\mathcal{U}$), and consider the case where the support of μ in $\mathcal{U}$ is finite, so that the preference functional V can be written as $V(x) = \sum_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + v(p_z) \right] \mu(u)$. Suppose that $V'(x) = \sum_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + v'(p_z) \right] \mu'(u)$ is another preference functional. Since beliefs and utilities are jointly identified, there exist functions (ζ, ξ) :

(17) Intuitively, the carrier of a charge is the smallest closed set that contains all the weight of the charge. If the charge is also a measure, the carrier is referred to as the support of the measure. See definition 20 in the appendix.

$\mathcal{U} \mapsto (\mathbb{R}_+, \mathbb{R})$, such that for each u' in the support of μ', we have $u'(p_k) + v'(p_z) = \zeta(u)\big[u(p_k) + v(p_z)\big] + \xi(u)$ for some u in the support of μ.

Consider two lotteries p and q such that $p_k = q_k$ but $v(p_z) \neq v(q_z)$. Then,

$$\begin{aligned}&\big(u'(p_k) + v'(p_z)\big) - \big(u'(q_k) + v'(q_z)\big)\\ &\quad = v'(p_z) - v'(q_z) = \zeta(u)\big(v(p_z) - v(q_z)\big)\end{aligned}$$

which implies that $\zeta(u)$ is independent of u and hence is constant. Disregarding the additive constants, $\xi(u)$, we can write $V'(x) = \sum_{\mathcal{U}} \max_{p\in x}\left[\zeta u(p_k) + \zeta v(p_z)\right]\mu'(\zeta u)$. Joint identification of beliefs and utilities implies that $\mu'(\zeta u) = \mu(u)$. That is, μ is unique up to scaling.

The final steps of the proof note that Stationarity (Axiom 6) and Singleton Indifference to Timing (Axiom 7) imply that $v(x) = \delta V(x)$, and that since V represents $\succsim$, it must be finite everywhere, so we must have $\delta \in (0, 1)$.

The proof implies that beliefs over $\mathcal{U}$ are identified up to scaling in any additive, separable representation, which does not require Stationarity (Axiom 6) and Singleton Indifference to Timing (Axiom 7). To appreciate the role that recursivity plays for identification in the PFC representation, consider again the finitely additive EU representation V above, where the support of μ is finite. Futher, consider any other measure μ' with the same support as μ. Let $u' := \frac{\mu(u)}{\mu'(u)}u$ and $v'_u := \frac{\mu(u)}{\mu'(u)}v$, which now depend on u. Then $V'(x) = \sum_{\mathcal{U}} \max_{p\in x}\left[u'(p_k) + v'_u(p_z)\right]\mu'(u')$ also represents $\succsim$. Thus, unique identification of μ up to scaling relies on the normalization that the utility of continuation problems should be the same for every u. This is the type of normalization that also underlies the identification of subjective probabilites in the Anscombe-Aumann model (though see Karni [2011] for a critique of this assumption). In the PFC representation, recursivity *implies* this normalization.

4. Markovian Utilities

In this section we suppose that DM is strategically rational with respect to continuation problems, but only contingent on consumption preferences.

An important feature of the PFC representation is that regardless of the current consumption utility, the beliefs about future consumption utilities remain the same. This rules out situations where consumption utilities are correlated over time as, for example, with persistent taste shocks. We show that given our other axioms, the weaker requirement of Consumption Contingent CSR permits the simplest non-trivial dependence of beliefs over utilities on past utilities by allowing utilities to follow a Markov process. Specifically, today's *ranking* of consumption prizes is a sufficient statistic for all the knowledge DM has about his utility tomorrow.

4.1. Choice Contingent CSR

In what follows, we will find particular use for menus with a product structure.

Definition 4. For $p \in \mathscr{P}(K \times Z)$, let $p_k \in \mathscr{P}(K)$ and $p_z \in \mathscr{P}(Z)$ be the corresponding marginals, and let (p_k, p_z) be the product lottery. For $c \in \mathscr{F}_K$ and $A \in \mathscr{F}_Z$, we write $(c, A) \in Z$ to denote the ***rectangular menu*** $\{(p_k, p_z) : p_k \in c, p_z \in A\}$.

For $\mathbf{x}, \mathbf{y} \subset Z$, let $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ be the continuation problem $\{\lambda x + (1-\lambda)y : x \in \mathbf{x}, y \in \mathbf{y}\}$. We maintain Axioms 1–6. Under CSR (Axiom 8), it is sufficient to require Singleton Indifference to Timing (Axiom 7). Without CSR, preference for flexibility may extend to continuation problems, in which case it is natural to strengthen Axiom 7 to apply to all collections of continuation problems.

Axiom 9 (Indifference to Timing)**.** $\{\lambda(k, \mathbf{x}) + (1-\lambda)(k, \mathbf{y})\} \sim \{(k, \lambda\mathbf{x} + (1-\lambda)\mathbf{y})\}$ for all $\lambda \in [0, 1]$, $k \in K$ and $\mathbf{x}, \mathbf{y} \subset Z$.

Separability (Axiom 5) allows us to consider an induced marginal preference relation $\succsim_K \subset \mathscr{F}_K \times \mathscr{F}_K$.

Definition 5. Fix $A \in \mathscr{F}_Z$. Let $a \succsim_K b$ if, and only if, $(a, A) \succsim (b, A)$.

Lemma 43 in appendix F shows that for a separable preference that has a finitely additive EU representation, $\succsim_K$ is independent of the choice of $A \in \mathscr{F}_Z$. In order to elicit continuation preferences contingent on any particular consumption ranking, we aim to identify two consumption menus a and b in $\mathscr{F}_K$ with $a \cup b \succ_K b$, such that the best alternative is in a only under that consumption ranking. This is only possible if the collection of relevant consumption rankings is finite.

Axiom 10 (Finiteness)**.** For all $a \in \mathscr{F}_K$, there is a finite set $b \subset a$ with $b \sim_K a$.

Intuitively, if every set a has a finite subset b that is as good as a itself, then only a finite collection of consumption rankings can be relevant. The formal statement of this result is provided by Riella [2011, Theorem 2], who establishes that Axiom 10 is the appropriate version of the finiteness assumption in Dekel, Lipman and Rustichini [2009] (their Axiom 11) when Monotonicity (Axiom 4) is assumed (but see also lemma 44 in appendix F for a direct proof).

The next axiom is a relaxation of CSR, and is the main behavioral assumption of this section. The goal is to ask DM, in terms of behavior, whether he is strategically rational with respect to continuation problems *contingent* on his consumption ranking, $\succsim_K$. Given a particular element of a finite collection of consumption rankings, it is straight forward to construct two consumption menus a and b in $\mathscr{F}_K$, such that the most preferred alternative from $a \cup b$ is in a only under that consumption ranking. Hence, it would be sufficient to ask DM, for all a, b with $a \cup b \succ_K b$, whether he is strategically rational contingent on his preferred alternative from $a \cup b$ being in a.

However, this requirement would be too strong, as for some a and b, the best alternative is in a for more than one consumption ranking.[18] In order to avoid such situations, we allow the addition of any collection of alternatives, c, to b, so long as $a \cup b \cup c \succ_K b \cup c$. If the best alternative from $a \cup b$ is in a for only one ranking, this must remain true for any such c. If the best alternative is in a for multiple rankings, then there is c such that the best alternative from $a \cup b \cup c$ is in a for only one ranking. Summing up, we would like to require the following:

> If $a \cup b \succ_K b$ then there is $c \in \mathscr{F}_K$, such that (i) $a \cup b \cup c \succ_K b \cup c$, and (ii) DM is strategically rational with respect to continuation problems, contingent on his preferred consumption choice from $a \cup b \cup c$ being in a.

To formalize (ii), consider two continuation menus A and B. Let C be a subset of $A \cup B$. Then, the most preferred lottery from $(a, C) \cup (b \cup c, A \cup B)$ is in (a, C) only if the most preferred consumption choice is in a. The following definition simplifies notation.

Definition 6. Given a menu z, let $\overset{\circ}{\succsim}_z$ denote the induced ranking over additional alternatives: $x \overset{\circ}{\succsim}_z y$ if, and only if, $x \cup z \succsim y \cup z$.

Axiom 11 (Choice Contingent CSR)**.** If $a \cup b \succ_K b$ then there is $c \in \mathscr{F}_K$, such that

(i) $$a \cup b \cup c \succ_K b \cup c$$

(ii) $$(a, A) \overset{\circ}{\succsim}_{(b \cup c, A \cup B)} (a, B) \text{ implies } (a, A) \overset{\circ}{\sim}_{(b \cup c, A \cup B)} (a, A \cup B)^{19}$$

In the setting where DM considers only finitely many consumption rankings relevant, the axiom is falsifiable in spite of the existential qualifier. To see this, note that $\succsim_K$ determines the finite set of relevant consumption rankings. As noted in the discussion preceding the axiom, the construction of a and b such that a outperforms b for only one given consumption ranking is straightforward. For such a and b we may assume, without loss of generality, that the c whose existence is guaranteed by the axiom is empty. This observation is formally established in Lemma 45.

Finally, we want to ensure that ex-ante preferences are a useful description of choice on the recursive domain of IHCPs, in the sense that no alternative that is relevant from the ex-ante perspective can become permanently irrelevant in the future. To avoid conditioning on entire consumption paths, we impose the stronger requirement that the collection of relevant alternatives is not choice contingent.

Axiom 12 (Persistent Preference for Flexibility)**.** For all $a, b \in \mathscr{F}_K$ such that $a \cup b \succ_K b$,

$$x \cup y \succ x \text{ implies } (a, \{x \cup y\}) \overset{\circ}{\succ}_{(b,\{x \cup y\})} (a, \{x\}).$$

(18) In particular, Continuation Strategic Rationality (Axiom 8) would be implied for a and b such that the best alternative is always in a.

The first qualifier considers two consumption menus a and b, where a is not dominated by b. Again, the axiom only has implications for the case where the preferred consumption choice is in a rather than b. The axiom says that, if $x \cup y \succ x$, then this must also be true contingent on the preferred consumption choice being in a. We emphasize that this requirement and Choice Contingent CSR (Axiom 11) are *not* mutually exclusive: contingent on next period's preferred consumption choice being in a, the singleton $\{x \cup y\}$ provides DM with additional alternatives for choice two periods from now, which he may value (Axiom 12), even though he is strategically rational with respect to the union $\{x\} \cup \{y\}$ (Axiom 11), which would force him to choose one of the two smaller continuation problems in the next period.

4.2. Preference for Flexibility with Ranking Persistent Utilities

In terms of the representation, weakening CSR allows for correlation of consumption utilities. At the same time, choice of a lottery in a over any lottery in $b \cup c$ only carries information about DM's *current* ranking of immediate consumption and, contingent on this information, preferences are required to satisfy strategic rationality with respect to continuation problems. Hence, today's consumption ranking must be a sufficient statistic for today's beliefs over future consumption utilities.

Definition 7. A ***ranking persistent*** Markov process $(\mathcal{U}_M, M)$ consists of a state space $\mathcal{U}_M \subset \mathcal{U}$ that is a finitely generated cone,[20] and a Markov kernel[21] M from $\mathcal{U}_M$ to itself, such that M is:

(a) ***ranking contingent:*** $M(u, \cdot) = M(\lambda u, \cdot)$ for all $\lambda > 0$, and

(b) ***persisent:*** $M\big(u, \{\lambda u' : \lambda > 0\}\big) > 0$ for all $u, u' \in \mathcal{U}_M$.

The ranking persistent Markov process $(\mathcal{U}_M, M)$ is ***nice*** if $M(u, \cdot)$ is a nice probability measure for each $u \in \mathcal{U}_M$.

We identify the Markov process $(\mathcal{U}_M, M)$ by its Markov kernel M, when the state space $\mathcal{U}_M$ is understood. The following lemma establishes that the class of ranking persistent Markov kernels is a desirable subclass of all Markov kernels on $\mathcal{U}$, as ranking persistence guarantees the existence of a unique invariant (and hence ergodic) measure of the Markov process.[22]

(20) A set $D \subset \mathcal{U}$ is a ***cone*** if for all $\lambda > 0$, $u \in D$ implies $\lambda u \in D$. The cone D is ***generated*** by a set $D_0 \subset \mathcal{U}$ if $D := \bigcup_{\lambda > 0} \lambda D_0$. It is ***finitely generated*** if it is generated by a finite set.

(21) Let $(X, \mathscr{X})$ be a measurable space. Then, $M : X \times \mathscr{X} \to [0, 1]$ is a ***Markov kernel*** from $(X, \mathscr{X})$ to itself if (i) for each $x \in X$, $M(x, \cdot)$ is a probability measure on $(X, \mathscr{X})$, and (ii) for each $D \in \mathscr{X}$, $M(\cdot, D)$ is a measurable function defined on X. The Markov kernel represents the transition probabilities for a Markov process with state space X.

(22) This is intuitive, because the induced Markov process on the relevant rankings is fully connected. For

Lemma 8. If $(\mathcal{U}_M, M)$ is a ranking persistent Markov process, then an invariant measure μ_0 of M exists and is unique, where $\mu_0(\mathrm{d}u) = \int_{\mathcal{U}} M(u', \mathrm{d}u)\, \mu_0(\mathrm{d}u')$.

A proof is in appendix F. As before, we represent integrals with respect to measures as extensions by linearity and continuity, and write $V(\cdot, \mu_0) := \int_{\mathcal{U}} V(\cdot, u)\ \mathrm{d}\mu_0(u)$. Similarly, $V(p_z, u)$ denotes the linear extension (by continuity) of $V(z, u)$ from Z to $\mathcal{P}(Z)$.

Definition 9. Let $\delta \in (0, 1)$ and let $(\mathcal{U}_M, M)$ be a nice ranking persistent Markov process. A preference $\succsim$ has a representation of ***Preference for Flexibility with Ranking Persistent Utilities*** (PFR), $\big((\mathcal{U}_M, M), \delta\big)$, if $V(\cdot, \mu_0)$ represents $\succsim$, where μ_0 is the invariant measure of M, and V is defined recursively as

$$V(x, u) = \int_{\mathcal{U}} \max_{p \in x} \left[u'(p_k) + \delta V(p_z, u')\right] M(u, \mathrm{d}u') \tag{4.1}$$

Since μ_0 is the invariant measure of the Markov process, $V(x, \mu_0)$ takes the intuitive form

$$V(x, \mu_0) = \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta V(p_z, u)\right]\ \mathrm{d}\mu_0(u)$$

To see this, consider some menu $x \in Z$. Then,

$$\begin{aligned}
\int_{\mathcal{U}} V(x, u)\ \mathrm{d}\mu_0(u) &= \int_{\mathcal{U}} \Big[\int_{\mathcal{U}} \max_{p \in x} \left[u'(p_k) + \delta V(p_z, u')\right] M(u, \mathrm{d}u') \Big]\ \mathrm{d}\mu_0(u) \\
&= \int_{\mathcal{U}} \Big[\max_{p \in x} \left[u'(p_k) + \delta V(p_z, u')\right] \underbrace{\int_{\mathcal{U}} \mu_0(\mathrm{d}u) M(u, \mathrm{d}u')}_{=\mu_0(\mathrm{d}u')} \Big] \\
&= \int_{\mathcal{U}} \max_{p \in x} \left[u'(p_k) + \delta V(p_z, u')\right]\ \mathrm{d}\mu_0(u')
\end{aligned}$$

where the first equality uses equation (4.1), the second equality uses Fubini's theorem to reverse the order of integration, and $\mu_0(\mathrm{d}u') = \int_{\mathcal{U}} M(u, \mathrm{d}u')\, \mu_0(\mathrm{d}u)$ because μ_0 is the invariant distribution of the Markov process. The ranking persistence of M implies that for all $u \in \mathcal{U}$, $V(x, u) = V(x, \lambda u)$ for all $\lambda > 0$.

Proposition 10. Each PFR representation $\big((\mathcal{U}_M, M), \delta\big)$ induces a unique continuous function $V \in C(Z \times \mathcal{U}_M)$ that satisfies equation (4.1) above.

The proof is in the supplementary appendix Krishna and Sadowski [2012a]. Generalising our notion for equivalence of measures on $\mathcal{U}$ (Definition 3), we shall say that two Markov processes $(\mathcal{U}_M, M)$ and $(\mathcal{U}_{M'}, M')$ are ***identical up to scaling*** if $\mathcal{U}_M = \mathcal{U}_{M'}$ and there exists $\lambda > 0$ such that $M(u, D) = M'(u, \lambda D)$ for all measurable $D \subset \mathcal{U}$.

the purpose of the identification results it would be sufficient to consider irreducible Markov processes on the relevant rankings, as discussed in footnote 38. Note that this generalization is small, in the sense that the class of fully connected Markov processes is dense in the class of irreducible processes. The small gain in generality does not seem to warrant imposing a weaker, but non-falsifiable, assumption that only requires the existence of some finite consumption path, contingent on which persistence is satisfied.

Theorem 2. *The binary relation $\succsim$ satisfies Axioms 1–6, and 9–12 if, and only if, it has a PFR representation $\big((\mathcal{U}_M, M), \delta\big)$. Moreover, $(\mathcal{U}_M, M)$ is unique up to scaling, and δ is unique.*

The proof of the theorem is in appendix F. In contrast to the PFC representation, a PFR representation can generate, for example, persistent taste shocks via the Markov process that governs the evolution of consumption tastes.

The next section provides a sketch of the most instructive steps in the proof of the theorem, and also demonstrates how the recursive structure of the PFR representation implies uniqueness of M up to scaling, even though the continuation value varies with the consumption utility.

4.3. Proof Intuition for the PFR Representation

Given the general representation in Theorem 5 (appendix B.1), Independence and Separability play the same role as before. Finiteness implies that there are only finitely many relevant consumption rankings. For any $u \in \mathcal{U}_M$, let $[u] := \{\lambda u : \lambda > 0\}$ denote the set of utilities in $\mathcal{U}_M$ that correspond to the same ranking as u. We refer to $[u]$ as a consumption ranking. Thus, there exist $u_1, \ldots, u_n \in \mathcal{U}$ such that $\mathcal{U}_M := \bigcup_i [u_i]$. We argued, when discussing Choice Contingent CSR (Axiom 11), that we can establish strategic rationality with respect to continuation problems for any one of those consumption rankings. Therefore the consumption ranking must be a sufficient statistic for beliefs. That is, for each ranking $[u_i]$, there exists a unique $v\big(\cdot, [u_i]\big) : \mathcal{F} \to \mathbb{R}$ that evaluates continuation problems, so that there are only finitely many valuations for continuation problems. This observation is key to showing that the representation is jointly identified – see proposition 31 – because this allows us to show that the subjective state space can be taken to be finite dimensional, which in turn allows us to use DLR's joint identification result.

Persistent Preference for Flexibility (Axiom 12) implies that the functions $v\big(\cdot, [u_i]\big)$ are monotone with respect to set inclusion and, in particular, are locally non-satiated. To show that the functions $v\big(\cdot, [u_i]\big)$ are linear, let $v_i(z) := v(z, [u_i])$, and consider the set $O := \bigcap_i v_i^{-1}\big(\operatorname{int} v_i(Z)\big)$. It is easy to see that O is open, and because the functions v_i are locally non-satiated, O is also dense in Z. Moreover, each v_i is uniformly continuous (because Z is compact). Therefore, to show that v_i is linear on Z, it suffices to show that v_i is linear on O.

Suppose now, there exist $x, y \in O$ and $\lambda \in (0, 1)$ such that $v_1\big(\lambda x + (1-\lambda) y\big) \neq \lambda v_1(x) + (1-\lambda) v_1(y)$. There exist sets $\mathbf{x} := \{x_1, \ldots, x_n\}$ and $\mathbf{y} := \{y_1, \ldots, y_n\}$ where (i) $x = x_1$, $y = y_1$, (ii) $v_i(x_i) > v_i(x_j)$ for all $i \neq j$ and similarly for $\mathbf{y}$, and (iii) $\mathbf{x}, \mathbf{y} \subset O$. It is easy to see that for each $\lambda \in [0, 1]$, $\lambda(k, x_i) + (1-\lambda)(k, y_i)$ is the unique lottery in the menu $\lambda(k, \mathbf{x}) + (1-\lambda)(k, \mathbf{y})$ that maximizes $u + \delta v_i$. Indifference to Timing (Axiom 9) allows us to conclude that, $\lambda x_i + (1-\lambda) y_i \in Z$ is the unique maximizer of

v_i from the set $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \subset Z$ for each i, and for $i = 1$ in particular, which is a contradiction. This establishes that v_i is linear on O.

Thus, each v_i has a finitely additive EU representation. By Stationarity (Axiom 6), we show that $\succsim$ has a recursive representation

$$V(x, \mu_0) = \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta(u) V(p_z, u)\right] \mu_0(\mathrm{d}u)$$

where the discount factor may only depend on the consumption ranking, that is, $\delta(u) = \delta(\lambda u)$ for all $\lambda > 0$, and $(\mathcal{U}_M, M)$ is a ranking contingent Markov process such that

$$V(x, u) = \int_{\mathcal{U}} \max_{p \in x} \left[u'(p_k) + \delta(u') V(p_z, u')\right] M(u, \mathrm{d}u')$$

for each $u \in \mathcal{U}_M$.

We need to establish that the value function V can be renormalized to make the discount factor independent of u. Suppose, for simplicity, that the support of μ_0 is $\mathcal{U}^* = \{u_1, ..., u_n\}$ where no u_i and u_j are collinear. Suppose also that there exists a value function

$$\hat{V}(x, \hat{\mu}_0) = \sum_{\hat{u} \in \hat{\mathcal{U}}^*} \max_{p \in x} \left[\hat{u}(p_k) + \hat{\delta} \hat{V}(p_z, \hat{u})\right] \mathrm{d}\hat{\mu}_0(\hat{u})$$

that represents $\succsim$ and features a constant discount factor. The uniqueness result in DLR implies that $\hat{\mathcal{U}}^*$ and $\mathcal{U}^*$ must correspond to the same collection of vN-M rankings. That is, there is a reordering of $\hat{\mathcal{U}}^*$, such that $\xi(u_i) := u_i / \hat{u}_i$ is well defined for $i = 1, \ldots, m$.

