Chapter 3

POTENTIALS AND WAVE SOLUTIONS

In this chapter (Chapter 6 in textbook) we consider again a general homogeneous conductive medium, but with a particular source excitation. There are two ways to solve the electromagnetic fields due to imposed sources. One is the solve the fields **E** and **H** directly by using Maxwell's equations or the derivable second-order partial differential equations. The other way is to express the electromagnetic fields in terms of potentials, and solve these potentials first.

In this chapter we will use the second way. We express the **E** and **H** fields in terms of vector potentials **A** (magnetic vector potential) and **F** (electric vector potential). PDE's for **A** and **H** can be derived from Maxwell's equations. And the boundary conditions for **E** and **H** can be converted into those for **A** and **F** so that these potential can be solved to yield the solutions for **E** and **H**.

Other potentials are possible. For example, sometimes the Hertz vector potentials Π_e and Π_h are introduced to solve the electromagnetic fields.

3.1 Principle of Superposition

In Chapter 3, we introduced the complex permittivity $\tilde{\epsilon}$ and permeability $\tilde{\mu}$ to include the effects of the conductivity in the medium. For simplicity, this complex permittivity and permeability will be denoted simply as ϵ and μ in the rest of the course, keeping in mind that it applies to both conductive and non-conductive media.

The time-harmonic EM fields satisfy Maxwell's curl equations (1.63) and (1.64), which can be cast into the following linear form,

$$\mathcal{L}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix} = \begin{bmatrix}-\mathbf{M}_i\\ \mathbf{J}_i\end{bmatrix} \tag{3.1}$$

where

$$\mathcal{L} = \begin{bmatrix} \nabla \times & j\omega\mu \\ -j\omega\epsilon & \nabla \times \end{bmatrix} \tag{3.2}$$

Equation (3.1) is a linear system, and hence the principle of linear superposition applies. Specifically, if ($\mathbf{E}_A$, $\mathbf{H}_A$) are due to the electric current density $\mathbf{J}_i$, and if ($\mathbf{E}_F$, $\mathbf{H}_F$) are due to the magnetic current density $\mathbf{M}_i$, i.e.,

$$\mathcal{L} \begin{bmatrix} \mathbf{E}_A \\ \mathbf{H}_A \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{J}_i \end{bmatrix} \tag{3.3}$$

$$\mathcal{L} \begin{bmatrix} \mathbf{E}_F \\ \mathbf{H}_F \end{bmatrix} = \begin{bmatrix} -\mathbf{M}_i \\ \mathbf{0} \end{bmatrix} \tag{3.4}$$

then the total fields ($\mathbf{E}$, $\mathbf{H}$) due to $\mathbf{J}$ and $\mathbf{M}$ together will be

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F, \quad \mathbf{H} = \mathbf{H}_A + \mathbf{H}_F \tag{3.5}$$

This principle of superposition can be easily verified by summing up equations (3.4) and (3.5).

Because of the principle of superposition, we can solve ($\mathbf{E}_A$, $\mathbf{H}_A$) and ($\mathbf{E}_F$, $\mathbf{H}_F$) separately, and then sum them up to obtain the total fields. Each of these set of fields will be obtained by the following vector potentials.

3.2 Magnetic Vector Potential A

In the absence of the magnetic sources (i.e., $\mathbf{M}_i = 0$, $\rho_{mi} = 0$), Maxwell's equations for fields ($\mathbf{E}_A$, $\mathbf{H}_A$, $\mathbf{D}_A$, $\mathbf{B}_A$) are

$$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A \tag{3.6}$$

$$\nabla \times \mathbf{H}_A = \mathbf{J}_i + j\omega\epsilon\mathbf{E}_A \tag{3.7}$$

$$\nabla \cdot \epsilon\mathbf{E}_A = \rho_{ei} \tag{3.8}$$

$$\nabla \cdot \mathbf{B}_A = 0 \tag{3.9}$$

From (3.9), it is clear that $\mathbf{B}_A = \mu\mathbf{H}_A$ is solenoidal (i.e., divergence-free). Since $\nabla \cdot \nabla \times \mathbf{A} = 0$ for any vector $\mathbf{A}$, The magnetic flux density can be written as

$$\mathbf{B}_A = \nabla \times \mathbf{A}, \quad \mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A} \tag{3.10}$$

Then from (3.6) and (3.10) we obtain

$$\nabla \times (\mathbf{E}_A + j\omega\mathbf{A}) = 0$$

That is, ($\mathbf{E}_A + j\omega\mathbf{A}$) is a curl-free (or irrotational) vector, and hence can be written as

$$(\mathbf{E}_A + j\omega\mathbf{A}) = -\nabla\phi_e$$

or

$$\mathbf{E}_A = -\nabla\phi_e - j\omega\mathbf{A} \tag{3.11}$$

Vector $\mathbf{A}$ is called the magnetic vector potential, and scalar ϕ_e is called the scalar electric potential.