With $\mu_i(\cdot) := M(u_i, \cdot)$, we write $\langle \mu_i, \xi \rangle$ to denote $\sum_j \mu_i(u_j)\xi(u_j)$. Then $\hat{\mu}_i(\hat{u}) = \frac{\mu_i(u)\xi(u)}{\langle \mu_i, \xi \rangle}$ must hold for all $\hat{\mu}_i$, $i = 0, 1, \ldots, m$ and clearly

$$\sum_{\hat{\mathcal{U}}^*} \max_{p \in x} \left[\hat{u}(p_k) + \frac{\delta(u)}{\xi(u)} V(p_z, u)\right] \hat{\mu}_0(\hat{u})$$

represents $\succsim$, as it is a renormalization of $V(x, \mu_0)$. Therefore, $\hat{\delta}\hat{V}(\cdot, \hat{u}) = \frac{\delta(u)}{\xi(u)} V(\cdot, u)$ must hold for all $u \in \mathcal{U}^*$. At the same time $\hat{V}(\cdot, \hat{u}_i) = \frac{V(\cdot, u_i)}{\langle \mu_i, \xi \rangle}$. Hence, $\hat{\delta} = \frac{\delta(u_i)}{\xi(u_i)} \langle \mu_i, \xi \rangle$. To establish that the value function $\hat{V}$ exists, we have to show that there is ξ such that $\frac{\delta(u_i)}{\xi(u_i)} \langle \mu_i, \xi \rangle$ is constant for all $u_i \in \mathcal{U}^*$.

Letting A denote the Markov transition matrix for the Markov process M, the condition amounts to finding ξ such that $\kappa\xi = \Delta\mathsf{A}\xi$ for some $\kappa > 0$, where Δ is the diagonal matrix[23]

$$\begin{bmatrix} \delta(u_1) & 0 & \ldots & 0 \\ 0 & \delta(u_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta(u_m) \end{bmatrix}$$

(23) It is easy to see that with Δ and A defined as above, $\Delta\mathsf{A}$ is the square matrix where the row of A that corresponds to u is multiplied by $\delta(u)$.

That is, we need a strictly positive eigenvector ξ of the matrix $\Delta\mathsf{A}$. Persistent Preference for Flexibility (Axiom 12) implies that A (and hence $\Delta\mathsf{A}$) is strictly positive. The Perron Theorem (Theorem 6 in appendix F.3) then implies that there is such an eigenvector ξ that is strictly positive, and hence there is a representation $\hat{V}$ as above. It also implies that the eigenvector ξ is unique up to scaling and the corresponding eigenvalue $\hat{\delta}$ is unique. This implies that beliefs in the representation $\hat{V}$, ie, the collection $\{\hat{\mu}_i : i = 0, \ldots, m\}$, must be unique up to a common scaling. The proof generalizes this argument to apply to the case where $\mathcal{U}^*$ may contain collinear elements (and hence may not be finite).

To see directly why uniqueness of beliefs up to scaling must hold, consider the example of two PFR representations of the same preferences, V and $\hat{V}$ as above, where $\mathcal{U}^*$ does not contain collinear elements, and δ and $\hat{\delta}$ are constant. Suppose, without loss of generality, that $1 > \delta \geqslant \hat{\delta}$. We argued above that $\xi(u_i) = \frac{\delta}{\hat{\delta}}\langle \mu_i, \xi\rangle$ must hold. The Markov process M is ranking persistent, and hence the support of μ_i is $\mathcal{U}^*$ for all $i \in \{0, \ldots, m\}$. Suppose $\xi(u_i) \neq \xi(u_j)$ for some $i, j \in \{0, \ldots, m\}$. Pick the i that minimizes $\xi(u_i)$ and observe that $\xi(u_i) < \langle \mu_i, \xi\rangle \leqslant \frac{\delta}{\hat{\delta}}\langle \mu_i, \xi\rangle$, a contradiction. Hence, $\xi(u_i) = \xi(u_j)$ for all $i, j \in \{0, \ldots, m\}$, which just says that $\{\mu_i : i = 0, \ldots, m\}$ and $\{\hat{\mu}_i : i = 0, \ldots, m\}$ must be identical up to a common scaling.

To better understand why Persistent Preference for Flexibility (Axiom 12) is necessary, we now provide an example of a preference that satisfies all axioms except Axiom 12, and show that it does not have a PFM representation. Suppose $\succsim$ can be represented by the value function V above, where $\mathcal{U}^* = \{u_1, u_2\}$ is the support of μ_0, $\mathsf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\delta(u_1) \neq \delta(u_2)$. The interpretation is that today DM is uncertain about tomorrow's consumption utility, but once he learns his utility, he does not expect it to ever change again. In that case, there is no representation with a constant discount factor, because $\frac{\delta(u_i)}{\xi(u_i)}\langle \mu_i, \xi\rangle = \delta(u_i)$ for all positive ξ.

5. State Contingent Beliefs

Many applications involve contractable objective states. Investors can, for example, contract on inflation, dividend growth or other measures of the state of the economy. As we point out in the introduction, risk premia that vary negatively with the state of the economy are essential in explaining a wide range of bond and stock market phenomena.[24] In this section we consider an environment where the objective state of the world may evolve over time (according to a subjective process), and where DM can make state contingent plans of action. The model we provide allows consumption utilities, and in particular risk aversion, to be correlated with the state of the world.

Let $S := \{1, \ldots, n\}$ be a finite set of ***states of the world***. For any metric space Y,

(24) Campbell and Cochrane [1999] suggest that this correlation might be due to habit forming consumption. Empirically, however, it is the correlation itself that is important, and not the particular mechanism that drives it, as Bekaert et al [2010] point out.

let $\mathscr{H}(Y) := Y^S$ denote the space of ***acts*** from S to Y. In a manner analogous to the approach taken above, we now define a State Contingent Infinite Horizon Consumption Problem (SIHCP) as an act that specifies, for each state of the world, a continuation problem. A continuation problem is a menu of lotteries that yield a consumption prize in the present period and a new SIHCP starting in the next period.

Let H be the collection of all SIHCPs.[25] H is a compact metric space and is also convex in the sense that for $f, g \in H$, $\lambda f + (1-\lambda)g \in H$ for all $t \in [0,1]$. Each $f \in H$ can be identified with an act that yields a compact set of probability measures over $K \times H$ in every state. In particular, it can be shown that H is linearly homeomorphic to the space of all acts that take values in $\mathscr{F}\left(\mathscr{P}(K \times H)\right)$. We shall denote this linear homeomorphism as $H \simeq \mathscr{H}\left(\mathscr{F}\left(\mathscr{P}(K \times H)\right)\right)$, and we will consider a preference relation $\succsim$ on H. Typical SIHCPs $f, g \in H$ are ***acts***, typical elements $x, y, z \in \mathscr{F}\left(\mathscr{P}(K \times H)\right)$ are ***menus*** of lotteries over consumption and SIHCPs, $p, q \in \mathscr{P}(K \times H)$ are typical lotteries, and p_k and p_h denote the marginal distributions of p on K and H respectively. In what follows, we will abuse notation and refer to the act that gives $x \in \mathscr{F}$ in every state by $x \in H$. Similarly, $(k, f) \in \mathscr{F}$ will refer to the degenerate lottery that yields the sure consumption prize $k \in K$ and the sure SIHCP $f \in H$.

To define the induced ranking of menus contingent on the state $s \in S$, $\succsim_s$, fix $x^* \in \mathscr{F}$. For any $x \in \mathscr{F}$, consider the act

$$f_s^x(s') := \begin{cases} x & \text{if } s = s' \\ x^* & \text{otherwise} \end{cases}$$

For any $x, y \in \mathscr{F}$, let $x \succsim_s y$ if, and only if, $f_s^x \succsim f_s^y$. In the presence of our other axioms, $\succsim_s$ is independent of the particular x^* for which it is defined.

We adapt the requirements on $\succsim$ encountered in section 2.2 to the new domain H. In the interests of space, we list them here, but state them formally in appendix A.[26] We require that $\succsim$ be complete, transitive, and continuous (Axiom X3), and satisfy Independence (Axiom X4) and Stationarity (Axiom X7). The remaining requirements on $\succsim$ are most easily stated as requirements on $\succsim_s$. We require that $\succsim_s$ satisfy Monotonicity (Axiom X5), Separability (Axiom X6), and Indifference to Timing (Axiom X8). A natural relaxation of strategic rationality on this domain is to consider the case where all uncertainty is captured by the objective state, in which case each $\succsim_s$ must be strategically rational. This results in a recursive version of the state dependent representation in Anscombe and Aumann [1963] (see corollary 13 below).

Suppose strategic rationality is violated, even contingent on the state. As before, one might conjecture that the DM is strategically rational with respect to continuation problems, in which case $\succsim$ must satisfy CSR. This assumption results in a representation

(25) We provide a sketch of the construction of H in appendix C.

(26) Axioms in this section are labelled with an ‘X’ to indicate the presence of exogenous states in the environment. The numbering of the axioms corresponds to those in the PFC representation.

where beliefs over utilities depend only on the state s, which evolves according to a subjective iid process (see corollary 14 below).

However, in many applications, exogenous states may be correlated over time. In such situations, CSR is likely to be violated. We are then lead to relax CSR by considering a state contingent version of CSR.

5.1. State Contingent CSR

The central assumption of this section requires DM to be strategically rational with respect to continuation problems, contingent on the state, in the sense that the state at every point in time is a sufficient statistic for the DM's preference over future continuation problems.

Axiom X9 (State Contingent CSR)**.**

(a) $\{(k, f)\} \succsim_s \{(k, g)\}$ implies $\{(k, f)\} \sim_s \{(k, f), (k, g)\}$.

(b) $\{(k, f_s^x)\} \succsim \{(k, f_s^y)\}$ implies $\{(k, f_s^x), (k, f_s^y)\} \sim \{(k, f_s^x)\}$, and

Property (a) of Axiom X9 requires CSR to hold contingent on the state s one period in the future, when DM first chooses an SIHCP. Property (b) requires CSR to hold, contingent on the state s two periods in the future, when DM gets to choose from the specified continuation problem.

We further require that every state $s' \in S$ be non-null two periods ahead, independent of the realization the state $s \in S$ one period ahead.

Axiom X1 (Persistent Non-triviality)**.** For all $s, s' \in S$ there exist $x, y \in \mathcal{F}$, such that $\{(k, f_{s'}^x)\} \succ_s \{(k, f_{s'}^y)\}$.

The axiom implies that regardless of the state realized in the first period, DM still believes that all states are possible in the subsequent period.

5.2. State Contingent Markovian Utilities

In this section, we provide a representation that features a subjective Markov process over objective states, as well as state contingent beliefs over consumption utilities.

Definition 11. Let $\mathcal{U}$ be defined as above, for each $s \in S$, let μ_s be a nice probability measure on (the Borel sigma-algebra of) $\mathcal{U}$, and $\delta \in (0, 1)$. Let Π represent the transition probabilities for a fully connected Markov process on S,[27] and let π_0 be the unique invari-

(27) The Markov process on S is fully connected if $\Pi(s, s') > 0$ for all $s, s' \in S$. A weaker requirement would be that every state is irreducible. The class of fully connected Markov processes is dense in the class of irreducible processes. The small gain in generality does not seem to warrant imposing a weaker, but unfalsifiable version of persistence, that only requires the existence of a finite path of state realizations, contingent on which persistence is satisfied. Footnote 22 made an analogous argument for the PFR representation.

ant measure of Π. A preference $\succsim$ has a representation of ***Preference for Flexibility with Exogeneous States*** (PFX), $\big((\mu_s)_{s\in S}, \Pi, \delta\big)$, if $V(\cdot, \pi_0) := \sum_s V(\cdot, s)\pi_0(s)$ represents $\succsim$, where V is defined recursively as

$$V(f, s) = \sum_{s\in S} \Pi(s, s') \left[\int_{\mathcal{U}} \max_{p\in f(s)} \big[u(p_k) + \delta V(p_h, s')\big] \, \mathrm{d}\mu_{s'}(u) \right] \tag{5.1}$$

In the representation above, V is linear on H, and $V(p_h, \cdot)$ denotes the linear extension (by continuity) of V from H to $\mathcal{P}(H)$, that is, $V(p_h, \cdot) = \int V(g, \cdot) \, \mathrm{d}p_h(g)$.

Proposition 12. Each PFX representation $\big((\mu_s)_{s\in S}, \Pi, \delta\big)$ induces a unique continuous function $V \in C(H \times S)$, that satisfies equation (5.1) above.

The proof is in appendix E.

Theorem 3. *The binary relation $\succsim$ satisfies Axioms X1 to X9 if, and only if, it has a PFX representation, $\big((\mu_s)_{s\in S}, \Pi, \delta\big)$. Moreover, the measures $(\mu_s)_{s\in S}$ are unique up to a common scaling, and Π and δ are unique.*

The proof is in appendix E. In analogy to the PFR representation, we have

$$V(f, \pi_0) = \sum_{s} \pi_0(s) \int_{\mathcal{U}} \max_{p\in f(s)} \big[u(p_k) + \delta V(p_h, s)\big] \, \mathrm{d}\mu_s(u)$$

which follows from the fact that $\pi_0(s') = \sum_s \pi_0(s)\Pi(s, s')$.

The PFC representation is a special case of the PFX representation when $|S| = 1$. Thus, Theorem 1 follows immediately from Theorem 3. As mentioned above, another interesting special case is the situation where each $\succsim_s$ is fully strategically rational. In that case the model reduces to a recursive version of the state dependent Anscombe and Aumann [1963] model, where DM has a preference for flexibility only because he is unsure about which state $s \in S$ will obtain and because his consumption utility $u_s \in U$ depends on the state.

Corollary 13. Suppose $\succsim$ has a PFX representation $\big((\mu_s)_{s\in S}, \Pi, \delta\big)$. Then, each $\succsim_s$ is strategically rational if, and only if, $\succsim$ has a recursive Anscombe-Aumann representation. That is, there is a collection $(u_s)_{s\in S} \subset \mathcal{U}$ where $\mu_s\big(u_s\big) = 1$ for all $s \in S$, such that $V(\cdot, \pi_0) := \sum_s \pi_0(s)V(\cdot, s)$ represents $\succsim$, where V is defined recursively as

$$V(f, s') = \sum_{s} \Pi(s', s) \max_{p\in f(s)} \big[u_s(p_k) + \delta V(p_h, s)\big]$$

Moreover, the set $(u_s)_{s\in S}$ is unique up to a common scaling, and Π and δ are unique.

Identification of subjective probabilities on the state space in the standard (static) Anscombe-Aumann model requires two assumptions. First, the ordinal ranking of lotteries should be independent of the state. This assumption is referred to as the State

Independence Axiom. Second, the cardinal representation of these rankings (the utility) should be state independent. In contrast, in the recursive Anscombe-Aumann representation of corollary 13, the uniqueness of (Markovian) subjective probabilities only relies on the recursive structure with a constant discount factor.

We end with the observation (without proof) that requiring full CSR would imply that the Markov process generated by Π is iid.

Corollary 14. Suppose $\succsim$ has a PFX representation $\big((\mu_s)_{s\in S}, \Pi, \delta\big)$. Then, the following are equivalent:

(a) $\succsim$ satisfies CSR, ie, $\{(k, f)\} \succsim \{(k, g)\}$ implies $\{(k, f)\} \sim \{(k, f), (k, g)\}$.

(b) Π is trivial, ie, Π represents an iid process wherein $\Pi(s, \cdot) = \Pi(s', \cdot)$ for all $s, s' \in S$.

5.3. Proof Intuition for the PFX Representation

Given the general representation in Theorem 5 (appendix B.1), Independence (Axiom X4) implies that there exists a representation of $\succsim$ of the form $W(f) := \sum_s \pi_0(s) U_s\big(f(s)\big)$, where $U_s : \mathscr{F} \to \mathbb{R}$ is linear, continuous, and monotone (with respect to set inclusion). Following the ideas sketched for the PFR representation, we establish that property (a) of State Contingent CSR (Axiom X9) implies that we can write $U_s(x) = \int_{\mathscr{U}} \max_{p\in x} \big[u(p_k)+v_s(p_h)\big]\, \mathrm{d}\mu_s(u)$, where every v_s is linear, continuous, and monotone (with respect to set inclusion). By the Mixture Space Theorem (see, for instance, Kreps [1988]), v_s is unique up to scaling and can be written as $v_s(f) := \sum_{\tilde{s}} \pi_s(\tilde{s}) w_{s,\tilde{s}}(f(\tilde{s}))$. By property (b) of State Contingent CSR (Axiom X9), $w_{s,\tilde{s}}$ must be a scaled version of $U_{\tilde{s}}$, and in particular independent of s. Using Persistent non-triviality (Axiom X1), we show that each π_s must have full support.

Hence, we have a recursive representation of $\succsim$, where the discount factor may only depend on the state:

$$V(f, \pi_0) = \sum_s \pi_0(s) \left[\int_{\mathscr{U}} \max_{p\in f(s)} \big[u(p_k) + \delta_s V(p_h, s)\big]\, \mu_s(\mathrm{d}u) \right]$$

represents $\succsim$, and Π is a fully connected Markov process on S, such that

$$V(f, s') = \sum_s \Pi(s', s) \left[\int_{\mathscr{U}} \max_{p\in f(s)} \big[u(p_k) + \delta_s V(p_h, s)\big]\, \mathrm{d}\mu_s(u) \right]$$

As in the proof of the PFR representation, the Perron theorem then allows us to establish the existence of a unique transformation of the parameters in V to attain a representation with a state independent discount factor.

In the appendix we prove Theorem 3 before establishing Theorem 2, as part of the argument is analogous and easier to explain in the context of objective states.

6. Behavioral Comparisons

Preference for flexibility is the preference for non-degenerate menus over singletons. Intuitively, one decision maker has more preference for flexibility than another if he has a stronger preference for menus over singletons. To make matters precise, let us consider the restricted domain $L \subset Z$ of ***Infinite Horizon Consumption Streams (IHCSs)***. This domain consists of lotteries that deliver consumption for the present period and an IHCS for the next period. It is easy to show that L is a closed and convex subset of Z. In a manner analogous to the characterisation of risk aversion where lotteries are compared to certain amounts of money, characterizing preference for flexibility requires a comparison between IHCPs and IHCSs. This comparison is meaningful only if the preference restricted to L is non-trivial.

Axiom 13 (Consumption Non-triviality)**.** There exist $\ell, \ell' \in L$ such that $\ell \succ \ell'$.

If $\succsim$ has a PFC representation, the restriction of $\succsim$ to L is generated by the vN-M function μu. (Recall our notational convention whereby $\mu u := \int_{\mathcal{U}} u \, \mathrm{d}\mu(u)$.) It then follows that if $\succsim$ has a PFC representation, it satisfies Consumption Non-triviality (Axiom 13) if, and only if, $\mu u \neq \mathbf{0}$ (where $\mathbf{0} \in \mathcal{U}$).[28]

Definition 15. $\succsim^*$ has ***greater preference for flexibility*** than $\succsim$ if

$$x \succsim \ell \ \text{ implies } \ x \succsim^* \ell$$

for all $\ell \in L$ and $x \in Z$.[29]

The comparison in the definition implies that $\succsim$ and $\succsim^*$ have the same ranking of IHCSs, ie, $\ell \succsim \ell'$ if, and only if, $\ell \succsim^* \ell'$. (This is lemma 55 in the appendix and also assumes Independence.) Since preference for flexibility is the behavioral manifestation of uncertainty about consumption utilities, it is ideally characterized in terms of beliefs, which requires the following definition.

Definition 16 (Dilation)**.** Let $Q(u, D)$ be a Markov kernel from $\mathcal{U}$ to itself. Then $Q(u, D)$ is a ***dilation*** if it is expectation preserving, ie, if $\int_{\mathcal{U}} u' \, Q(u, \mathrm{d}u') = u$ for each $u \in \mathcal{U}$. If μ and μ^* are probability measures on $\mathcal{U}$, then μ^* is a ***dilation*** of μ if there exists a dilation Q, such that $\mu^* = Q\mu$, ie, $\mu^*(\mathrm{d}u') := \int Q(u, \mathrm{d}u') \, \mu(\mathrm{d}u)$.

DM's expected vN-M function is μu, representing his expected utility preference over lotteries over K. If μ^* is a dilation of μ, then $\mu^* u = \mu u$, because a dilation preserves expectations.

(28) The "if" part of the claim is easy to see. To see the "only if" part, let μ have $\mu u = \mathbf{0}$, and consider $W \in C(Z)$ such that $W(\ell) = 0$ for all $\ell \in L$ and $W(x) \geqslant 0$ for all $x \in Z$. Then, with $\Phi : C(Z) \to C(Z)$ given by $\Phi W(x) := \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta W(p_z) \right] \mathrm{d}\mu(u)$, we have $\Phi W(x) \geqslant 0$ for all $x \in Z$, with equality on L. Therefore, the unique fixed point of Φ, namely the value function V representing $\succsim$, must also have this property.

(29) The definition is identical to the definition of *more averse to commitment* in HHT.

The PFC representation (μ, δ) only identifies the measure μ up to scaling. In order to facilitate a comparison of measures, we shall say that a PFC representation (μ, δ) is ***canonical*** if $\|\mu u\|_2 = 1$. Obviously, $\succsim$ admits a canonical PFC representation if, and only if, $\mu u \neq \mathbf{0}$ if, and only if, $\succsim$ satisfies Axiom 13.

Theorem 4. *Let $\succsim$ and $\succsim^*$ have canonical PFC representations (μ, δ) and (μ^*, δ^*), respectively. Then, the following are equivalent:*

(a) *$\succsim^*$ has greater preference for flexibility than $\succsim$.*

(b) *$\delta = \delta^*$ and μ^* is a dilation of μ.*

The proof is in appendix G. Intuitively, DM* with preference $\succsim^*$ has greater preference for flexibility than DM with preference $\succsim$ precisely because he expects more uncertainty to resolve before making a choice, which increases the option value of waiting to make a choice from the menu. Some asset pricing implications of Theorem 4 are explored in Krishna and Sadowski [2012b].

It is worthwhile to consider the case where $\succsim$ has a PFC representation but does not satisfy Consumption Non-triviality (Axiom 13). In that case DM is indifferent between all IHCSs, but nevertheless has a preference for flexibility, presumably because he expects to learn about his consumption utility in the future. It follows from Monotonicity (Axiom 4) that every menu x is preferred to every IHCS $\ell \in L$. Obviously, our definition of greater preference for flexibility has no bite in that case.

As mentioned above, subjective uncertainty in the dynamic model of HHT only concerns the discount factor. Thus, their model is a special case of the PFC representation, where the support of the measure μ can be written as $\{\lambda u : \lambda > 0\}$ for some $u \in \mathcal{U}$. Consequently, HHT's characterisation of greater preference for commitment (their Theorem 4.2) is a special case of our Theorem 4.

Finally, we emphasise that the behavioral comparison in terms of beliefs provided by Theorem 4 is possible only because beliefs are identified up to a scaling in our dynamic setting. In contrast, the notion of 'greater preference for flexibility' proposed in DLR for the static context cannot rely on beliefs because those are not identified in their model. Hence, instead of characterizing whether one DM has a stronger preference for flexibility than another, DLR characterize whether one DM has any preference for flexibility, whenever the other does. Neither ranking is complete. While ours can only compare preferences that agree on the ranking of singletons, the ranking in DLR can only compare preferences with representations for which the support of the measure is ordered by set inclusion.

In the context of the PFX representation we can also compare DM's strength of preference for flexibility *across* states. We define a state contingent consumption stream as an act that gives, in each state, a lottery over the pair consisting of instantaneous consumption and a contingent continuation stream. A formal recursive definition is given in Appendix C. This allows us to adapt Consumption Non-triviality (Axiom 13) to the domain H.

Proposition 17. Suppose $\succsim$ has a PFX representation and satisfies the appropriate version of Axiom 13. Then, for $s, s' \in S$, the following are equivalent:

(a) $\succsim_s$ exhibits greater preference for flexibility than $\succsim_{s'}$.

(b) μ_s is a dilation of $\mu_{s'}$ and $\Pi(s, \cdot) = \Pi(s', \cdot)$.

The fact that $\succsim_s$ and $\succsim_{s'}$ must be the same on the subdomain of consumption streams implies $\Pi(s, \cdot) = \Pi(s', \cdot)$. Hence, if all states are comparable in the sense of greater preference for flexibility, then objective states evolve according to a process that is iid.

Proposition 17 describes how DM trades off flexibility across states. For example, if condition (b) is satisfied for states s and s', and if $\pi_0(s) \geqslant \pi_0(s')$, then DM prefers flexibility in state s over flexibility in state s'.

7. Conclusion

We provide foundations for three qualitatively different recursive representations of choice between IHCPs by investigating three relaxations of the standard assumption of strategic rationality. In particular, we require strategic rationality with respect to continuation problems unconditionally (CSR), conditional on the choice of current consumption from a sufficiently large menu (Choice Contingent CSR), and contingent on the objective state of the world (State Contingent CSR). The subjective Markov processes that govern the evolution of beliefs over consumption utilities in the corresponding representations are uniquely identified.

While we suggest the three versions of CSR above as a natural starting point when trying to understand how strategic rationality is violated, these could obviously be relaxed further. For example, CSR might only be satisfied contingent on finite histories of consumption choices and states. Modelling the corresponding evolution of beliefs as a Markov process would require a larger subjective state space, but as long as preferences also satisfied a weak version of Persistence, one could obtain a recursive representation. Beyond that, our proofs of identification only require that the implied Markov process over preferences have a unique stationary distribution. Given a suitable version of Persistence, such uniqueness can be ensured by Stationarity (via an application of the Perron Theorem).

In that sense, in addition to providing foundations for the three models presented in the paper, which are as close as possible to the standard model, we view our work as illustrative of how to achieve a fully identified recursive representation of choice when the assumption of strategic rationality is further weakened.

Appendices

A. Axioms from Section 5

Axiom X2 (Non-triviality)**.** $\succsim$ is non-trivial, in the sense that there exist $f, g \in H$ such that $f \succ g$.

Axiom X3 (Continuous Order)**.** $\succsim$ satisfies the following:

(a) $\succsim$ is complete and transitive.

(b) $\succsim$ is continuous, in the sense that the sets $\{f : f \succsim g\}$ and $\{f : g \succsim f\}$ are closed.

Axiom X4 (Independence)**.** $f \succ g$ implies $\lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h$ for all $\lambda \in (0, 1)$.

Axiom X5 (Monotonicity)**.** $x \cup y \succsim_s x$ for all $x, y \in \mathscr{F}$.

Axiom X6 (Separability)**.** If, for $p, q \in \mathscr{P}(K \times H)$, the marginal distributions satisfy $p_k = q_k$ and $p_h = q_h$, then $\{p, q\} \sim_s \{p\}$.

Recall that (k, f) denotes the degenerate lottery that gives $(k, f) \in K \times H$ with probability one in every state.

Axiom X7 (Aggregate Stationarity)**.** $f \succsim g$ implies $\{(k, f)\} \succsim \{(k, g)\}$.

Axiom X8 (Indifference to Timing)**.** $\{\lambda(k, f) + (1-\lambda)(k, g)\} \sim_s \{(k, \lambda f + (1-\lambda)g)\}$ for all $\lambda \in [0, 1]$.

B. Abstract Representations

We shall first construct a general representation in the spirit DLRS where the prize space is infinite. We then study the effect of assuming Strategic Rationality. We begin by defining and collecting some facts about support functions.

Let Y be a compact metric space. Let $\mathscr{P}(Y)$ be the space of probability measures on Y, endowed with the topology of weak convergence, which makes $\mathscr{P}(Y)$ compact, metrizable, and let $C(Y)$ denote the Banach space of uniformly continuous functions on Y. In what follows, for $f \in C(Y)$ and $p \in \mathscr{P}(Y)$, we will frequently denote $\int f \, \mathrm{d}p$ by $f(p)$. Fix $p^* \in \mathscr{P}(Y)$, and let $X := \{f \in C(Y) : f(p^*) = 0\}$.

For any weak* closed, convex subset G of $\mathscr{P}(Y)$, let $\overline{h}_G : X \to \mathbb{R}$ be its *extended support function*, given by $\overline{h}_G(f) := \sup_{p \in G} f(p)$. The support function is sublinear, ie, subadditive and positively homogeneous, and Mackey continuous [Theorem 5.102, Aliprantis and Border, 1999], and hence is also norm continuous, since the Mackey and norm topology coincide on normed spaces [Corollary 6.27, Aliprantis and Border, 1999].

Let $\mathfrak{U}_Y := \{f \in X : \|f\|_\infty = 1\}$, and notice that an extended support function is completely defined by the values it takes on $\mathfrak{U}_Y$. Therefore, for any sublinear and norm continuous function $\overline{h} : X \to \mathbb{R}$, we shall consider its restriction to $\mathfrak{U}_Y$, denoted by h. We call a function

$h : \mathfrak{U}_Y \to \mathbb{R}$ a ***support function*** if its unique extension to X by positive homogeneity is sublinear and norm continuous, ie, if it is an extended support function in the sense described above.

Support functions have the following duality: For any weak* compact, convex subset G of aff $(\mathcal{P}(Y))$, $G_{h_G} = G$ where $G_{h_G} := \{p \in \text{aff}(\mathcal{P}(Y)) : f(p) \leqslant h_G(f) \text{ for all } f \in \mathfrak{U}_Y\}$. Support functions also have the following useful properties[30] for weak* compact, convex subsets G, H of $\mathcal{P}(Y)$: (i) $G \subset H$ if and only if $h_G \leqslant h_H$, (ii) $h_{tG+(1-t)H} = th_G + (1-t)h_H$ for all $t \in (0,1)$, (iii) $h_{G\cap H} = h_G \wedge h_H$, and (iv) and $h_{\text{conv}(G\cup H)} = h_G \vee h_H$. (Since G and H are convex, it follows from Lemma 5.14 of Aliprantis and Border [1999] that conv$(G \cup H)$ is also compact.) Notice also that $h_{\{p^*\}} = \mathbf{0}$.

B.1. Constructing a General Representation

Let Y be a compact metric space of prizes, so $\mathcal{P}(Y)$ is a compact metric space. Let $\mathcal{F}(\mathcal{P}(Y))$ denote the space of compact subsets of $\mathcal{P}(Y)$, and $\mathcal{K}(\mathcal{P}(Y))$ the space of closed, convex subsets of $\mathcal{P}(Y)$, both endowed with the Hausdorff metric. Then, $\mathcal{K}(\mathcal{P}(Y))$ is a closed subspace of $\mathcal{F}(\mathcal{P}(Y))$. We consider a preference $\succsim$ over $\mathcal{F}(\mathcal{P}(Y))$. For notational ease, we shall write $\mathcal{F}$ and $\mathcal{K}$ for $\mathcal{F}(\mathcal{P}(Y))$ and $\mathcal{K}(\mathcal{P}(Y))$ respectively. Typical elements of $\mathcal{F}$ will be denoted by G, G' etc. The following axiom is the standard version of Independence, where $\alpha G + (1-\alpha)G' \in \mathcal{F}$ is the Minkowski sum.

Axiom 14 (Independence)**.** $G \succ G'$ implies $\alpha G + (1-\alpha)G'' \succ \alpha G' + (1-\alpha)G''$ for all $\alpha \in (0,1]$.

The space of all vN-M utility functions is simply $C(Y)$, the Banach space of uniformly continuous functions on Y. In general, in a state dependent additive EU representation, the vN-M utility functions need only be identified up to positive affine transformation. Fix $p^* \in \mathcal{P}(Y)$ as above and, analogous to DLRS, let the subjective state space be given by $\mathfrak{U}_Y := \{f \in C(Y) : f(p^*) = 0 \text{ and } \|f\|_\infty = 1\}$, the space of all vN-M utility functions that (i) take the value 0 at p^*, (ii) are nontrivial on $\mathcal{P}(Y)$, and (iii) lie on the boundary of the unit ball of $C(Y)$.

We refer to the state space $\mathfrak{U}_Y$ as the ***canonical state space***. Let $\mathcal{A}_{\mathfrak{U}_Y}$ be the Borel algebra of sets in $\mathfrak{U}_Y$, and μ a *normal*[31] *charge* on $(\mathfrak{U}_Y, \mathcal{A}_{\mathfrak{U}_Y})$. A pair $(\mathfrak{U}_Y, \mu)$ is a ***finitely additive EU representation*** of $\succsim$ if $V(x) = \int_{\mathfrak{U}_Y} \max_{p\in x} f(p)\, \mathrm{d}\mu(f)$ represents $\succsim$.

Theorem 5. *A preference, $\succsim$, satisfies Non-triviality, Continuous Order, Monotonicity*[32] *and Independence (Axiom 14) if, and only if, it admits a finitely additive EU representation.*

Note the key differences between Theorem 5 and the additive EU representation theorem of DLR. They establish that given the subjective state space, the measure is unique and countably additive, while the theorem above establishes neither. There are two reasons for these differences.

(30) See, for instance, p 226 of Aliprantis and Border [1999].

(31) A charge is *outer regular* if every set in $\mathcal{A}_{\mathfrak{U}_Y}$ can be approximated from without by open sets; *inner regular* if it can be approximated from within by closed sets; and *normal* if it is both outer and inner regular – see also definition 10.2 in Aliprantis and Border [1999].

(32) These are the obvious extensions of Axioms 1, 2, and 4 to this general domain.

The first is that our subjective state space is not compact, therefore, the Riesz Representation Theorem only guarantees that the measure is finitely additive. (Notice that this would remain the case even if one were able to establish uniqueness of the representation.) The second difference is that we are unable to show that the span of the space of all support functions is dense in the space of all continuous bounded functions on the subjective state space, a result that holds when the subjective state space is finite dimensional (and hence compact). Before the formal details are presented, we provide some intuition for the proof.

The proof of the representation naturally extends ideas in DLRS to the infinite dimensional setting. The first step is to show that each menu can be identified (isometrically) with its support function, and that support functions live in the space of twice normalized, non-trivial vN-M functions on the prize space Y. (This is exactly as in DLRS.) This allows us to define the subjective state space as a space of all twice normalized, continuous, non-constant, non-trivial functions on Y. Instead of looking at the space of menus, we can look at the space of support functions, a subset of all continuous bounded functions on the subjective state space.

The second step of the proof shows that any linear functional on the space of menus induces a continuous linear functional on the corresponding space of support functions. Moreover, since the preference $\succsim$ is monotone, this linear functional is Lipschitz, and can therefore be extended to the space of all continuous bounded functions on the subjective state space. (This step uses the Hahn-Banach Theorem.) The final step uses the Riesz Representation Theorem to show that any linear functional on the space of all continuous bounded functions can be written as an integral with respect to a finitely additive measure.

Proof of Theorem 5. By the continuity of $\succsim$, and since $\succsim$ satisfies Independence (Axiom 14), an adaptation of Lemmas 1 and 2 from DLR implies that $G \sim \overline{\text{conv}}(G)$ for each $G \in \mathcal{F}$. (In particular, we have $\overline{\text{conv}}\left(tG + (1-t)G'\right) = t\,\overline{\text{conv}}(G) + (1-t)\,\overline{\text{conv}}(G')$.) Following DLR, it suffices to restrict attention to $\mathcal{K}$, the space of all weak* compact, convex subsets of $\mathcal{P}(Y)$. Here, $\mathcal{P}(Y)$ is endowed with a metric inducing the weak* topology, and $\mathcal{K}$ is endowed with the Hausdorff metric. Let $K_0 := \{h \in C_b(\mathfrak{U}_Y) : h = h_G \text{ for some } G \in \mathcal{K}\}$ denote the affine embedding of $\mathcal{K}$ in $C_b(\mathfrak{U}_Y)$, where $C_b(\mathfrak{U}_Y)$ is the space of bounded and continuous functions on $\mathfrak{U}_Y$, endowed with the supremum norm, and recall that by construction, $\mathbf{0} \in K_0$.

Define the induced preference $\succsim^*$ on K_0, so that $G \succsim G'$ if, and only if, $h_G \succsim^* h_{G'}$. It is easily seen that $\succsim^*$ is complete and transitive, and satisfies continuity, Independence (Axiom 14), and Monotonicity[33] (Axiom 4), since $\succsim$ has these properties. By the Mixture Space Theorem (see, for instance, Fishburn [1970] or Kreps [1988]), there exists a function $\varphi : K_0 \to \mathbb{R}$ that represents $\succsim^*$, is linear in the sense that $\varphi\big(\alpha h^1 + (1-\alpha)h^2\big) = \alpha\varphi(h^1) + (1-\alpha)\varphi(h^2)$, and is *positive* so that $h^1 \geqslant h^2$ implies $\varphi(h^1) \geqslant \varphi(h^2)$.

Let $K_1 := \bigcup_{r \geqslant 0} rK_0$ be the cone generated by K_0. Extend φ to K_1 by positive homogeneity, and let Φ_0 denote this extension. Notice that Φ_0 is linear and positive.

Let us now consider $K_1 - K_1$, and notice first that any $f \in K_1 - K_1$ can be written as $r_1(h_1 - h_1')$. Then, for any $f, g \in K_1$, we have $f - g = [r_1(h_1 - h_1')] - [r_2(h_2 - h_2')] = \big(r_1h_1 + r_2h_2\big) - \big(r_1h_1' + r_2h_2'\big)$. It follows that $f - g \in K_1 - K_1$ since $\big(r_1h_1 + r_2h_2\big), \big(r_1h_1' + r_2h_2'\big) \in K_1$.

(33) Recall that $G \supset G'$ if, and only if, $h_G \geqslant h_{G'}$. Monotonicity of $\succsim$ requires that if $G \supset G'$, then $G \succsim G'$, which implies that we must have $h_G \succsim^* h_{G'}$. Thus, $\succsim^*$ satisfies Monotonicity in the sense that for $h^1, h^2 \in K_0$, $h^1 \geqslant h^2$ implies $h^1 \succsim^* h^2$.

In particular, there exist $h_{G_1}, h_{G_2} \in K_0$ and $r \geqslant 0$ such that $f - g = r\big[h_{G_1} - h_{G_2}\big]$, from which it follows that $K_1 - K_1$ is a vector subspace of $C_b(\mathfrak{U}_Y)$.

The linear function Φ_0 can be extended to $K_1 - K_1$ by linearity. As in DLRS, we claim that Φ_0 on $K_1 - K_1$ is Lipschitz. To see this, consider $f - g \in K_1 - K_1$ such that $f - g \geqslant \mathbf{0}$. Then, $\Phi_0(f - g) = r\Phi_0\big(h_{G_1} - h_{G_2}\big) = r\Phi_0(h_{G_1}) - r\Phi_0(h_{G_2})$. But by the positivity of φ, it follows that $\varphi(h_{G_1}) \geqslant \varphi(h_{G_2})$, since $h_{G_1} - h_{G_2} \geqslant \mathbf{0}$ (ie, $G_1 \supset G_2$). Therefore, it must be the case that $\Phi_0(f) \geqslant \Phi_0(g)$. Finally, notice that for any $f \in K_1 - K_1$, we have $f \leqslant \|f\|_\infty \mathbf{1}$ (indeed, $|f| \leqslant \|f\|_\infty \mathbf{1}$, where $\mathbf{1}$ is the constant function equal to 1), so that $\Phi_0(f) \leqslant \|f\|_\infty \Phi_0(\mathbf{1})$, ie, Φ_0 is Lipschitz with constant $\Phi_0(\mathbf{1})$.

Since Φ_0 is Lipschitz on a vector subspace $K_1 - K_1$, the Hahn-Banach Theorem allows us to extend it to $C_b(\mathfrak{U}_Y)$, with the extension being denoted by Φ. The Riesz Representation Theorem [Theorem 13.9, Aliprantis and Border, 1999] allows us to represent the linear functional Φ on $C_b(\mathfrak{U}_Y)$ as an integral with respect to a normal charge μ on $\mathscr{A}_{\mathfrak{U}_Y}$, the algebra generated by the open sets of $\mathfrak{U}_Y$, as desired. □

B.2. Strategic Rationality

Let $\succsim$ be a preference on $\mathscr{F}$.

Definition 18. A preference, $\succsim$, is ***strategically rational*** if $G \succsim G'$ implies $G \sim G \cup G'$.

In this section, we show that if $\succsim$ has a finitely additive representation $(\mathfrak{U}_Y, \mu)$, it is strategically rational if, and only if, μ is carried by a singleton. We begin with a lemma.

Lemma 19. For any $\mathfrak{U}_0 \subset \mathfrak{U}_Y$, the following are equivalent.

(a) $|\mathfrak{U}_0| = 1$.

(b) For all $p, q \in \mathscr{P}(Y)$ and for all $f_1, f_2 \in \mathfrak{U}_0$, $f_1(p) \geqslant f_1(q)$ if and only if $f_2(p) \geqslant f_2(q)$.

Proof. It is clear that (a) implies (b). To see that (b) implies (a), notice that by the definition of $\mathfrak{U}_Y$, there exists $p^* \in \mathscr{P}(Y)$ such that $f(p^*) = 0$ for all $f \in \mathfrak{U}_Y$. Moreover, $\|f\|_\infty = 1$ for all $f \in \mathfrak{U}_Y$. Therefore, no $f_1 \in \mathfrak{U}_Y$ is a positive affine transformation of some other f_2 in $\mathfrak{U}_Y$.

Now suppose $|\mathfrak{U}_0| > 1$, and let $f_1, f_2 \in \mathfrak{U}_Y$ be distinct. But by the definition of $\mathfrak{U}_Y$, f_1 and f_2 represent the same expected utility preference $\succsim^*$ on $\mathscr{P}(Y)$. Therefore, by the expected utility theorem and since $f_1(p^*) = f_2(p^*) = 0$, $f_1 = \alpha f_2$ for some $\alpha > 0$, which contradicts the definition of $\mathfrak{U}_Y$, wherein $\|f\|_\infty = 1$ for all $f \in \mathfrak{U}_Y$. This proves our claim. □

Definition 20. The ***carrier*** of the charge μ is defined as the set $\mathfrak{U}_\mu := \bigcap\{N : N \text{ is closed}, \mu(N^c) = 0\}$.

The carrier of the charge always exists, and is clearly well defined. If the charge is also a measure, then the carrier is referred to as the ***support*** of the measure. Moreover, given the definition of μ, we have $\mathfrak{U}_\mu \subset \mathfrak{U}_Y$. For any $p, q \in \mathscr{P}(Y)$, define $\mathfrak{U}_{p,q} := \{f \in \mathfrak{U}_Y : f(p) > f(q)\}$ and $\mathfrak{U}^\circ_{p,q} := \{f \in \mathfrak{U}_Y : f(p) = f(q)\}$. Notice that $\mathfrak{U}_{p,q}$ is always open and $\mathfrak{U}^\circ_{p,q}$ is always closed, since p is a continuous (linear) functional on $\mathfrak{U}_Y$, which is a closed set.

Lemma 21. If $\succsim$ on $\mathscr{F}$ is strategically rational, then $\min\{\mu(\mathfrak{U}_{p,q}), \mu(\mathfrak{U}_{q,p})\} = 0$ for all $p, q \in \mathscr{P}(Y)$.

Proof. Suppose to the contrary there exist p, q with $\min\{\mu(\mathfrak{U}_{p,q}), \mu(\mathfrak{U}_{q,p})\} = \mu(\mathfrak{U}_{p,q}) > 0$. It is clear that $\{p, q\} \not\sim \{p\}, \{q\}$, which violates Strategic Rationality. ☐

Lemma 22. If for all $p, q \in \mathscr{P}(Y)$, $\min\{\mu(\mathfrak{U}_{p,q}), \mu(\mathfrak{U}_{q,p})\} = 0$, then $|\mathfrak{U}_\mu| = 1$.

Proof. If for all $p, q \in \mathscr{P}(Y)$, $\min\{\mu(\mathfrak{U}_{p,q}), \mu(\mathfrak{U}_{q,p})\} = 0$, then either (i) $\mathfrak{U}_\mu \subset \mathfrak{U}_{p,q} \cup \mathfrak{U}^\circ_{p,q}$, or (ii) $\mathfrak{U}_\mu \subset \mathfrak{U}_{q,p} \cup \mathfrak{U}^\circ_{p,q}$, but not both (by the definition of $\mathfrak{U}_\mu$ and since $\mathfrak{U}_{p,q}$ and $\mathfrak{U}_{q,p}$ are open).

In other words, for all $p, q \in \mathscr{P}(Y)$, and for all $f_1, f_2 \in \mathfrak{U}_Y$, $f_1(p) \geqslant f_1(q)$ if, and only if, $f_2(p) \geqslant f_2(q)$. Lemma 19 now implies that $|\mathfrak{U}_\mu| = 1$, as required. ☐

We may now put all this together, as follows.

Proposition 23. Let $\succsim$ be a preference on $\mathscr{F}(\mathscr{P}(Y))$ that has a finitely additive representation $(\mathfrak{U}_Y, \mu)$. Then, the following are equivalent.

(a) $\succsim$ is strategically rational.

(b) The carrier of the charge μ is a singleton, ie, $|\mathfrak{U}_\mu| = 1$.

Proof. It is easy to see that (b) implies (a). We shall now establish that (a) implies (b). By Lemma 21, strategic rationality implies $\min\{\mu(\mathfrak{U}_{p,q}), \mu(\mathfrak{U}_{q,p})\} = 0$ for all $p, q \in \mathscr{P}(Y)$. By Lemma 22, we can then conclude that $|\mathfrak{U}_\mu| = 1$, as desired. ☐

C. The Domain of IHCPs

We now sketch the construction of State Contingent Infinite Horizon Consumption Problems (SIHCPs) introduced in section 5. The IHCPs introduced in section 2 are a special case of this construction. The construction adapts ideas from GP and so we omit details. As in the text, K is a finite set of consumption prizes in any period, and S is a finite set of states.

Let $H_1 := \mathscr{H}(\mathscr{F}(\mathscr{P}(K)))$ denote the set of acts that give a closed subset of $\mathscr{P}(K)$ in each state $s \in S$. It is a compact metric space when endowed with the (product) Hausdorff metric. For each $t > 1$, inductively define $H_t := \mathscr{H}(\mathscr{F}(\mathscr{P}(K \times H_{t-1})))$, which is also a compact metric space. Thus, each $f_t \in H_t$ is an act that gives a closed set of probability measures over $K \times H_{t-1}$ in each state $s \in S$. This allows us to define the space $H^* := \bigtimes_{t=1}^\infty H_t$, which is also a compact metric space.

Thus, an element in H^* is a collection (f_t), where $f_t \in H_t$ for each $t \geqslant 1$. Each f_t contains information about some of the $f_\tau \in H_\tau$ for all $\tau < t$ since each $p_t \in f_t(s)$ is a probability measure over $K \times H_{t-1}$, and each $p_{t-1} \in f_{t-1}(s)$ is a probability measure over $K \times H_{t-2}$, and so on. A sequence (f_t) is ***consistent*** if, roughly speaking, the information in f_t about f_{t-2} is in accord with the information in f_{t-1} about f_{t-2}.[34] What is relevant for us is the

(34) See GP for a precise definition of consistency. It is easy to construct examples of sequences in H^* that are *not* consistent.

space of *all* consistent sequences, denoted by $H \subset H^*$. The space H is the space of *IHCP*s. A simple adaptation of Theorem A1 of GP shows that there is a homeomorphism between H and $\mathscr{H}\big(\mathscr{F}\big(\mathscr{P}(K \times H)\big)\big)$. In fact, it can be shown that this homeomorphism is an *affine* or *linear* homeomorphism, so that we can define a linear preference on H (ie, a preference that satisfies Independence) and study the naturally induced preference on $\mathscr{H}\big(\mathscr{F}\big(\mathscr{P}(K \times H)\big)\big)$.

There are two special cases that are of importance to us. The first is the space of State Contingent Infinite Horizon Consumption Streams (SIHCSs). This is the subset $L \subset H$ that delivers a singleton menu in each state an in every period. It can be shown that $L \simeq \mathscr{H}\big(\mathscr{P}(K \times L)\big)$. The second important case is where S is a singleton, so that the agent always chooses between menus rather than acts. This defines a domain $Z \simeq \mathscr{F}\big(\mathscr{P}(K \times Z)\big)$, which is the domain of IHCPs introduced in section 2 and used in the PFC and PFM representations. This is also the domain constructed in GP.

D. A Separable Representation

In the process of obtaining our representations, we will frequently find it useful to obtain an intermediate representation on one state space, and then transform the representation so it is defined on another state space. The following lemma is an abstract version of this idea. In what follows, we consider the prize space $K \times Y$. If Y is compact, $K \times Y$ is also compact. This allows us to define the canonical state space $\mathfrak{U}_{K \times Y}$ and with a typical state given by $\mathfrak{u} \in \mathfrak{U}_{K \times Y}$. Recall that $\mathfrak{U}_{K \times Y} = \big\{\mathfrak{u} \in C(K \times Y) : \|\mathfrak{u}\|_\infty = 1,\ \sum_{k \in K} \mathfrak{u}(k, p^*) = 0\big\}$ for some lottery $p^* \in \mathscr{P}(Y)$. We say that a finitely additive EU representation $(\mathfrak{U}_Y, \mu)$ is ***jointly identified*** if, given the state space $\mathfrak{U}_Y$, the charge μ is unique.[35]

Lemma 24 (Change of State Space)**.** Let $(\mathfrak{U}_{K \times Y}, \mu)$ be a finitely additive EU representation of $\succsim$. A sufficient condition for $(\mathfrak{U}'_{K \times Y}, \mu')$ to be another finitely additive EU representation of $\succsim$ is that there exist functions $\Psi : \mathfrak{U}_{K \times Y} \to \mathfrak{U}'_{K \times Y}$ and $(\zeta, \xi) : \mathfrak{U}_{K \times Y} \to \mathbb{R}_{++} \times \mathbb{R}$ such that Ψ is a measurable bijection and (ζ, ξ) are integrable, and that satisfy:

(a) for all $\mathfrak{u}'$ in the image of Ψ, $\mathfrak{u}'(p) = \zeta(\mathfrak{u})(\Psi\mathfrak{u})(p) + \xi(\mathfrak{u})$ for all $p \in \mathscr{P}(K \times Y)$, and

(b) the bijection Ψ is measure preserving, ie, for all measurable $D' \subset \mathfrak{U}'_{K \times Y}$, $\mu'(D') = \mu\big(\Psi^{-1} D'\big)$, and for all measurable $D \subset \mathfrak{U}_{K \times Y}$, $\mu(D) = \mu'(\Psi D)$.

If the finitely additive EU representation $(\mathfrak{U}_{K \times Y}, \mu)$ is jointly identified, then the condition is also necessary.

The proof of the sufficiency part of the lemma merely amounts to a change of variable, and is an instance of the nonuniqueness encountered in DLR. The necessary part of the lemma is also not difficult, and a statement and proof (albeit, in a slightly different setting) can be found in Schenone [2010]. It is worthwhile to note that in DLR, the additive EU representation is jointly identified, while our abstract representation theorem, Theorem 5, allows for different finitely additive EU representations of $\succsim$, given the canonical state space.

(35) This is the sense in which the representation in DLR is identified.

Separability (Axiom 5) says that if $p, q \in \mathscr{P}(K \times Y)$ are such that their marginals are identical, ie, $p_k = q_k$ and $p_y = q_y$, then $\{p, q\} \sim \{p\}$. This gives us the following lemma.

Lemma 25. Let $(\mathfrak{U}_{K\times Y}, \mu)$ be a finitely additive representation and satisfy Separability (Axiom 5). For p and q that induce the same marginals, ie, $p_k = q_k$ and $p_y = q_y$, $\mu\{\mathrm{u} \in \mathfrak{U}_{K\times Y} : \mathrm{u}(p) > \mathrm{u}(q)\} = 0$.

Proof. If the lemma were not true, we would have $V(\{p, q\}) - V(\{q\}) > 0$, which contradicts Separability (Axiom 5). □

Definition 26. Let $(\mathfrak{U}_{K\times Y}, \mu)$ be a finitely additive EU representation. The representation is ***finitely additive, separable*** if for each u in the carrier of μ, there exist $u(\mathrm{u}) \in \mathscr{U}$ and $v(\mathrm{u}) \in C(Y)$ such that $\mathrm{u}(p) := u(p_k; \mathrm{u}) + v(p_y; \mathrm{u})$ for each $p \in \mathscr{P}(K \times Y)$, and if the mapping $\mathrm{u} \mapsto (u, v)$ is measurable. A finitely additive, separable representation $(\mathfrak{U}_{K\times Y}, \mu)$ is ***additive, separable*** if μ is a countably additive probability measure on the Borel sigma-algebra of $\mathfrak{U}_{K\times Y}$. For ease of notation, we shall suppress the dependence of u and v on u.

Lemma 27. A finitely additive EU representation satisfies Separability if, and only if, it is also a finitely additive separable representation as in

$$V(G) = \int_{\mathfrak{U}_{K\times Y}} \max_{p\in G} \left[u(p_k) + v(p_y)\right] \mathrm{d}\mu(\mathrm{u}) \tag{D.1}$$

Proof. The 'if' part is clear. We now prove the 'only if' part. Fix $\tilde{k} \in K$ and $\tilde{y} \in Y$. Since $\frac{1}{2}(k, y) + \frac{1}{2}(\tilde{k}, \tilde{y})$ and $\frac{1}{2}(k, \tilde{y}) + \frac{1}{2}(\tilde{k}, y)$ have the same marginals, lemma 25 implies that $\mathrm{u}\big(\frac{1}{2}(k, y) + \frac{1}{2}(\tilde{k}, \tilde{y})\big) = \mathrm{u}\big(\frac{1}{2}(k, \tilde{y}) + \frac{1}{2}(\tilde{k}, y)\big)$ for all u in the carrier of μ. Since each u is linear in probabilities, $\mathrm{u}\big((k, y)\big) = \mathrm{u}\big((k, \tilde{y})\big) + \mathrm{u}\big((\tilde{k}, y)\big) - \mathrm{u}\big((\tilde{k}, \tilde{y})\big)$ must be satisfied for each individual state u in the carrier of μ.[36] Following GP, we define $u(k) := \mathrm{u}\big((k, \tilde{y})\big)$ and $v(y) := \mathrm{u}\big((\tilde{k}, y)\big) - \mathrm{u}\big((\tilde{k}, \tilde{y})\big)$ to find $\mathrm{u}\big((k, \tilde{y})\big) = u(k) + v(y)$. This allows us to write $V(G) = \int_{\mathfrak{U}_{K\times Y}} \max_{p\in G} \left[u(p_k) + v(p_y)\right] \mathrm{d}\mu(\mathrm{u})$ which is the desired separable representation. □

D.1. Reduction of the State Space

Recall that $\mathscr{U} := \left\{u \in \mathbb{R}^K : \sum_i u_i = 0\right\}$ be the collection of all vN-M utility functions on K modulo additive constants. It is useful to define the space of all twice-normalized, vN-M utility functions on K as $\mathfrak{U}_K := \left\{r \in \mathscr{U} : \|r\|_2 = 1\right\}$. Since $\mathfrak{U}_K \subset \mathscr{U}$, we must have $\sum_i r_i = 0$, so that $r(p_k^*) = 0$ for all $r \in \mathfrak{U}_K$. We now use lemma 24 to show that for any finitely additive separable representation, the state space can be viewed as (a subset of) $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$.

Proposition 28. A preference $\succsim$ over the compact subsets of $\mathscr{P}(K \times Y)$ with a finitely additive representation $(\mathfrak{U}_{K\times Y}, \mu)$ satisfies Separability (Axiom 5) if, and only if, there is a change of state space as in lemma 24 that allows to write a finitely additive representation of $\succsim$ of the form

$$\int_{\mathfrak{U}_K\times[0,1]\times\mathfrak{U}_Y} \max_{p\in G} \left[\gamma r(p_k) + (1-\gamma) v(p_y)\right] \mathrm{d}\mu'(r, \gamma, v)$$

where μ' is a charge on $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$.

(36) In comparison, the weaker separability axiom in GP would only imply this condition for the preference functional V that aggregates over all states.

Proof. The 'if' part is immediate, so we only prove the 'only if' part of the proposition. Let $(\mathfrak{U}_{K\times Y}, \mu)$ be a finitely additive representation. (Recall that Theorem 5 guarantees the existence of a finitely additive representation, though we have not ruled out the possibility that there could be many such representations.) By lemma 27, every such representation must have the property that for every $\mathfrak{u}$ in the carrier of μ, $\mathfrak{u}(p) = u(p_k) + v(p_y)$, where $u \in \mathscr{U}$ and $v \in C(Y)$ are vN-M functions and the mapping $\mathfrak{u} \mapsto (u, v)$ is measurable.

Every separable utility function $u(p_k) + v(p_y)$ is of the form $\alpha r(p_k) + \beta w(p_y)$, where $r \in \mathfrak{U}_K$, $w \in \mathfrak{U}_Y$, and $(\alpha, \beta) \in \mathbb{R}^2_+ \setminus (0, 0)$. Let $X := (\mathfrak{U}_K \times \mathbb{R}_+) \times (\mathfrak{U}_Y \times \mathbb{R}_+)$, and consider $\mathfrak{U}_{K\times Y} \cap X$. A finitely additive separable representation is one where $\mu(\mathfrak{U}_{K\times Y} \cap X) = 1$.

An even smaller state space is $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$, wherein the utility in state (r, γ, v) is $\gamma r(p_k) + (1 - \gamma)w(p_y)$.

Define $\Psi : \mathfrak{U}_{K\times Y} \cap X \to \mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$ as follows: $\Psi : \big((r, \alpha), (w, \beta)\big) \mapsto \big(r, \alpha/(\alpha + \beta), w\big)$. It is clear that Ψ is continuous, and hence measurable. Moreover, Ψ is also a bijection. To see this, suppose there are $\alpha, \alpha', \beta, \beta'$ such that $\alpha/(\alpha + \beta) = \alpha'/(\alpha' + \beta')$. This implies $\alpha' = \rho\alpha$ and $\beta' = \rho\beta$ for some $\rho > 0$, which is impossible since, as mentioned above, $\mathfrak{U}_{K\times Y} \cap X$ only contains functions that are unique up to scaling.

By lemma 24, we may define the charge $\mu'(D) := \mu(\Psi^{-1} D)$ on $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$, and restrict attention to a separable representation on this state space. □

In sum, a ***finitely additive, separable, EU representation*** of $\succsim$ is given by a charge on $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$, with a typical vN-M utility function of the form $\gamma r(p_k) + (1 - \gamma)w(p_y)$.[37] Notice that since Ψ is continuous, the charge μ' is normal (ie, both inner and outer regular).

D.2. Additively Separable Representation

We now show that in the presence of some additional assumptions, the normal probability charge μ' in proposition 28 can be replaced by a regular probability measure, leading to an ***additive, separable EU representation***. Recall that μ' is a *regular probability measure* if it is (i) regular (ie, outer regular and tight), and (ii) a probability measure, ie, is countably additive, and has $\mu'(\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y) = 1$, and is defined on the Borel sigma-algebra of $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$. For any $D \subset \mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y$, denote its *indicator function* by $\mathbb{I}(D)$.

Definition 29. Let $(\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Y, \mu')$ be a finitely additive separable EU representation. The representation is ***Y-simple*** if the marginal charge of μ' on $\mathfrak{U}_Y$ — given by $\int_{\mathfrak{U}_K \times [0,1] \times D} \mathrm{d}\mu(r, \gamma, w)$ for any $D \in \mathscr{A}_{\mathfrak{U}_Y}$ — has a finite carrier. It is ***Y-trivial*** if the carrier of the marginal on $\mathfrak{U}_Y$ is a singleton.

Lemma 30. Every finitely additive, separable representation that is Y-simple can be extended uniquely to an additive, separable, Y-simple representation

$$V(G) = \int_{\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}} \max_{p \in G} \big[\gamma r(p_k) + (1 - \gamma)w(p_y)\big] \ \mathrm{d}\mu(r, \gamma, w) \tag{D.2}$$

(37) In our terms, the representation of HHT is a finitely additive, separable, EU representation, where there is no uncertainty about $r \in \mathfrak{U}_K$ or $v \in \mathfrak{U}_Y$, so that all the uncertainty is about the *stochastic discount factor* $\gamma \in [0, 1]$.

Proof. As μ' is Y-simple, there must exist an $n \in \mathbb{N}$ with $n > 0$ such that the carrier of μ' is a closed subset of $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$, where $\{w_1, \ldots, w_n\} \subset \mathfrak{U}_Y$. Since $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$ is compact, it follows immediately that the carrier of μ' is also compact.

The charge μ' is normal (see footnote 31), so for any Borel (algebra) measurable $A \subset \mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$, the inner regularity of μ' implies that $\mu'(A) = \sup\{\mu(C) : C \subset A \text{ and closed}\}$. Since $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$ is compact, it follows that any closed $C \subset A$ is also compact. Therefore, the charge μ' is 'tight' relative to a compact class of sets, namely the collection of all compact subsets of $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$. By Theorem 9.12 of Aliprantis and Border [1999], μ' is also countably additive (on the algebra of open sets).

By the Carathéodory Extension Procedure Theorem, μ' can be uniquely extended from the algebra of open sets to the Borel sigma-algebra of $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$. (The Carathéodory Theorem is Theorem 9.22 of Aliprantis and Border [1999]. The extension of the measure is unique since μ' is a finite measure.) The unique extension of μ' will be written as μ.

Finally, Theorem 10.7 of Aliprantis and Border [1999] says that a measure on a Polish space is regular, and because $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$ is a compact subset of Euclidean space, it follows that μ is regular. □

D.3. Identification of the Representation

Recall that in the original, abstract EU representation theorem, Theorem 5, we are unable to establish that μ is a regular measure or that the representation is jointly identified. We have seen that in the presence of additional assumptions, it is possible to show that μ is a regular measure. We now show that in that case, the representation can be jointly identified.

Proposition 31. Suppose a continuous preference $\succsim$ has an additive, separable, Y-simple representation as in equation (D.2). Then, the representation is jointly identified, ie, the measure μ is unique given the state space $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}$.

Proof. Let $\succsim$ have a utility representation V as in (D.2), and let $\mu_0 = \mu$. Suppose there is another regular measure μ_1 (that need not be a probability measure) on $\mathfrak{U}_K \times [0,1] \times \{w_{n+1}, \ldots, w_{n+m}\}$ such that

$$V(G) = \int_{\mathfrak{U}_K \times [0,1] \times \{w_{n+1}, \ldots, w_{n+m}\}} \max_{p \in G} \left[\gamma r(p_k) + (1-\gamma) w(p_y)\right] \, \mathrm{d}\mu_1(r, \gamma, w)$$

It is without loss of generality to consider μ_0 and μ_1 as measures on $\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_{n+m}\}$. We show that $\mu_1 = \mu_0$.

Let $Y_0 \subset Y$ be a finite set such that (i) w_i is nonconstant on Y_0 for all $i = 1, \ldots, n$, and (ii) for each $i, j \in 1, \ldots, n$ where $i \neq j$, there exists $y_{ij} \in Y_0$ such that $w_i(y_{ij}) \neq w_j(y_{ij})$. Since each w_i is nonconstant on Y, the first requirement is easily satisfied. The second requirement is also satisfied, since w_i is a continuous function on Y and $i \neq j$ implies $w_i \neq w_j$, which in turn implies that the two functions must disagree somewhere.

Now consider the set $B := K \times Y_0$, and the domain $\mathscr{F}\big(\mathscr{P}(B)\big)$. Then, each measure μ_j, $j = 0, 1$, induces the preference functional W_j on $\mathscr{F}\big(\mathscr{P}(B)\big)$ as follows:

$$W_j(G) = \int_{\mathfrak{U}_K \times [0,1] \times \{w_1, \ldots, w_n\}} \max_{p \in G} \left[\gamma r(p_k) + (1-\gamma) w(p_y)\right] \, \mathrm{d}\mu_j(r, \gamma, w)$$

Define, as in DLR, $\mathfrak{U}_B := \{r \in \mathbb{R}^B : \sum_i r_i = 0, \sum r_i^2 = 1\}$. In that case, $\mathfrak{U}_K \times [0,1] \times \{w_1, \dots, w_n\}$ is isomorphic to a subset of $\mathfrak{U}_B$. Indeed, for any vN-M utility function on $K \times Y_0$ of the form $\alpha r(q_k) + (1-\alpha) w(q_y)$, the two normalizations that map the function into $\mathfrak{U}_B$ are continuous.

Thus, μ_0 and μ_1 are transformed into measures on $\mathfrak{U}_B$ in the obvious way, and have supports on sets (in $\mathfrak{U}_B$) that are isomorphic to $\mathfrak{U}_K \times [0,1] \times \{w_1, \dots, w_n\}$. From the definition of the functionals W_0 and W_1, we know that for each menu $G \in \mathscr{F}(\mathscr{P}(B))$, we have $W_0(G) = W_1(G)$. The uniqueness part of the additive EU representation theorem in DLR now says that the two transformed measures agree on $\mathfrak{U}_B$, and hence $\mu_0 = \mu_1$, as desired. □

It is useful to write Y-trivial, additive, separable EU representations of a $\succsim$ by the collection $(\mathscr{U}, v, \mu)$ where $v : Y \to \mathbb{R}$ is a vN-M function, and μ is a probability measure on (the Borel sigma-algebra of) $\mathscr{U}$. It induces the preference functional

$$V(G) = \int_{\mathscr{U}} \max_{p \in G} \left[u(p_k) + v(p_y) \right] \, \mathrm{d}\mu(u) \tag{D.3}$$

that represents the preference $\succsim$ over menus. The transformation from the state space with $(r, \alpha) \in \mathfrak{U}_K \times [0,1]$ to the state space with $u \in \mathscr{U}$ is achieved by setting $u := \frac{\alpha}{(1-\alpha)} r$ and by applying the appropriate transformation to the measure, as in lemma 24.

E. Proofs from Section 5.2

We begin with an alternate characterisation of nice probability measures. We then proceed to prove Theorem 3.

Lemma 32. A nice probability measure on $\mathscr{U}$ satisfies $\int_{\mathscr{U}} \|u\|_2 \, \mathrm{d}\mu(u) < \infty$.

Proof. Let $W : \mathscr{F}_K \to \mathbb{R}$ be defined as $W(a) := \int_{\mathscr{U}} \max_{\alpha \in a} u(\alpha) \, \mathrm{d}\mu(u)$ for $a \in \mathscr{F}_K$. Recall that p_k^* is the uniform lottery over K. Let $a := \{\alpha \in \mathscr{P}(K) : \|\alpha - p_k^*\|_2 \leqslant \varepsilon$ for some $\varepsilon > 0$ so that $a \in \mathscr{F}_K$. Then, for each $u \in \mathscr{U}$, $\max_{\alpha \in a} u(\alpha) = \varepsilon \|u\|_2$. Therefore, $0 \leqslant W(a) = \varepsilon \int_{\mathscr{U}} \|u\|_2 \, \mathrm{d}\mu(u)$. But $W((\mathscr{P}(K)) = W(K) \leqslant \sum_k \mu \, |u_k| < \infty$ because μ is nice. As $a \subset \mathscr{P}(K)$, it follows that $0 \leqslant W(a) < \infty$. □

E.1. Proof of Theorem 3

In section E.1.1, we establish that for each $s \in S$, the preference $\succsim_s$ has a H-trivial, separable, additive representation as in (D.3). This amounts to showing that the relevant state space is isomorphic to $\mathscr{U}$. In section E.1.2, we show that the parameter of the representation of $\succsim_s$, namely the measure μ_s on $\mathscr{U}$, is uniquely identified up to a scaling. In section E.1.3, we show that $\succsim$ has a recursive representation, possibly with state-dependent discount factors. In section E.1.4, we show that there exists an equivalent representation of $\succsim$ that with a constant discount factor. We also show that the collection of measures (μ_s) is unique up to a common scaling, and that the Markov chain on S with transition probabilities Π has the unique invariant distribution π_0.

E.1.1. *Separable Representation*

It is easy to see that $\succsim_s$ is continuous, satisfies Independence, Monotonicity, and Separability, so that by proposition 28, $\succsim_s$ has a finitely additive separable representation. We shall now show that the representation is also H-trivial.

Proposition 33. Suppose $\succsim_s$ has a finitely additive separable representation. If $\succsim_s$ also satisfies State Contingent Strategic Rationality (Axiom X9), then the representation is H-trivial.

Proof. As in Proposition 28, we know that a separable, finitely additive separable representation has the form

$$U_s(x) = \int_{\mathfrak{U}_K \times [0,1] \times \mathfrak{U}_H} \max_{p \in x} \left[\gamma r(p_k) + (1-\gamma) v(p_h) \right] \, \mathrm{d}\mu_s(r, \gamma, v)$$

where $(r, \gamma, v) \in \mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_H$.

For an arbitrary consumption prize $k \in K$, and let (k, A) be a rectangular menu (see definition 4). Define a utility function $W_s : \mathscr{F}_H \to \mathbb{R}$ as $W_s(A) = U_s(k, A)$. It follows from the separability of the representation that the choice of $k \in K$ only affects W_s up to a constant. Let $\mathrm{d}\mu_s^*(v) = \iint_{\mathfrak{U}_K \times [0,1]} \mathrm{d}\mu_s(r, \gamma, v)$ be the marginal of μ on $\mathfrak{U}_H$. Then, $W_s(A) = \int_{\mathfrak{U}_H} \max_{p_h \in A} v(p_h) \, \mathrm{d}\mu_s^*(v) + \text{constant}$.

By proposition 23 above, property (a) of State Contingent CSR (Axiom X9) implies that $W_s(A) = \max_{p_h \in A} v_s(p_h)$, where $v_s \in C(H)$ is given by $v_s(f) = W_s(\{f\})$. But this implies that $\max_{p_h \in A} v_s(p_h) = \int \max_{p_h \in A} v(p_h) \, \mathrm{d}\mu^*(v)$. Therefore, the carrier of μ^* must be a singleton. □

It follows from lemma 30 that μ_s can be extended to a measure in a unique way. Thus, as mentioned at the end of section D.3, $\succsim_s$ has a separable, additive, H-trivial EU representation that, after a change of the state space, can be written as

$$U_s(x) = \int_{\mathscr{U}} \max_{p \in x} [u(p_k) + v_s(p_h)] \, \mathrm{d}\mu_s(u) \tag{E.1}$$

where, abusing notation, the transformed measure on $\mathscr{U}$ is also denoted by μ_s. We end with the observation that μ_s is nice. To see this, note that there exists $p_h^* \in \mathscr{P}(H)$ such that $v_s(p_h^*) = 0$. Now consider the menu (k, p_h^*). It is easy to see that $U_s\big((k, p_h^*)\big) = \mu_s u(k) := \int_{\mathscr{U}} u(k) \, \mathrm{d}\mu_s(u)$. But $U_s\big((k, p_h^*)\big)$ is finite, which implies $\mu_s u(k)$ is finite for every $k \in K$, which proves that μ_s is nice.

The preference $\succsim$ on H is continuous and satisfies Independence (Axiom X4). Therefore, there exists a representation $W(f) := \sum_s \pi_0(s) U_s\big(f(s)\big)$ of $\succsim$, where $\pi_0 \in \mathscr{P}(S)$.

E.1.2. *Identification*

Section E.1.1 shows that the state space relevant for a finitely additive, separable, H-trivial representation is $\mathscr{U}$. Our goal in this section is to show that the measure μ_s on $\mathscr{U}$ and v_s are unique up to a common scaling.

Proposition 34. If there are two separable, H-trivial, additive EU representations, (v_s, μ_s) and (v_s', μ_s'), of $\succsim_s$, then there exists $\zeta > 0$ such that the following properties hold.

(a) $\mu_s(\zeta D) = \mu'_s(D)$ for all measurable $D \subset \mathcal{U}$,

(b) $v'_s = \zeta v_s + \text{constant}$.

Proof. By lemma 24, our result on the change of state space, we know that there exists a measurable bijection $\Psi : \mathcal{U} \to \mathcal{U}$, and integrable functions $(\zeta, \xi) : \mathcal{U} \mapsto (\mathbb{R}_{++}, \mathbb{R})$ such that for each u' in the support of μ'_s, we have $u'(p_k) + v'_s(p_h) = \zeta(u)\big[(\Psi u)(p_k) + v_s(p_h)\big] + \xi(u)$.

Consider two lotteries p, q such that $p_k = q_k$ but where $v_s(p_h) \neq v_s(q_h)$. Then,

$$\begin{aligned}\big[u'(p_k) + v'_s(p_h)\big] - \big[u'(q_k) + v'_s(q_h)\big] &= v'_s(p_h) - v'_s(q_h)\\ &= \zeta(u)[v_s(p_h) - v_s(q_h)]\end{aligned}$$

But this implies $\zeta(u) = \dfrac{v'_s(p_h) - v'_s(q_h)}{v_s(p_h) - v_s(q_h)}$ for all u, and hence $\zeta(u)$ is constant, which proves (b).

Let V'_s be the functional induced by (v'_s, μ'_s). Then,

$$\begin{aligned}V'_s(x) &= \int_{\mathcal{U}} \max_{p \in x} \big[u'(p_k) + v'_s(p_h)\big] \,\mathrm{d}\mu'_s(u')\\ &= \int_{\mathcal{U}} \max_{p \in x} \zeta\big[(\Psi u)(p_k) + v_s(p_h)\big] \,\mathrm{d}\tilde{\mu}'_s(u) + \int \xi \mu'_s\\ &= \zeta \int_{\mathcal{U}} \max_{p \in x} \big[\tilde{u}(p_k) + v_s(p_h)\big] \,\mathrm{d}\mu''_s(\tilde{u}) + \text{constant}\end{aligned}$$

where $\tilde{\mu}'_s$ and μ''_s each obtain from a change of state space. Proposition 31 says that $\mu''_s = \mu_s$, which implies that Ψ is the identity mapping. Thus, it must be that $\mu'_s(D) = \mu_s(\zeta D)$ for all measurable $D \subset \mathcal{U}$. These observations prove part (a). □

E.1.3. *Recursive Representation*

Thus far, we have established that $\succsim$ has a separable representation of the form

$$W(f, \pi_0) = \sum_s \pi_0(s) \left[\int_{\mathcal{U}} \max_{p \in f(s)} \big[u(p_k) + v_s(p_h)\big] \,\mathrm{d}\mu_s(u)\right] \tag{E.2}$$

In the representation above, each $v_s : H \to \mathbb{R}$ is continuous (recall that H is a convex set).

Proposition 35. Suppose $\succsim$ has a representation as in (E.2) and satisfies Indifference to Timing (Axiom X8). Then, each $v_s : H \to \mathbb{R}$ is linear.

Proof. As above, we have $U_s : \mathcal{F} \to \mathbb{R}$ given by $U_s(x) = \int_{\mathcal{U}} \max_{p \in x} \big[u(p_k) + v_s(p_h)\big] \,\mathrm{d}\mu_s(u)$. Let $f, g \in H$ such that $v_s(f) > v_s(g)$. Then, $U_s\big(\{p^*_k, f\}\big) > U_s\big(\{p^*_k, g\}\big)$. Let $f' \in H$, and fix $\lambda \in (0, 1]$. By the linearity of U_s, we have $U_s\big(\lambda\{p^*_k, f\} + (1-\lambda)\{p^*_k, f'\}\big) > U_s\big(\lambda\{p^*_k, g\} + (1-\lambda)\{p^*_k, f'\}\big)$. But $\succsim_s$ satisfies Indifference to Timing (Axiom X8), which implies that $U_s\big(\lambda\{p^*_k, f\} + (1-\lambda)\{p^*_k, f'\}\big) = U_s\big(\{p^*_k, \lambda f + (1-\lambda) f'\}\big)$, so that $U_s\big(\{p^*_k, \lambda f + (1-\lambda) f'\}\big) > U_s\big(\{p^*_k, \lambda g + (1-\lambda) f'\}\big)$, which in turn implies that $v_s\big(\lambda f + (1-\lambda) f'\big) > v_s\big(\lambda g + (1-\lambda) f'\big)$. Notice that $v_s : H \to \mathbb{R}$ is continuous and hence induces a preference $\succsim^*_s$ on H that is continuous. We have just established that $\succsim^*_s$ also satisfies Independence (Axiom X4), from which it follows that v_s is linear, as desired. □

The following is a useful property of $\succsim$.

Axiom X10 (Uniform Persistence). $f_s^x \succsim f_s^y$ implies $\{(k, f_s^x)\} \succsim_{s'} \{(k, f_s^y)\}$ for all $s, s' \in S$.

Proposition 36. Suppose $\succsim$ has a representation as in (E.2). Suppose also that $\succsim$ satisfies State Contingent CSRb (Axiom X9), Persistent non-triviality (Axiom X1), Aggregate Stationarity (Axiom X7). Then, $\succsim$ satisfies Uniform Persistence (Axiom X10).

Proof. Fix $s \in S$ and let $x, y \in \mathcal{F}$ be such that $f_s^x \succsim f_s^y$. By Aggregate Stationarity (Axiom X7), it follows that $\{(k, f_s^x)\} \succsim \{(k, f_s^y)\}$.

Then, by State Contingent CSRb (Axiom X9), we have $\{(k, f_s^x), (k, f_s^y)\} \sim \{(k, f_s^x)\}$. Using the representation in (E.2), we see that

$$\sum_{s'} \pi_0(s') \max\left[v_{s'}(f_s^x), v_{s'}(f_s^y)\right] = \sum_{s'} \pi_0(s') v_{s'}(f_s^x)$$

which implies that $v_{s'}(f_s^x) \geqslant v_{s'}(f_s^y)$ for all $s' \in S$. Therefore, $\{(k, f_s^x)\} \succsim_{s'} \{(k, f_s^y)\}$, which establishes Uniform Persistence (Axiom X10). $\square$

In what follows, to simplify exposition, we will often say that $\succsim$ satisfies Uniform Persistence (Axiom X10) instead of stating the conditions under which this is true. Recall that each v_s induces a preference $\succsim_s^*$ on H.

Proposition 37. If $\succsim$ satisfies Uniform Persistence (Axiom X10), then each $\succsim_s^*$ also satisfies Monotonicity, ie, $x \cup y \succsim_s^* x$ for all $x, y \in \mathcal{F}$.

Proof. As v_s is linear and continuous, it follows that there exist utility functions $w_{s,s'} : \mathcal{F} \to \mathbb{R}$ for all $s' \in S$ such that $v_s(f) = \sum_{s'} w_{s,s'}\big(f(s')\big)$. Suppose contrary to the proposition there exist $x, y \in \mathcal{F}$ and $s' \in S$ such that $x \subset y$ but $w_{s,s'}(x) > w_{s,s'}(y)$. In that event, we have $v_s(f_{s'}^x) > v_s(f_{s'}^y)$, which implies $\{(k, f_{s'}^x)\} \succ_s \{(k, f_{s'}^y)\}$. By Uniform Persistence (Axiom X10), we must have $f_{s'}^x \succ f_{s'}^y$ or $x \succ_s^* y$, which contradicts the assumption that $\succsim$ satisfies Monotonicity (Axiom X5) (which is reflected by the fact that π_0 and μ_s, for each $s \in S$, are probability measures). Therefore, $\succsim_s^*$ must also satisfy Monotonicity (Axiom X5). $\square$

We now establish the existence of a *recursive* representation.

Proposition 38. Let $\succsim$ have a representation as in (E.2), and suppose $\succsim$ satisfies Persistent non-triviality (Axiom X1), State Contingent CSRb (Axiom X9), and Indifference to Timing (Axiom X8). Then, there is a value function

$$V(f, s') = \sum_{s} \Pi(s', s) \left[\int_{\mathcal{U}} \max_{p \in f(s)} \left[u(p_k) + \delta_s V(p_h, s)\right] \mathrm{d}\mu_s(u)\right] \tag{E.3}$$

where Π governs transition probabilities for a Markov process on S and $\Pi(s', s) > 0$ for all $s', s \in S$, such that

$$V(f, \pi_0) = \sum_{s} \pi_0(s) \left[\int_{\mathcal{U}} \max_{p \in f(s)} \left[u(p_k) + \delta_s V(p_h, s)\right] \mathrm{d}\mu_s(u)\right] \tag{E.4}$$

represents $\succsim$.

Proof. Fix $s' \in S$ and consider the act $f^x_{s'}$. Then,

$$W(f^x_{s'}, \pi_0) = \pi_0(s') \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + v_{s'}(p_h)\right] \mathrm{d}\mu_{s'}(u) + (\cdot)$$

By proposition 36, Uniform Persistence (Axiom X10) holds; for all $s \in S$, $\{(k, f^x_{s'})\} \succsim_s \{(k, f^y_{s'})\}$ if $f^x_{s'} \succsim f^y_{s'}$. Recall that $U_s(\cdot)$ in (E.1) represents $\succsim_s$, and $U_s\big(\{(k, f^x_{s'})\}\big) = \int_{\mathcal{U}} u(k)\ \mathrm{d}\mu_s(u) + v_s(f^x_{s'})$.

As in Propositions 35 and 37, we note that $v_s(\cdot)$ induces a preference $\succsim^*_s$ on H such that (i) $\succsim^*_s$ is continuous, (ii) $\succsim^*_s$ satisfies Independence, and (iii) by Uniform Persistence (Axiom X10), $\succsim^*_s$ also satisfies Monotonicity. Then,

$$v_s(f^x_{s'}) = \delta_s \left[\pi_s(s') \int_{\mathfrak{U}_{K\times H}} \max_{p\in x} \mathrm{u}(p)\ \mathrm{d}\tilde{\mu}^s_{s'}(\mathrm{u}) + \sum_{t \neq s'} \pi_s(t) \int_{\mathfrak{U}_{K\times H}} \max_{p\in f(t)} \mathrm{u}(p)\ \mathrm{d}\tilde{\mu}^s_t(\mathrm{u})\right]$$

where δ_s is a scaling factor chosen so that (i) $\pi_s \in \mathscr{P}(S)$ and (ii) $\tilde{\mu}^s_t$ is a probability charge for all $t \in S$. Such a choice can be made as follows: If $\tilde{\mu}^s_t(\mathfrak{U}_{K\times H}) > 1$, define $\pi'_s(t) := \tilde{\mu}^s_t(\mathfrak{U}_{K\times H})$, so that we can take $\tilde{\mu}^s_t$ to be a probability measure. Because $\pi'_s(t) > 0$ for all $t \in S$, it follows that we may let $\delta_s = \sum_t \pi'_s(t)$, so that the scaling factor $\pi'_s(t)$ can be replaced by $\pi_s(t) := \pi'_s(t)/\delta_s$.

By Indifference to Timing (Axiom X8), $\pi_0(s') \int_{\mathcal{U}} \max_{p\in x} \left[u(p_k) + v_{s'}(p_h)\right] \mathrm{d}\mu_{s'}(u)$ and $\delta_s \pi_s(s') \int_{\mathfrak{U}_{K\times H}} \max_{p\in x} \mathrm{u}(p)\ \mathrm{d}\tilde{\mu}^s_{s'}(\mathrm{u})$ are (positive) affine transformations of each other. Each $\pi_s(s') > 0$ by Persistent non-triviality (Axiom X1), and every state is also non-null under π_0, ie, $\pi_0(s') > 0$. Hence the measures $\{\pi_0, \pi_s : s \in S\}$ have full support.

Recall that $\mu_{s'}$ defined on $\mathcal{U}$ is the marginal of a measure on $\mathcal{U} \times \mathfrak{U}_H$, where the marginal on $\mathfrak{U}_H$ has a singleton carrier. The transformations of the state space used in obtaining a separable representation are an instance of those considered in lemma 24, so that we may regard $\mu_{s'}$ as a measure on $\mathfrak{U}_{K\times H}$.

Since both $\tilde{\mu}^s_{s'}$ and $\mu_{s'}$ are probability charges, we must have $\tilde{\mu}^s_{s'} = \mu_{s'}$ for all $s \in S$. Therefore, each v_s can be written as

$$v_s(f) = \delta_s \sum_{s'} \pi_s(s') \int_{\mathcal{U}} \max_{p\in f(s')} \left[u(p_k) + v_{s'}(p_h)\right] \mathrm{d}\mu_{s'}(u)$$

Define the Markov transition probabilities on S to be $\Pi(s, s') := \pi_s(s')$, and let $V(f, s) := v_s(f)/\delta_s$ so that

$$V(f, s) = \sum_{s'} \Pi(s, s') \int_{\mathcal{U}} \max_{p\in f(s')} \left[u(p_k) + \delta_{s'} V(p_h, s')\right] \mathrm{d}\mu_{s'}(u)$$

Finally, define

$$V(f, \pi_0) := W(f, \pi_0) = \sum_{s} \pi_0(s) \int_{\mathcal{U}} \max_{p\in f(s)} \left[u(p_k) + \delta_s V(p_h, \pi_s)\right] \mathrm{d}\mu_s(u)$$

to attain the desired recursive representation. □

E.1.4. *Representation with a Constant Discount Factor — Existence and Uniqueness*

Proposition 38 establishes the existence of a recursive value function as in (E.3), which can be described by the parameters $\big((\mu_s, \pi_s, \delta_s)_{s\in S}\big)$. We will now show that there exists an equivalent representation with a *constant* discount factor. Let $\xi \in \mathbb{R}^S_{++}$, and let $\langle \pi, \xi \rangle := \sum_s \pi(s)\xi(s))$ for any $\pi \in \mathcal{P}(S)$.

Proposition 39. Let $\succsim$ have two recursive representations $\big((\mu_s, \pi_s, \delta_s)_{s\in S}\big)$ and $\big((\hat{\mu}_s, \hat{\pi}_s, \hat{\delta}_s)_{s\in S}\big)$ as in (E.3). Then, there exists $\xi \in \mathbb{R}^S_{++}$ such that the parameters are related by the following transformations.

- $\hat{\pi}_0(s) := \pi_0(s)\xi(s)/\langle \pi_0, \xi \rangle$
- $\hat{\pi}_s(s') := \pi_s(s')\xi(s')/\langle \pi_s, \xi \rangle$
- $\hat{\mu}(D) := \mu\big(D/\xi(s)\big)$
- $\hat{\delta}_s := \delta_s \langle \pi_s, \xi \rangle / \xi(s)$
- $\hat{V}(\cdot, \hat{\pi}_s) := V(\cdot, \pi_s)/\langle \pi_s, \xi \rangle$

Moreover, given a representation $\big((\mu_s, \pi_s, \delta_s)_{s\in S}\big)$ of $\succsim$, a transformed set of parameters $\big((\hat{\mu}_s, \hat{\pi}_s, \hat{\delta}_s)_{s\in S}\big)$ also represents $\succsim$.

Proof. The last part of the proposition is immediate. By our identification result, proposition 34, there exists $\xi \in \mathbb{R}^S_{++}$ such that (i) $\hat{\mu}_s(D) = \mu_s\big(D/\xi(s)\big)$ for all $s \in S$ and for all Borel measurable $D \subset \mathcal{U}$, and (ii) $\hat{\delta}_s \hat{V}(\cdot, s) := \delta_s V(\cdot, s)/\xi(s)$. By construction, $\hat{V}(\cdot, s) = V(\cdot, s)/\langle \pi_s, \xi \rangle$, and hence we must have $\hat{\delta}_s := \delta_s \langle \pi_s, \xi \rangle / \xi(s)$. Finally, the identification from the Mixture Space Theorem implies that $\hat{\pi}_s(s') := \pi_s(s')\xi(s')/\langle \pi_s, \xi \rangle$. □

Proposition 40. There exists $\xi \in \mathbb{R}^S_{++}$ such that $\hat{\delta}_s$ is independent of $s \in S$. Moreover, ξ is unique up to scaling, so that $\hat{\delta}$ is unique, and the corresponding measures $(\hat{\mu}_s)_{s\in S}$ are unique up to scaling.

Proof. A representation with a constant discount factor will obtain immediately if we can establish that there exists a vector $\xi \gg \mathbf{0}$ and a number $\hat{\delta} > 0$ such that $\hat{\delta}\xi(s) = \delta_s \langle \pi_s, \xi \rangle$ for all $s \in S$.

For $S = \{1, \dots, n\}$, consider the stochastic matrix Π, whose row s is π_s. Define the diagonal matrix Δ as follows:

$$\Delta := \begin{bmatrix} \delta_1 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \delta_n \end{bmatrix}$$

In matrix notation, our problem amounts to finding a $\xi \gg \mathbf{0}$ and $\hat{\delta} > 0$ such that $\hat{\delta}\xi = \xi\Delta\Pi$. This amounts to showing that (i) ξ is a (left) eigenvector of the matrix $\Delta\Pi$, and (ii) $\hat{\delta}$ is the corresponding eigenvalue.

We shall say that a matrix is ***positive*** if each of its entries is strictly positive. By proposition 38 above, the matrix Π is positive, so the matrix $\Delta\Pi$ is also positive. Thus, by the Perron theorem below, such ξ and $\hat{\delta}$ exist, ξ is unique up to scaling, and $\hat{\delta}$ is unique.

By proposition 39, these are the only transformations that we need consider, which establishes that the measures $(\hat{\mu}_s)_{s\in S}$ are unique up to a common scaling. □

The Perron Theorem is standard and can be found, for instance, as Theorem 1 in chapter 16 of Lax [2007].[38] For completeness, we state the relevant part of the theorem.

Theorem 6 (Perron)**.** *Every positive matrix* A *has a* ***dominant eigenvalue*** *denoted by* $\hat{\delta}$ *which has the following properties:*

(a) $\hat{\delta} > 0$ *and the associated eigenvector* $\xi \gg \mathbf{0}$.

(b) $\hat{\delta}$ *is a simple eigenvalue, and hence has algebraic and geometric multiplicity one.*

(c) A *has no other eigenvector with nonnegative entries.*

Thus, we have the established the existence of the following representation

$$V(f,s') = \sum_s \Pi(s',s)\left[\int_{\mathcal{U}} \max_{p\in f(s)} \big[u(p) + \delta V(f,s)\big]\, \mathrm{d}\mu_s(u)\right] \tag{E.5}$$

and

$$V(f,\pi_0) = \sum_s \pi_0(s)\left[\int_{\mathcal{U}_K} \max_{p\in f(s)} \big[u(p) + \delta V(f,s)\big]\, \mathrm{d}\mu_s(u)\right] \tag{E.6}$$

where $V(\cdot,\pi_0)$ represents $\succsim$.

Proposition 41. Let $\succsim$ have a recursive representation as in (E.6). If $\succsim$ satisfies Aggregate Stationarity (Axiom X7), then π_0 is the unique stationary distribution of the Markov process with transition matrix Π.

Proof. Recall that $V\big(\{(k,f)\},\pi_0\big) = \sum_s \pi_0(s)\big[\int u(k) + \delta V(f,s)\big]\,\mathrm{d}\mu_s(u)$. Letting $\kappa := \sum_s \pi_0(s)\mu_s u(k)$, we see that

$$\begin{aligned} V\big(\{(k,f)\},\pi_0\big) &= \kappa + \delta \sum_s \pi_0(s)\left[\sum_{s'} \Pi(s,s')\left[\int_{\mathcal{U}} \max_{p\in f(s')} \big[u(p_k) + \delta V(p_h,s')\big]\,\mathrm{d}\mu_{s'}(u)\right]\right] \\ &= \kappa + \delta \sum_{s'}\Big[\sum_s \pi_0(s)\Pi(s,s')\Big]\left[\int_{\mathcal{U}} \max_{p\in f^x_s} \big[u(p_k) + \delta V(p_h,s')\big]\,\mathrm{d}\mu_{s'}(u)\right] \end{aligned}$$

By Aggregate Stationarity (Axiom X7), $V\big(\{(k,f)\},\pi_0\big)$ and $V(f,\pi_0)$ represent the same preference. Moreover, by Indifference to Timing (Axiom X8), they must be affine transformations of each other. Therefore, $\pi_0(s') = \sum_s \pi_0(s)\Pi(s,s')$ for all $s'\in S$. In other words, π_0 is a stationary distribution of Π. Since Π is positive, the stationary distribution is unique. □

(38) The Frobenius-Perron theorem generalises the Perron theorem by only requiring that the Markov process be irreducible, ie, for each $s,s'\in S$, there exists $n>0$ such that $\Pi^n(s,s')>0$. As discussed in footnote 27, irreducibility of the Markov process corresponds to a weaker version of Persistent non-triviality (Axiom X1).

Proposition 42. If $\succsim$ has a recursive representation of the form in (E.5) with constant δ, then $\delta \in (0, 1)$.

Proof. It follows immediately from the non-triviality of $\succsim$ and from Aggregate Stationarity (Axiom X7) that $\delta > 0$. We shall now show that $\delta < 1$.
Step 1: Stationarity. As π_0 is the unique stationary distribution of Π, we have for any $f \in H$, $V(f, \pi_0) = \sum_s \pi_0(s)V(f, s)$.
Step 2: Constructing Menus. Let $a \in \mathscr{F}_K$ be the closed ε-ball around the p_k^*, the uniform lottery over K. It is clear that $0 < \int_{\mathscr{U}} \max_{p \in a} u(p_k)\, \mathrm{d}\mu_s(u)$, and as each μ_s is nice, it follows from lemma 32 that $\int_{\mathscr{U}} \max_{p \in a} u(p_k)\, \mathrm{d}\mu_s(u) < \infty$.
Step 3: $\delta < 1$. By the recursive construction of H, it follows that there exists a unique act f^* that gives the menu a in each period and in every state, ie, $f^* \simeq (a, f^*)$. Letting $\eta := \sum_s \pi_0(s) \left[\int_{\mathscr{U}} \max_{p_k \in a} u(p_k)\, \mathrm{d}\mu_s(u)\right]$, we see that $V(f^*, \pi_0) = \eta + \delta V(f^*, \pi_0) = \eta \sum_{\tau \geqslant 0} \delta^\tau$. Since that $\eta > 0$ and because $V(f^*, \pi_0)$ is finite, we conclude that $\delta < 1$, as required. □

E.1.5. *Putting it all together*

Proof of Theorem 3. That the representation satisfies all the axioms is straightforward. Consider a preference $\succsim$ that satisfies Axioms X1–X9. By Theorem 5, $\succsim$ has a finitely additive, EU representation. Separability (Axiom X6) implies, according to lemma 27 and proposition 28, that any such finitely additive EU representation also has a representation based on a regular countably additive measure.

Proposition 33 says that since $\succsim$ satisfies State Contingent Strategic Rationality (Axiom X9), the marginal of μ_s on $\mathfrak{U}_H$ is a singleton, and hence by proposition 31, it follows that each μ_s is identified up to a scaling. Propositions 35, 37, and 38 show that Indifference to Timing (Axiom X8), Uniform Persistence (Axiom X10), and Aggregate Stationarity (Axiom X7) imply the existence a recursive representation. Proposition 40 shows that there exists an equivalent recursive representation with a constant discount factor, and transition probabilities Π on S, and this representation is unique in the sense of Theorem 3. Proposition 41 shows that because of Aggregate Stationarity (Axiom X7), it must be that π_0 is the unique stationary or invariant distribution of Π. Proposition 42 shows that $\delta \in (0, 1)$ because all utilities from feasible menus are finite. This proves the theorem. □

E.2. Proof of Proposition 12

Let $W \in C(H \times S)$ and consider the function $\Phi W(f, s)$, given by

$$\Phi W(f, s) := \sum_{s' \in S} \Pi(s, s') \left[\int_{\mathscr{U}} \max_{p \in f(s')} \left[u(p_k) + \delta W(p_h, s')\right] \mathrm{d}\mu_{s'}(u)\right]$$

for all $s \in S$. It is easy to see that Φ is monotone, ie, $W \leqslant W'$ implies $\Phi W \leqslant \Phi W'$, and satisfies discounting, ie, $\Phi(W + \rho) \leqslant \Phi W + \delta\rho$ when $\rho \geqslant 0$. If we assume that $\Phi W \in C(H \times S)$ for all $W \in C(H \times S)$, it follows that Φ is a contraction mapping (with modulus δ), and has a unique fixed point, which establishes the proposition. All that remains is to show that Φ is an operator on $C(H \times S)$.

For each $x \in \mathcal{F}$, $u \in \mathcal{U}$, $W \in C(H \times S)$, and $s \in S$, define

$$\varphi(x, u, s) = \max_{p \in x} \left[u(p_k) + \delta W(p_h, s) \right]$$

Then,

$$\begin{aligned} |\varphi(x, u, s)| &\leqslant \max_{p \in x} |u(p_k) + \delta W(p_h, s)| \\ &\leqslant \|u\|_2 \max_{p \in x} \left| \frac{u(p_k)}{\|u\|_2} \right| + \max_{p \in x} \delta \, |W(p_h, s)| \\ &\leqslant \|u\|_2 \, M_1 + M_{2,s} \end{aligned}$$

where $M_1 := \max_{x \in \mathcal{F}} \max_{p \in x} \left| \frac{u(p_k)}{\|u\|_2} \right|$, $M_{2,s} > 0$, and the bounds follow from the definition of $u \in \mathcal{U}$, the compactness of H, and the continuity of W.

As W is continuous, the function $u(p_k) + \delta W(p_h, s) \in C(K \times H)$ is a continuous, linear functional on $\mathcal{P}(K \times H)$, when the latter is endowed with the topology of weak convergence (which is metrisable). Therefore, by the Maximum Theorem, for each $u \in \mathcal{U}$ and $s \in S$, $\varphi(x, u, s)$ is continuous in x.

We will now show that if $(f_n) \in H^\infty$ is a sequence that converges to $f \in H$, then $\Phi(W)(f_n, s') \to \Phi(W)(f, s')$ whenever $W \in C(H \times S)$, which establishes that $\Phi W \in C(H \times S)$. (Since S is finite, any convergent sequence in S must eventually be constant, which we take to be s'.)

Consider any sequence (f_n) that converges to f. By the bounds established above, $|\varphi(f_n(s), u, s)| \leqslant \|u\|_2 \, M_1 + M_{2,s}$, and $\|u\|_2 \, M_1 + M_{2,s}$ is μ_s-integrable since μ_s is nice (see lemma 32). Moreover,

$$\begin{aligned} \lim_{n \to \infty} \Phi W(f_n, s') &= \lim_{n \to \infty} \sum_s \Pi(s', s) \left[\int_{\mathcal{U}} \varphi(f_n(s), u, s) \, \mathrm{d}\mu_s(u) \right] \\ &= \sum_s \Pi(s', s) \left[\int_{\mathcal{U}} \lim_{n \to \infty} \varphi(f_n(s), u, s) \, \mathrm{d}\mu_s(u) \right] \\ &= \sum_s \Pi(s', s) \left[\int_{\mathcal{U}} \varphi(f(s), u, s) \, \mathrm{d}\mu_s(u) \right] \\ &= \Phi W(f, s') \end{aligned}$$

As f and (f_n) are arbitrary, we conclude that $\Phi W \in C(H \times S)$ whenever $W \in C(H \times S)$. The equalities above rely on the Dominated Convergence Theorem to interchange the order of limits and integration, and the continuity of $\varphi(\cdot, u, s)$ for each u and s to establish the pointwise limit. This completes the proof.

F. Proofs from Section 4

Proof of lemma 8. By definition of the Markov process $(\mathcal{U}_M, M)$, there exist $u_1, u_2, \ldots, u_n \in \mathcal{U}$ such that $\mathcal{U}_M := \text{cone}\left(\{u_1, \ldots, u_n\}\right)$. Denoting $[u_i] = \{\lambda u_i : \lambda > 0\}$, we see an induced

Markov chain on the rankings, with state space $\{[u_i] : i = 1, \ldots, n\}$, and with transition probabilities $M\big([u_i], [u_j]\big)$, where we have abused notation, because $M(u_i, \cdot) = M(\lambda u_i, \cdot)$ for all $\lambda > 0$. This is a Markov chain on a finite state space, and because all transition probabilities are strictly positive, there is a unique invariant distribution $\nu_0\big([u_i]\big)$.

Let ν be any measure on $\mathcal{U}_M$ such that $\nu\big([u_i]\big) = \nu_0\big([u_i]\big)$ for $i = 1, \ldots, n$. Define the measure μ_0 on $\mathcal{U}_M$ as follows: for any measurable $D \subset \mathcal{U}_M$,

$$\begin{aligned}\mu_0(D) &:= \sum_{i=1}^{n} \Big[\int_{[u_i]} M(u, D)\, \nu(\mathrm{d}u) \Big] \\ &= \sum_i M(u_i, D) \Big[\int_{[u_i]} \nu(\mathrm{d}u) \Big] \\ &= \sum_i M(u_i, D) \nu\big([u_i]\big)\end{aligned}$$

It is easy to see that $\mu_0\big([u_i]\big) = \nu_0\big([u_i]\big)$ for $i = 1, \ldots, n$. Moreover, $\mu_0(D)$ is independent of the choice of the measure ν, as long as ν satisfies $\nu\big([u_i]\big) = \nu_0\big([u_i]\big)$ for $i = 1, \ldots, n$. Therefore, μ_0 is the unique invariant measure of the Markov process $(\mathcal{U}_M, M)$. □

F.1. Separable Representation: Existence and Identification

The relevant domain here is $Z \simeq \mathcal{F}\big(\mathcal{P}(K \times Z)\big)$. Lemma 27 establishes that if $\succsim$ has a finitely additive EU representation (Theorem 5) and satisfies Separability (Axiom 5), then it has a separable representation of the form

$$W(x) = \int_{\{\mathrm{u} \in \mathfrak{U}_{K \times Z} : \mathrm{u}(p) = u(p_k) + v(p_z)\}} \max_{p \in x} \big[u(p_k) + v(p_z) \big]\, \mathrm{d}\mu(\mathrm{u}) \tag{F.1}$$

Lemma 43. Let $\succsim$ have a separable representation as in (F.1). If $\succsim_K$ is as in definition 5, then $\succsim_K$ is independent of the choice of $A \in \mathcal{F}_Z$.

Proof. Let W be as in (F.1). Then, for any $a, b \in \mathcal{F}_K$, and $A, B \in \mathcal{F}_Z$, we have

$$\begin{aligned}W((a, A)) &= \int_{\{\mathrm{u} \in \mathfrak{U}_{K \times Z} : \mathrm{u}(p) = u(p_k) + v(p_z)\}} \max_{p_k \in a; p_z \in A} \big[u(p_k) + v(p_z) \big]\, \mathrm{d}\mu(\mathrm{u}) \\ &=: \varphi_K(a) + \varphi_Z(A)\end{aligned}$$

where $\varphi_K : \mathcal{F}_K \to \mathbb{R}$ and $\varphi_Z : \mathcal{F}_Z \to \mathbb{R}$. By construction, φ_K represents $\succsim_K$, which is independent of the choice of A, which completes the proof. □

By a variation of proposition 28, we can transform the state space in (F.1) to be $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Z$. In what follows, let ν be the marginal of μ on $\mathfrak{U}_K$.

Lemma 44. Let $\succsim$ have a separable representation as in (F.1). If $\succsim_K$ satisfies Finiteness (Axiom 10), then the support of the marginal of μ on $\mathfrak{U}_K$ is finite.[39]

(39) Riella [2011] offers a different proof of this claim.

Proof. Let φ_K represent $\succsim_K$. By lemma 43, it follows that φ_K is well defined. As φ_K is continuous, linear, and monotone, it has an additive EU representation of the form $\varphi_K(a) = \int_{\mathfrak{U}_K} \max_{p_k \in a} u(p_k)\, \mathrm{d}\nu(u)$ where ν is countably additive and is the marginal of μ on $\mathfrak{U}_K$.

To see that ν must have finite support, suppose not. Let a be the ball of radius ε around p_k^*, the uniform lottery over K. It is easy to see that because ν has infinite support, there can be no finite $b \subset a$ such that $\varphi_K(b) = \varphi_K(a)$, which contradicts Finiteness (Axiom 10). □

Lemma 45. Suppose $\succsim$ has a separable representation as in (F.1), and also satisfies Finiteness (Axiom 10). Fix $r^* \in \mathfrak{U}_K$ such that $\nu(r^*) > 0$. Let $a, b \in \mathscr{F}_K$ be such that $\max_{\alpha \in a} r^*(\alpha) > \max_{\alpha \in b} r^*(\alpha)$ and $\max_{\alpha \in a} r(\alpha) < \max_{\alpha \in b} r(\alpha)$ for all $r \neq r^*$. For such $a, b \in \mathscr{F}_K$, we may take $c = b$ in Choice Contingent CSR (Axiom 11).

Proof. By hypothesis, $\succsim$ has a representation of the form

$$W(x) = \int_{\mathfrak{U}_K \times [0,1] \times \mathfrak{U}_Z} \max_{p \in x} \left[\gamma r(p_k) + (1-\gamma) v(p_z, r)\right] \mathrm{d}\mu(r, \gamma, v)$$

where the marginal of μ on $\mathfrak{U}_K$ has finite support. The representation implies that if $a \cup b \cup c \succ_K b \cup c$, then a dominates $b \cup c$ in state r^*, while c possibly dominates b in relevant states $r \neq r^*$. However, in those states, b already dominates a, and thus

$$(a, A) \overset{\circ}{\succsim}_{(b \cup c, A \cup B)} (a, B) \quad \text{implies} \quad (a, A) \overset{\circ}{\sim}_{(b \cup c, A \cup B)} (a, A \cup B)$$

if, and only if,

$$(a, A) \overset{\circ}{\succsim}_{(b, A \cup B)} (a, B) \quad \text{implies} \quad (a, A) \overset{\circ}{\sim}_{(b, A \cup B)} (a, A \cup B)$$

Thus, under the conditions stated, we may take $c = b$ in Axiom 11. □

Proposition 46. Let $\succsim$ have a separable representation as in equation (F.1), and suppose it also satisfies Finiteness (Axiom 10) and Choice Contingent CSR (Axiom 11). In that case, for all u in the carrier of μ, the induced marginal charge $\mu(\cdot|u)$ on $\mathfrak{U}_Z$ has singleton support. Moreover, if $\lambda > 0$ and $u, \lambda u$ are in the carrier of μ, then $\mu(\cdot|u) = \mu(\cdot|\lambda u)$.

Proof. As before, we may regard μ as a charge on $\mathfrak{U}_K \times [0,1] \times \mathfrak{U}_Z$. We will show that for each $r^* \in \mathfrak{U}_K$, the induced marginal charge $\mu(\cdot|r^*)$ has singleton support in $[0,1] \times \mathfrak{U}_Z$. Proposition 28 implies that the separable representation can be written as

$$W(x) = \int_{\mathfrak{U}_K \times [0,1] \times \mathfrak{U}_Z} \max_{p \in x} \left[r(p_k) + \tfrac{1-\gamma}{\gamma} v(p_z)\right] \gamma \, \mathrm{d}\mu(r, \gamma, v)$$

Fix r^* in the finite support of the marginal of μ on $\mathfrak{U}_K$. Let $\varepsilon > 0$ be such that $b := \{\alpha \in \mathfrak{U}_K : \|\alpha - p_k^*\|_2 \leqslant \varepsilon\}$ is contained in the interior of $\mathscr{P}(K)$.[40] Then, there exists $\alpha \in \mathscr{P}(K)$ that is superior to all alternatives in b in consumption taste r^*, while for all other relevant consumption utilities, some alternative from b is preferred to α. Letting $a := \{\alpha\}$, we see that $a \cup b \succ_K b$. Choice Contingent CSR (Axiom 11) now implies that there exists $c \in \mathscr{F}_K$ such that (i) $a \cup b \cup c \succ_K b \cup c$ and (ii) $(a, A) \overset{\circ}{\succsim}_{(b \cup c, A \cup B)} (A, B)$ implies $(a, A) \overset{\circ}{\sim}_{(b \cup c, A \cup B)} (A, B)$.

(40) Recall that p_k^* is the uniform lottery over K.

By the construction of a and b, and by lemma 45, we may assume, without loss of generality, that Choice Contingent Strategic Rationality (Axiom 11) is satisfied for $c = b$.

Property (i) of the axiom is trivially satisfied. To check (ii), consider arbitrary $A, B \in \mathscr{F}_Z$ and define $x := (a, A) \cup (b, A \cup B)$ and $y := (a, B) \cup (b, A \cup B)$. We may assume, without loss of generality, that $x \succsim y$. Then, $W(x) = r^*(\alpha) + \psi(A) + \kappa \geqslant r^*(\alpha) + \psi(B) + \kappa = W(y)$, where the κ is the utility from consumption in all states other than r^*, and is the same in both menus x and y, and where for any menu $A \in \mathscr{F}_Z$,

$$\psi(A) := \int_{[0,1]\times\mathfrak{U}_Z} \tfrac{1-\gamma}{\gamma} \max_{p_z \in A} v(p_z)\, \gamma \,\mathrm{d}\mu(\gamma, v|r^*)$$

Notice that $W(x) \geqslant W(y)$ if, and only if,

$$\psi(A) \geqslant \psi(B) \tag{F.2}$$

Consider the menu $y' := (a \cup b, A \cup B)$. Since the menus $A, B \in \mathscr{F}_Z$ are such that $x \succsim y$, Choice Contingent CSR (Axiom 11) implies that $x \sim y'$, which requires $W(x) = r^*(\alpha) + \psi(A) + \kappa = r^*(\alpha) + \psi(A \cup B) + \kappa = W(y')$ and is equivalent to

$$\psi(A) = \psi(A \cup B) \tag{F.3}$$

The function ψ is a utility function on $\mathscr{F}_Z$. It is easily seen that ψ is linear, and monotone, and induces a preference on $\mathscr{F}_Z$ that is continuous, satisfies Independence (Axiom 3), and Monotonicity (4). Displays (F.2) and (F.3) now imply that the preference induced by ψ is *strategically rational*, and so for each r^*, the marginal of μ on $\mathfrak{U}_Z$ has singleton support (see proposition 23), which completes the proof. □

The proposition implies that there exists a continuous $v : Z \times \mathscr{U} \to \mathbb{R}$ such that for all $(u, v') \in \mathscr{U} \times \mathfrak{U}_Z$, $v'(z) = \kappa(u)v(z, u)$ (up to adding a constant), μ-almost surely. Moreover, for $u, \lambda u$ in the carrier of μ, where $\lambda > 0$, it follows that $v(\cdot, u) = v(\cdot, \lambda u)$.

The proposition implies that W is Z-simple. It follows from proposition 31 that the representation is jointly identified and μ is a regular probability measure on $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Z$. We can rewrite the representation, transforming $\mathfrak{U}_K \times [0, 1] \times \mathfrak{U}_Z$ to become $\mathscr{U} \times \mathfrak{U}_Z$. Proposition 46 implies that each consumption state $u \in \mathscr{U}$, corresponds to a unique continuation utility function $v(\cdot, u)$. Hence, we may assume that the state space is $\mathscr{U}$ and let μ_0 be the corresponding measure on $\mathscr{U}$.

Let $\mathscr{U}^* := \mathrm{supp}(\mu_0)$ be the support of μ_0. By proposition 46, there exists $\{u_1, \dots, u_n\} \subset \mathscr{U}$ such that $\mathscr{U}^* \subset \mathscr{U}_M = \bigcup_{\lambda>0} \lambda\{u_1, \dots, u_n\}$. As before, we write $[u] := \{\lambda u : \lambda > 0\}$ for all $u \in \mathscr{U}_M$. Intuitively, $[u]$ is an equivalence class of consumption utilities, all of which induce the same continuation utility. Thus, we may write $W(x)$ as

$$W(x) = \int_{\mathscr{U}} \max_{p \in x} \big[u(p_k) + v(p_z, [u])\big]\, \mathrm{d}\mu_0(u) \tag{F.4}$$

Proposition 47. In the representation in F.4, the marginal of μ_0 on each $[u_i]$ is identified uniquely up to scaling.

Proof. Define

$$W_i(x) = \int_{[u_i]} \max_{p \in x} \left[u(p_k) + v(p_z, [u_i]) \right] \mathrm{d}\mu_0(u)$$

so that $W(x) = \sum_i W_i(x)$. Arguments analogous to those in proposition 34 establish that the marginal of μ_0 on each $[u_i]$ is identified up to scaling, which proves the proposition. □

F.2. Recursive, Uniformly Ranking Persistent Representation

To show that a recursive representation exists, we need to first show that the functions $v(\cdot, [u_i]) : Z \to \mathbb{R}$ in (F.4) are linear. Towards this end, consider the mapping $v : Z \to \mathbb{R}^n$, defined as follows: $v_i(x) := v(x, [u_i])$. Let $z^* \simeq (\mathscr{P}(K), z^*)$ be the IHCP that gives all the lotteries in each period. By definition of S_Z, we have $v_i(z^*) = 1$ and $v_i(x^*) = 0$ for all i (where $x^* \simeq (p_k^*, x^*)$ and p_k^* is the uniform lottery over K so that x^* gives this lottery in each period).

Let $O_i := v_i^{-1}(\operatorname{int} v_i(Z))$, and let $O := \bigcap_i O_i$. Since each v_i is continuous, it follows that each O_i and hence O, is open. Let $D_i := Z \setminus \operatorname{cl} O_i$. To show that each $v(\cdot, [u_i])$ is linear on Z, we need to show that it is locally non-satiated, which amounts to $D_i = \varnothing$. To see this, we need the following lemma.

Lemma 48. For every $x \in Z$ and for every open neighbourhood $N \ni x$, there exist $y, z \in N$ such that (i) $y \subset z$, and (ii) $z \succ y$.

Proof. Fix $x \in Z$ and let $N \ni x$ be open. Then, there exists $y \in N$ such that y is not a $\succsim$-maximal or $\succsim$-minimal element in Z. Recall that $z^* \simeq \big(\mathscr{P}(K), z^*\big)$, and by definition, $z^* \supset y$. Moreover, $z^* \succ y$. Therefore, there exists $\lambda \in (0, 1)$ such that (i) $\lambda z^* + (1-\lambda)y \subset N$ (by the definition of the Hausdorff metric), and (ii) $\lambda z^* + (1-\lambda)y \succ y$ (by Independence, Axiom 3). Setting $z := \lambda z^* + (1-\lambda)y$ completes the proof. □

Lemma 49. Let $\succsim$ have a representation as in (F.4), and suppose $\succsim$ satisfies Persistent Preference for Flexibility (Axiom 12), Then, $D_i = \varnothing$ for each $i = 1, \ldots, n$.

Proof. Notice that by definition, D_i is the union of two disjoint, connected sets that are open. The function v_i achieves its maximum and minimum values on these components, and is therefore constant on each of the components of D_i.

Suppose $D_i \neq \varnothing$, ie, suppose $x \in D_i$, and let N be an open neighbourhood of x such that $x \in N \subset D_i$ and v_i is constant on N. Then, by lemma 48, there exist $y, z \in N$ where $y \subset z$ and $z \succ y$. By Persistent Preference for Flexibility (Axiom 12), we must then have $v_i(z) > v_i(y)$, which contradicts the fact that v_i must be constant on N, completing the proof. □

If each D_i is empty, then each O_i is open and dense in Z, which implies that $O = \bigcap_i O_i$ is also dense in Z. Therefore, to show that each v_i is linear in Z, it suffices to show that v_i is linear on O. We establish this next.

Definition 50. A pair of sets $(\mathbf{x}, \mathbf{y}) \in \mathscr{F}_Z^2$ is ***amenable*** if $\mathbf{x} := \{x_i \in Z : i = 1, \ldots, n\}$ and $\mathbf{y} := \{y_i \in Z : i = 1, \ldots, n\}$ and if the following hold:

- $v_i(x_i) > v_i(x_j)$ for all $j \neq i$, and

- $v_i(y_i) > v_i(y_j)$ for all $j \neq i$

Because O is open, given $x, y \in O$, we can construct an amenable pair $(\mathbf{x}, \mathbf{y})$ such that $x = x_i \in \mathbf{x}$ and $y = y_i \in \mathbf{y}$. In particular, this demonstrates that amenable sets exist.

For each $\lambda \in [0, 1]$, define $\mathbf{z}_\lambda := \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$. As $\mathbf{x}$ and $\mathbf{y}$ consist of degenerate lotteries, $\mathbf{z}_\lambda$ also consists of degenerate lotteries. For each $i = 1, \ldots, n$, define $\Psi_i : [0, 1] \twoheadrightarrow \mathbf{x} \times \mathbf{y}$ as follows:

$$\Psi_i(\lambda) := \left\{(x, y) : x \in \mathbf{x},\ y \in \mathbf{y},\ \lambda x + (1-\lambda)y \in \arg\max_{z \in \mathbf{z}_\lambda} v_i(z)\right\}$$

Let $\Psi_i(\lambda) = (\Psi_{i,\mathbf{x}}, \Psi_{i,\mathbf{y}})$. We shall establish some properties of Ψ_i for amenable pairs $(\mathbf{x}, \mathbf{y})$.

Proposition 51. Let $\succsim$ have a representation as in (F.4), and suppose $\succsim$ satisfies Indifference to Timing (Axiom 9). Then, the correspondence Ψ has the following properties:

(a) Ψ_i is 'onto'. For each $\lambda \in [0, 1]$ and $x_i \in \mathbf{x}$, there exists j such that $x_i \in \Psi_{j,\mathbf{x}}(\lambda)$, with a similar claim for $y_i \in \mathbf{y}$.

(b) Ψ_i is a function.

(c) Ψ_i is continuous.

(d) Ψ_i is constant, ie, is independent of λ.

Proof. In the proof, we shall make repeated use of the fact that by Indifference to Timing (Axiom 9), $\lambda W((k, \mathbf{x})) + (1-\lambda)W((k, \mathbf{y})) = W((k, \mathbf{z}_\lambda))$ for all $\lambda \in [0, 1]$.

(a) Suppose not, so that $x_i \notin \Psi_{j,\mathbf{x}}$ for any $j = 1, \ldots, n$. Then, by perturbing x_i, we obtain a contradiction to the equality $\lambda W((k, \mathbf{x})) + (1-\lambda)W((k, \mathbf{y})) = W((k, \mathbf{z}_\lambda))$ and the fact that $(\mathbf{x}, \mathbf{y})$ is an amenable pair.

(b) Suppose not, so for some $\lambda \in (0, 1)$, $\Psi_i(\lambda) = \{(x_i, y_i) : i = 1, \ldots, m\}$. Let $\lambda^* := \inf\{\lambda : \Psi_i(\lambda) \text{ is not a singleton}\}$. It is easy to see that $\lambda^* > 0$. It is also easy to see that we may choose, without loss of generality, $\mathbf{x}$ and $\mathbf{y}$ such that there exists a unique i where $\Psi_i(\lambda^*)$ is not a singleton. Since Ψ is onto (as established above), we can perturb one of the elements of $\Psi_i(\lambda^*)$ without affecting $W((k, \mathbf{z}_{\lambda^*}))$, but affecting $\lambda^* W((k, \mathbf{x})) + (1-\lambda^*)W((k, \mathbf{y}))$, which is a contradiction.

(c) This is a simple consequence of the Theorem of the Maximum.

(d) This follows because Ψ_i is continuous, $[0, 1]$ is connected, and $\mathbf{x}$ and $\mathbf{y}$ are finite sets. □

Proposition 52. Let $\succsim$ have a representation as in (F.4), and suppose $\succsim$ satisfies Indifference to Timing (Axiom 9) and Persistent Preference for Flexibility (Axiom 12). Then, each v_i is linear on Z.

Proof. Recall that Z is compact and v_i is continuous on Z, which implies that v_i is, in fact, uniformly continuous on Z. Lemma 49 says that because $\succsim$ satisfies Persistent Preference for Flexibility (Axiom 12), O is dense in Z. Therefore, it suffices to show that v_i is linear on O, because by lemma 3.8 of Aliprantis and Border [1999], v_i has a unique continuous extension to Z (which must also be linear).

Suppose now that there exist $x, y \in O$ such that $v_i\big(\lambda x+(1-\lambda)y\big) \neq \lambda v_i(x)+(1-\lambda)v_i(y)$. By Indifference to Timing (Axiom 9), we have

$$(\bigstar) \qquad W\big((k, \mathbf{z}_\lambda)\big) = \lambda W((k,\mathbf{x})) + (1-\lambda)W((k,\mathbf{y}))$$

for an amenable pair $(\mathbf{x}, \mathbf{y})$ with $x = x_i \in \mathbf{x}$ and $y = y_i \in \mathbf{y}$. By the properties of the function Ψ, we see that

$$\begin{aligned} W((k,\mathbf{z}_\lambda)) &= \kappa + v_i\big(\lambda x + (1-\lambda)y\big) \\ \lambda W((k,\mathbf{x})) + (1-\lambda)W((k,\mathbf{y})) &= \iota + \lambda v_i\,(x) + (1-\lambda)v_i\,(y) \end{aligned}$$

where κ and ι do not depend on x and y (locally). It is easy to see that we can now perturb x (say) such that ($\bigstar$) no longer holds, which proves our claim. $\square$

Proposition 53. Let $\succsim$ have a representation as in (F.4) where $v(\cdot,[u])$ is linear, and suppose $\succsim$ satisfies Stationarity (Axiom X7) and Persistence (Axiom 12). Then, there is a value function

$$\text{(F.5)} \qquad V(x,u) = \int_{\mathcal{U}} \max_{p\in x}\big[u'(p_k) + \delta(u')V(p_z,u')\big]\, M(u,\mathrm{d}u')$$

for all $u \in \mathcal{U}_M$, where $(\mathcal{U}_M, M)$ is a ranking contingent Markov process where $\operatorname{supp}(\mu_0) \subset \mathcal{U}_M$, such that

$$\text{(F.6)} \qquad V(x,\mu_0) := \int_{\mathcal{U}} \max_{p\in x}\big[u(p_k) + \delta(u)V(p_z,u)\big]\, \mathrm{d}\mu_0(u)$$

is a representation of $\succsim$.

Proof. For rectangular menus of the form $\{(k,x)\}$, (F.4) becomes

$$\begin{aligned} W\big(\{(k,x)\}\big) &= \int_{\mathcal{U}_M} \big[u(k) + v(x,[u])\big]\, \mathrm{d}\mu_0(u) \\ &= \sum_{[u]\subset\mathcal{U}_M} v(x,[u])\,\mu_0([u]) + \int_{\mathcal{U}_M} u(k)\ \mathrm{d}\mu_0(u) \end{aligned}$$

where $v(\cdot,[u])$ is continuous and linear. The last equality follows because $\mathcal{U}_M = \bigcup_{i=1}^n [u_i]$. Notice that the second term does not depend on x. By Theorem 5, each $v(\cdot,[u])$ induces a preference $\succsim_{[u]}$ on $\mathcal{F}\big(\mathcal{P}(K\times Z)\big)$ that is continuous and satisfies Independence (Axiom 14). We claim:

Claim. For each $u \in \mathcal{U}_M$, $\succsim_{[u]}$ also satisfies Monotonicity (Axiom 4).

Proof of Claim. Let $a := \{\alpha \in \mathcal{P}(K) : \|\alpha - p_k^*\|_2 \leqslant \varepsilon\}$ for some sufficiently small $\varepsilon > 0$. Let $k' \in K$ be such that $u_1\big(\lambda p_k^* + (1-\lambda)k'\big) > \max_{\alpha\in a} u_1(\alpha)$ for all $\lambda \in (0,1)$ such that $\lambda p_k^* + (1-\lambda)k' \notin a$. Define $b_\lambda := \{\lambda p_k^* + (1-\lambda)k'\} \cup a$ and $\lambda^* := \min\{\lambda > 0 : \lambda p_k^* + (1-\lambda)k' \in a\}$, so that $b_{\lambda^*} = a$. By construction, $a \cup b_\lambda \succ_K a$, for all $\lambda > \lambda^*$.

Suppose, contrary to the claim, that $x \succ_{[u_i]} x \cup y$ for $i = 1,\ldots,\ell$ but $x \cup y \succsim_{[u_i]} x$ for $i = \ell+1,\ldots,n$ for some $\ell \in \{1,\ldots,n\}$. By appropriately perturbing y if necessary, we may take y such that $x \cup y \succ x$ (because $\succsim$ is continuous). Observe that $(a, \{x, x\cup y\}) \succ (a, \{x \cup y\})$

because $W\big((a,\{x,x\cup y\})\big)-W\big((a,\{x\cup y\})\big)=\sum_{i=1}^{\ell}\big[v(x,[u_i])-v(x\cup y,[u_i])\big]\mu([u_i])>0$, where we have used the fact that $v(x,[u_i])>v(x\cup y,[u_i])$ for $i=1,\dots,\ell$.

Since $a\cup b_\lambda \succ_K a$ and $x\cup y\succ x$, Persistent Preference for Flexibility (Axiom 12) implies that $(a\cup b_\lambda,\{x\cup y\})\succ(a,\{x\})\cup(b_\lambda,\{x\cup y\})$. In the limit as $\lambda\to\lambda^*$ so that $b_\lambda\to a$, we must have $(a,\{x\cup y\})\succsim(a,\{x,x\cup y\})$, because $\succsim$ is continuous, which contradicts our earlier observation that $(a,\{x,x\cup y\})\succ(a,\{x\cup y\})$, thereby proving the claim. ▲

Theorem 5 then implies that because $v(\cdot,[u])$ represents $\succsim_{[u]}$, and because $v(\cdot,[u])$ is continuous, linear, and monotone with respect to set inclusion, $v(\cdot,[u])$ can be written as

$$v(x,[u])=\delta([u])\int_{\mathfrak{U}_{K\times Z}}\max_{p\in x}\mathrm{u}(p)\ \mathrm{d}\mu_{[u]}(\mathrm{u})$$

where $\mu_{[u]}$ is a probability charge on $\mathfrak{U}_{K\times Z}$, and $\delta([u])$ is the scaling factor that allows us to take $\mu_{[u]}$ to be a probability charge.

Let $W'(x):=W\big(\{(k,x)\}\big)$ for some fixed $k\in K$. By Stationarity (Axiom 6), $W'(x)\geqslant W'(y)$ if, and only if, $W(x)\geqslant W(y)$. Indifference to Timing (Axiom 7) implies that W' is an affine transformation of W. Therefore, we have

$$\begin{aligned}W(x)&=\int_{\mathcal{U}_M}\max_{p\in x}\big[u(p_k)+v(p_z,[u])\big]\,\mathrm{d}\mu_0(u)\\&\propto W'(x)=W\big(\{(k,x)\}\big)\\&=\sum_{[u]\subset\mathcal{U}_M}v(x,[u])\mu_0([u])+\text{ constant}\\&=\sum_{[u]\subset\mathcal{U}_M}\Big[\delta([u])\int_{\mathfrak{U}_{K\times Z}}\max_{p\in x}\mathrm{u}(p)\ \mathrm{d}\mu_{[u]}(\mathrm{u})\Big]\mu_0([u])+\text{ constant}\end{aligned}$$

Define the charge σ^* on $\mathfrak{U}_{K\times Z}$ as $\sigma^*(\mathrm{du}):=\sum_{[u]\in\mathcal{U}_M}\delta([u])\mu_0([u])\mu_{[u]}(\mathrm{du})$. Then, the last line in the display above can be written as

$$W(x)\propto\int_{\mathfrak{U}_{K\times Z}}\max_{p\in x}\mathrm{u}(p)\ \mathrm{d}\sigma^*(\mathrm{u})$$

where σ^* on $\mathfrak{U}_{K\times Z}$ is not necessarily a probability charge.

Following the arguments in proposition 38, and using the fact that $[u]$ is a sufficient statistic for $v\in\mathfrak{U}_Z$, we may take μ_0 to be defined on $\mathfrak{U}_{K\times Z}$. By virtue of proposition 31, which says that the state space and measure are jointly identified, it must be that $\mu_0\propto\sigma^*$.

This implies the carrier of σ^* coincides with the carrier of μ_0 and is finite. But σ^* is the positive linear combination of charges $\mu_{[u]}$ on $\mathfrak{U}_{K\times Z}$, and therefore, the carrier of each $\mu_{[u]}$ (where $u\in\mathcal{U}_M$) must be a subset of the carrier of σ^*, and hence of the carrier of μ_0, which is a subset of $\mathcal{U}_M$.

This allows us to write

$$v(x,[u])=\delta([u])\int_{\mathcal{U}_M}\max_{p\in x}\big[u'(p_k)+v(p_z,[u'])\big]\ \mathrm{d}\mu_{[u]}(u')$$

for each $u\in\mathcal{U}_M$. Define the Markov kernel M on $\mathcal{U}_M$ as follows: $M(u,\mathrm{d}u'):=\mu_{[u]}(\mathrm{d}u')$ for all $u\in\mathcal{U}_M$. It is clear that M is ranking contingent. Also define $\delta(u):=\delta([u])$, and

$V(x,u) := v(x,[u])/\delta(u)$ to find the recursive value function (F.5). Finally, plugging into equation (F.4), and defining $V(x,\mu_0) := W(x)$, we see that $V(x,\mu_0)$ as in (F.6) represents $\succsim$, as desired. □

We now show that the Markov kernel M is uniformly ranking persistent.

Proposition 54. Let $\succsim$ have a recursive representation as in (F.6), and suppose $\succsim$ satisfies Persistent Preference for Flexibility (Axiom 12). Then, $M\big(u,[u']\big) > 0$ for all $u,u' \in \mathcal{U}_M$.

Proof. Suppose $M(u',[u]) = 0$ for some $u,u' \in \mathcal{U}_M$. Construct menus x, y such that

$$\int_{[u]} \max_{p\in y} \big[\hat{u}(p_k) + \delta(\hat{u})V(p_z,\hat{u})\big]\, \mathrm{d}\mu_0(\hat{u}) \\ > \int_{[u]} \max_{p\in x} \big[\hat{u}(p_k) + \delta(\hat{u})V(p_z,\hat{u})\big]\, \mathrm{d}\mu_0(\hat{u})$$

but

$$\int_{\mathcal{U}_M\setminus[u]} \max_{p\in x} \big[\hat{u}(p_k) + \delta(\hat{u})V(p_z,\mu_{\hat{u}})\big]\, \mathrm{d}\mu_0(\hat{u}) \\ > \int_{\mathcal{U}_M\setminus[u]} \max_{p\in y} \big[\hat{u}(p_k) + \delta(\hat{u})V(p_z,\mu_{\hat{u}})\big]\, \mathrm{d}\mu_0(\hat{u})$$

This immediately implies $x \cup y \succ x$.

Similarly, construct consumption menus a and b, such that $\max_{\alpha\in a} u'(\alpha) > \max_{\alpha\in b} u'(\alpha)$ and $\max_{\alpha\in b} \hat{u}(\alpha) \geqslant \max_{\alpha\in a} \hat{u}(\alpha)$ for all $\hat{u} \in \mathcal{U}_M \setminus [u']$. Then, the best element from $(a\cup b,\{x\cup y\})$ is in $(a,\{x\cup y\})$ only in states in $[u']$. At the same time, contingent on being in a consumption state in $[u']$, the best choice of a continuation menu is surely $\{x\}$, since $M(u',[u]) = 0$ by assumption. Hence, $(a,\{x\}) \overset{\circ}{\sim}_{(b,\{x\cup y\})} (a,\{x\cup y\})$, which contradicts Persistent Preference for Flexibility (Axiom 12), which requires $(a,\{x\cup y\}) \overset{\circ}{\succ}_{(b,\{x\cup y\})} (a,\{x\})$. □

F.3. Representation with Constant Discount Factor: Existence and Uniqueness

Propositions 53 and 54 establish that $\succsim$ has a recursive representation as in (F.5) and (F.6), with a ranking persistent Markov process. Our goal is to show that there exists a unique equivalent representation with a *constant* discount factor. We now describe, in brief, how the approach parallels the construction of a representation with a constant discount factor that lead to the PFX representation above. The details of the construction are provided in an online appendix, Krishna and Sadowski [2012a].

In the PFX representation, the states are $S = \{1,\ldots,n\}$ and uncertain utilities are described by measures μ_s that depend on the state $s \in S$, while the Markov process on states S is given by the transition probabilities $\Pi(s',s)$.

In the PFR representation, the uncertain utilities are described by the ranking contingent Markov kernel $M(u,\cdot)$, wherein $M(u,\cdot) = M(\lambda u,\cdot)$. Moreover, the probability of the ranking $[u_i]$ is $M(u,[u_i])$. Consider the following correspondence:

PFX		PFR
$S = \{1, \dots, n\}$	States for Markov Process	$\{[u_i] : i = 1, \dots, n\}$
$\Pi(s, s')$	Probabilities of States/Rankings	$M(u, [u_i])$
$\mu_s(D),\ D \subset \mathcal{U}$	Probabilities of utilities given states	$M(u, E)/M(u, [u_i]),\ E \subset [u_i]$
π_0 (of Π)	Invariant Measures	μ_0 (of M)

In the PFX representation, at each instant in time, DM's uncertainty about utilities in the next period can be decomposed into uncertainty about the state $s \in S$ and the conditional probabilities over utilities in $\mathcal{U}$ given by μ_s. Formally, this determines a probability measure over $S \times \mathcal{U}$.

In the PFR representation, uncertainty about utilities in $\mathcal{U}_M$ can be decomposed into uncertainty about the ranking $[u_i]$, given by $M(u, [u_i])$, and uncertainty about the intensity of the utility, conditional on the ranking, given by $M(u, E)/M(u, [u_i])$ where $E \subset [u_i]$ is measurable. Thus, the formal structure of the two representations is essentially the same.

In order to obtain a representation with a constant discount factor, one can apply the same transformations that we used in the case of the PFX representation as in proposition 40. The Perron Theorem implies that such a representation exists and is unique. In particular, the distribution of intensities given the ranking $[u_i]$ is uniquely identified. To identify the stationary distribution μ_0 of M, it suffices then to calculate the stationary distribution of the induced Markov process on rankings.

Once we have a representation with a constant discount factor, all that remains is to show that the discount factor is less than one. This is done in a manner parallel to proposition 42 for the PFX representation.

Finally, the proof of proposition 10 parallels the proof of proposition 2.

G. Proofs from Section 6

We present here proofs concerning the behavioral comparison 'greater preference for flexibility'. We begin with some preliminary lemmas. Recall that L is the subdomain of IHCSs.

Lemma 55. Suppose $\succsim^*$ has a greater preference for flexibility than $\succsim$ and both $\succsim$ and $\succsim^*$ satisfy Independence (Axiom 3). Then, for all $\ell, \ell' \in L$, $\ell \succsim^* \ell'$ if, and only if, $\ell \succsim \ell'$.

Proof. By hypothesis, we have $\ell \succsim \ell'$ implies $\ell \succsim^* \ell'$, or equivalently, $\ell' \succ^* \ell$ implies $\ell' \succ \ell$. Therefore, it suffices to show that $\ell' \sim^* \ell$ implies $\ell' \sim \ell$. So, let us suppose $\ell' \sim^* \ell$ and, without loss of generality, assume that $\ell' \succ \ell$. Let $\ell^\dagger \succ^* \ell' \sim^* \ell$ — by Independence (Axiom 3), it suffices to consider ℓ and ℓ' for which such an $\ell^\dagger$ exists — so that for some sufficiently small $\lambda \in (0, 1)$, we have $\lambda\ell^\dagger + (1 - \lambda)\ell \succ^* \ell'$ but $\ell' \succ \lambda\ell^\dagger + (1 - \lambda)\ell$, which contradicts the hypothesis, thereby completing the proof. □

Lemma 56. Suppose $\succsim^*$ has a greater preference for flexibility than $\succsim$, and suppose $\succsim$ and $\succsim^*$ have canonical PFC representations (μ, δ) and (μ^*, δ^*) respectively. Then, $\delta = \delta^*$, and $V(\ell) = V^*(\ell)$ for all $\ell \in L$. Moreover, $V^*(x) \geqslant V(x)$ for all $x \in Z$.

Proof. By lemma 55, we know that $\ell \succsim^* \ell'$ if, and only if, $\ell \succsim \ell'$. Thus, $\succsim$ and $\succsim^*$ represent the same preference on the restricted domain L. Let V_L and V_L^* denote the value functions for the respective canonical PFC representations, restricted to L. As V_L and V_L^* represent the same preference, they are affine transformations of each other. Let $\ell^* \simeq (p_k^*, \ell^*)$ denote the IHCS that gives the uniform lottery in each period, so $V_L(\ell^*) = V_L^*(\ell^*) = 0$. For each $k \in K$, $V_L\big(\{(k, \ell^*)\}\big) = \mu u_k + 0 = \mu u_k$ and $V_L^*\big(\{(k, \ell^*)\}\big) = \mu^* u_k$. It follows that $\mu u \propto \mu^* u$. But μ and μ^* are canonical measures, ie, $\|\mu u\|_2 = \|\mu^* u\|_2 = 1$, so it must be that $\mu u = \mu^* u$. Hence, $V(\ell) = V^*(\ell)$ for all $\ell \in L$, ie, $V_L = V_L^*$.

Now let $\ell_k \simeq \{(k, \ell_k)\}$, which gives us $V(\ell_k) = \mu u_k + \delta V(\ell_k)$. Then, $(1-\delta)V(\ell_k) = \mu u_k = \mu^* u_k = (1-\delta^*)V^*(\ell_k)$, which implies $\delta = \delta^*$, because $\|\mu u\|_2 = 1$ means that $\mu u_k \neq 0$ for some $k \in K$.

For any probability measure μ on $\mathscr{U}$ with $\|\mu u\|_2 = 1$, there exist $k, k' \in K$ such that $\mu u_k > 0 > \mu u_{k'}$. That is, there exist $\ell, \ell' \in L$ such that $\ell \succ \ell^* \succ \ell'$ where ℓ^* is as above, and hence $V(\ell^*) = 0 = V^*(\ell^*)$. Consequently, for any $x \in Z$, there exists $\lambda \in (0,1)$ and $\ell^\dagger \in L$ such that $\lambda x + (1-\lambda)\ell^* \sim \ell^\dagger$, which means that $\lambda x + (1-\lambda)\ell^* \succsim^* \ell^\dagger$ because $\succsim^*$ has a greater preference for flexibility than $\succsim$. This implies $V^*\big(\lambda x + (1-\lambda)\ell^*\big) \geqslant V^*(\ell^\dagger) = V(\ell^\dagger) = V\big(\lambda x + (1-\lambda)\ell^*\big)$, from which it follows that $V^*(x) \geqslant V(x)$ for all $x \in Z$. □

For the canonical PFC representation (μ, δ), let V be the corresponding value function over menus. Notice that since there is no preference for flexibility with respect to continuation problems, for the purpose of assigning utilities, we may restrict attention to the domain $\mathscr{F}\big(\mathscr{P}(K \times \{z_\circ, z^\circ\})\big)$, where $V(z_\circ) \leqslant V(z) \leqslant V(z^\circ)$ for all $z \in Z$, and $V(z_\circ) < V(z^\circ)$. We are now able to state a preliminary lemma.

Lemma 57. Let $\varphi : \mathscr{U} \to \mathbb{R}$ be convex and $D \subset \mathscr{U}$ be compact, convex such that $\varphi|_D$ is Lipschitz. Then, there exists a compact $x \subset \operatorname{aff}\mathscr{P}(K \times \{z_\circ, z^\circ\})$ such that $\max_{p \in x}\big[u(p_k) + \delta V(p_z)\big] \leqslant \varphi(u)$ with equality for all $u \in D$.

Proof. Let $u^* \in D$, and let $h_{u^*}(u) : \mathscr{U} \to \mathbb{R}$ be an affine function such that $h_{u^*} \leqslant \varphi$ with equality at u^*. (The function h_{u^*} represents the hyperplane supporting φ at u^*.) Then, there exist $d \in \mathbb{R}^K$ and $d' \in \mathbb{R}$ such that $h_{u^*}(u) = \langle d, u\rangle + d'$. We shall now construct the corresponding menu.

Notice that $h_{u^*}(u) = \sum_i d_i u_i + d' = \sum_{i=1}^{K-1} u_i(d_i - d_K) + d'$ where we have used the fact that $\sum_i u_i = 0$ for all $u \in \mathscr{U}$. Let $\alpha \in \operatorname{aff}\mathscr{P}(K)$ be such that $\alpha_i = d_i - d_K$ for $i = 1, \dots, K-1$, and $\alpha_K = 1 - \sum_{i=1}^{K-1} \alpha_i$. Also, let $q_z \in \operatorname{aff}\mathscr{P}(\{z_\circ, z^\circ\})$ be such that $d' = \delta V(q_z)$ (where V has been extended to $\operatorname{aff}\mathscr{P}(\{z_\circ, z^\circ\})$ by linearity and hence uniquely). Now consider $q(u^*) = (\alpha, q_z)$, the signed measure on $K \times \{z_\circ, z^\circ\}$ (with marginals α and q_z). Then, $h_{u^*}(u) = u(\alpha) + \delta V(q_z)$.

Recall that φ restricted to D is Lipschitz. This implies that the set $x := \overline{\operatorname{conv}}\{q(u) : u \in D\}$ is compact. (Intuitively, the set of 'slopes' and intercepts of the supporting hyperplanes of $\varphi|_D$ is compact.) It is clear that $x \subset \operatorname{aff}\mathscr{P}(K \times \{z_\circ, z^\circ\})$. For each $q \in x$, $u(q_k) + \delta V(q_z)$ is an affine function of u. Therefore, $\max_{q \in x}\big[u(q_k) + \delta V(q_z)\big]$ is a convex function of u. By construction, this function is always dominated by φ and is equal to φ on D, which completes the proof. □

The proof of Theorem 4 relies on the following Theorem, which characterises dilations.

Theorem 7. *Let μ and μ^* be probability measures on $\mathscr{U}$. Then, the following are equivalent:*

(a) μ^* *is a dilation of* μ.

(b) $\mu^*\varphi \geqslant \mu\varphi$ *for every continuous convex function* $\varphi : \mathcal{U} \to \mathbb{R}$ *that is* $\mu + \mu^*$ *integrable.*

Theorem 7 is Theorem 7.2.17 in Torgersen [1991]. It was proved by Blackwell [1953] for the case where the supports of μ and μ^* are bounded.

Proof of Theorem 4. (a) implies (b). Suppose $\succsim^*$ has greater preference for flexibility than $\succsim$. By Theorem 7, it suffices to show that for every continuous convex function $\varphi : \mathcal{U} \to \mathbb{R}$ that is $\mu + \mu^*$ integrable, we have $\mu^*\varphi \geqslant \mu\varphi$.

Fix such a function φ. We shall show that there exist continuous, convex functions $\varphi_n : \mathcal{U} \to \mathbb{R}$ that are $\mu + \mu^*$ integrable, such that (i) $\varphi_n \leqslant \varphi_{n+1}$, (ii) $\varphi_n \uparrow \varphi$, and (iii) $\mu\varphi_n \leqslant \mu^*\varphi_n$. Since φ_1 is $(\mu+\mu^*)$ integrable, it follows that $\varphi-\varphi_1$ is integrable. Therefore, $0 \leqslant \varphi-\varphi_n \leqslant \varphi-\varphi_1$, so the Dominated Convergence Theorem and (iii) above imply $\mu\varphi \leqslant \mu^*\varphi$.

Let (D_n) be an increasing sequence of compact subsets of $\mathcal{U}$ such that $\bigcup_n D_n$ covers the effective domain of φ. By standard arguments (because the effective domain has a relative interior in our finite dimensional setting), we may assume that for all n, $\varphi|_{D_n}$ is Lipschitz. Then, by lemma 57, there exists $x_n \subset \operatorname{aff}\mathcal{P}(K \times \{z_\circ, z^\circ\})$ so that $\varphi_n(u) := \max_{q \in x_n} \left[u(q_k) + \delta V(q_z)\right]$ satisfies $\varphi_n \leqslant \varphi$, and $\varphi_n|_{D_n} = \varphi|_{D_n}$. It follows immediately from the construction (in lemma 57) that $\varphi_n \leqslant \varphi_{n+1}$ (since $x_n \subset x_{n+1}$). All that remains is to show that for all n, $\mu\varphi_n \leqslant \mu^*\varphi_n$.

Let W and W^* represent the restrictions of V and V^* to $\mathcal{F}\left(\mathcal{P}(K \times \{z_\circ, z^\circ\})\right)$. Thus, for any $x \in Z$, there exists $x' \in \mathcal{F}\left(\mathcal{P}(K \times \{z_\circ, z^\circ\})\right)$ such that $V(x) = W(x')$. Moreover, two PFC representations differ if, and only if, they differ on the domain $\mathcal{F}\left(\mathcal{P}(K \times \{z_\circ, z^\circ\})\right)$. Abusing notation again, let W and W^* denote the respective extensions to all compact subsets of $\mathcal{P}(K \times \{z_\circ, z^\circ\})$.

Recall that by lemma 56, $V^*(y) \geqslant V(y)$ for all $y \in Z$ and $V(z^\circ) > 0 > V(z_\circ)$. Then, by linearity, we must have $W^*(x) \geqslant W(x)$ for all compact $x \subset \mathcal{P}(K \times \{z_\circ, z^\circ\})$. But $W(x_n) = \int_{\mathcal{U}} \varphi_n(u)\, \mathrm{d}\mu(u)$ (which implies, in particular, that each φ_n is integrable, since x_n is compact and $\max\{|W(x_n)|, |W^*(x_n)|\} < \infty$), so that we have $\mu\varphi_n \leqslant \mu^*\varphi_n$ as required.

(b) implies (a). Consider the operators $\Phi, \Phi^* : C(Z) \to C(Z)$ defined as follows:

$$\Phi W(x) := \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta W(p_z)\right] \mathrm{d}\mu(u) \quad \text{and}$$

$$\Phi^* W(x) := \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta W(p_z)\right] \mathrm{d}\mu^*(u)$$

As observed in the proof of Proposition 12, for instance, both Φ and Φ^* are monotone and also satisfy discounting, and are therefore contractions. For each $x \in Z$, $\max_{p \in x} \left[u(p_k) + \delta W(p_z)\right]$ is a continuous and convex function of u. Therefore, by Theorem 7, for any $W \in C(Z)$, $\Phi W \leqslant \Phi^* W$. Let Φ^n and Φ^{*n} denote the n-th iterates of Φ and Φ^* respectively. We claim that $\Phi^n W \leqslant \Phi^{*n} W$ for all $W \in C(Z)$ and for all $n \geqslant 1$. We have already established this for $n = 1$. Suppose this is true for $n-1$, ie, $\Phi^{n-1} W \leqslant \Phi^{*(n-1)} W$. Then, $\Phi(\Phi^{n-1} W) \leqslant \Phi^*(\Phi^{n-1} W) \leqslant \Phi^*(\Phi^{*(n-1)} W)$, ie, $\Phi^n W \leqslant \Phi^{*n} W$, as claimed. Finally, let V and V^* be the unique fixed points of Φ and Φ^* respectively, so that $V \leqslant V^*$, which completes the proof. □

Proof of Proposition 17. It is easy to see, from the arguments in footnote 28 for instance, that $\succsim$ satisfies the appropriate version of Consumption non-triviality (Axiom 13) if, and only if, there exists $s \in S$ such that $\mu_s u \neq \mathbf{0}$. Moreover, the recursivity of the representation implies that if $\mu_s u \neq \mathbf{0}$, then for all $s' \in S$, $V(\cdot, s')$ is non-trivial when restricted to consumption streams.

Let U_s and $U_{s'}$ respectively represent $\succsim_s$ and $\succsim_{s'}$ over $\mathcal{F}$ where, as before, $U_s(x) = \int_{\mathcal{U}} \max_{p \in x} \left[u(p_k) + \delta V(p_h, s)\right] \, \mathrm{d}\mu(u)$ and similarly for s'. Following the arguments in lemma 56, we see that when restricted to consumption streams, $U_s|_L$ is a positive affine transformation of $U_{s'}|_L$. It is easy to see that the constant term must be zero, so let us suppose $U_s|_L = \lambda U_{s'}|_L$ for some $\lambda > 0$. Then, it must be that μ and μ^* differ by a scaling of λ, $\mu u = \lambda \mu^* u$, and $V_L(\cdot, s) = V_L(\cdot, s')$.

Now consider the SIHCS ℓ_t^k that delivers in each period, and in every state, p_k^*, the uniform lottery over K, except at time $t+1$, where it delivers the prize $k \in K$. Then, $U_s(\ell_t^k)/U_{s'}(\ell_t^k) = \lambda > 0$. Define $\pi_s(\cdot) := \Pi(s, \cdot)$. The probability distribution over states S at time $t+1$, conditional on being in state s in date 1 is $\pi_s \Pi^t$. Therefore, $U_s(\ell_t^k) = \delta^t \sum_{\tilde{s}} \pi_s \Pi^t(\tilde{s}) \mu_{\tilde{s}} u_k$, which implies that $\lambda = U_s(\ell_t^k)/U_{s'}(\ell_t^k) = \left[\sum_{\tilde{s}} \pi_s \Pi^t(\tilde{s}) \mu_{\tilde{s}} u_k\right] / \left[\sum_{\tilde{s}} \pi_{s'} \Pi^t(\tilde{s}) \mu_{\tilde{s}} u_k\right]$ (or at least, such a $k \in K$ can be chosen because $\mu_s u \neq \mathbf{0}$ for some $s \in S$). But Π is fully connected and has a unique invariant distribution π_0, which implies $\lim_{t \to \infty} \pi_s \Pi^t = \pi_0 = \lim_{t \to \infty} \pi_{s'} \Pi^t$, which means we must have $\lambda = 1$, ie, $U_s|_L = U_{s'}|_L$ and, in particular, $\mu_s u = \mu_{s'} u$.

Consider now, the state $\tilde{s} \in S$, and let $\ell \in L$ be such that $V(\ell, \tilde{s}) \neq 0$. Such an ℓ exists because $V(\cdot, \tilde{s})$ is non-trivial over consumption streams. Also, recall that ℓ^* is the SIHCS that gives the uniform lottery p_k^* in each state and in every period. Now, consider the consumption stream $\ell^\dagger$ that gives the consumption stream (p_k^*, ℓ) in state $\tilde{s}$ and the SIHCS ℓ^* in every other state. Then, $V(\ell^\dagger, s) = \Pi(s, \tilde{s}) V(\ell, \tilde{s}) = \Pi(s', \tilde{s}) V(\ell, \tilde{s}) = V(\ell^\dagger, s')$, which implies $\Pi(s, \tilde{s}) = \Pi(s', \tilde{s})$ because $V(\ell, \tilde{s}) \neq 0$.

Although $\|\mu_s u\|$ and $\|\mu_{s'} u\|$ need not be equal to 1, all that matters in using the arguments in the proof of Theorem 4 is that $\mu_s u = \mu_{s'} u$. Therefore, we can adapt the arguments from the proof of Theorem 4 and show that (a) is equivalent to (b), which proves the proposition. □

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