To uniquely determine a vector $\mathbf{A}$, one needs to specify both its curl and divergence. The curl of $\mathbf{A}$ is given by (3.10), and so we need the divergence condition. We may choose the so-called *Lorentz condition (gauge)*

$$\nabla\cdot\mathbf{A} + j\omega\mu\epsilon\phi_e = 0 \tag{3.12}$$

as the condition for its divergence. Alternatively, the Coulomb's gauge ($\nabla\cdot\mathbf{A} = 0$) can be chosen for the divergence.

Using the Lorentz condition together with (3.7) and (3.11), we obtain from (3.7) the PDE

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}_i \tag{3.13}$$

This is the vector wave equation for the magnetic vector potential. Once $\mathbf{A}$ is solve, the electric potential ϕ_e can be solved by (3.12), and ($\mathbf{E}_A$, $\mathbf{H}_A$) can be obtained by (3.10) and (3.11). In summary,

$$\mathbf{H}_A = \frac{1}{\mu}\nabla\times\mathbf{A} \tag{3.14}$$

$$\mathbf{E}_A = -j\omega\mathbf{A} - \nabla\phi_e = -j\omega\mathbf{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) \tag{3.15}$$

or

$$\mathbf{E}_A = -\frac{j}{\omega\epsilon}(\nabla\times\mathbf{H}_A - \mathbf{J}_i) \tag{3.16}$$

Equation (3.15) is more convenient for far-field evaluation since the second term can be neglected as it decays faster than $1/r$; whereas equation (3.16) is more convenient for near-field evaluation since the the source singularity term $j\mathbf{J}_i/\omega\epsilon$ is explicitly given.

3.3 Electric Vector Potential F

Similarly, in the absence of electric sources, Maxwell's equations for fields ($\mathbf{E}_F$, $\mathbf{H}_F$, $\mathbf{D}_F$, $\mathbf{B}_F$) are

$$\nabla\times\mathbf{E}_F = -\mathbf{M}_i - j\omega\mu\mathbf{H}_F \tag{3.17}$$

$$\nabla\times\mathbf{H}_F = j\omega\epsilon\mathbf{E}_F \tag{3.18}$$

$$\nabla\cdot\mathbf{D}_F = 0 \tag{3.19}$$

$$\nabla\cdot\mathbf{B}_F = \rho_{mi} \tag{3.20}$$

It is clear that $\mathbf{D}_F = \epsilon\mathbf{E}_F$ is solenoidal (i.e., divergence-free), and can be written as

$$\mathbf{D}_F = -\nabla\times\mathbf{F}, \quad \mathbf{E}_F = -\frac{1}{\epsilon}\nabla\times\mathbf{F} \tag{3.21}$$

Also, $(\mathbf{H}_F + j\omega\mathbf{F})$ is a curl-free (or irrotational) vector, and hence can be written as

$$(\mathbf{H}_F + j\omega\mathbf{F}) = -\nabla\phi_m$$

or

$$\mathbf{H}_F = -\nabla\phi_m - j\omega\mathbf{F} \tag{3.22}$$

Vector $\mathbf{F}$ is called the electric vector potential, and scalar ϕ_m is called the scalar magnetic potential.

The Lorentz condition (gauge) is

$$\nabla \cdot \mathbf{F} + j\omega\mu\epsilon\phi_m = 0 \tag{3.23}$$

as the condition for its divergence. Alternatively, the Coulomb's gauge ($\nabla \cdot \mathbf{F} = 0$) can be chosen for the divergence.

Using the Lorentz condition together with (3.17) and (3.20), we obtain the PDE

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M}_i \tag{3.24}$$

This is the vector wave equation for the magnetic vector potential. Once $\mathbf{F}$ is solve, the magnetic potential ϕ_m can be solved by (3.23), and $(\mathbf{E}_F, \mathbf{H}_F)$ can be obtained by (3.21) and (3.22). In summary,

$$\mathbf{E}_F = -\frac{1}{\epsilon}\nabla \times \mathbf{F} \tag{3.25}$$

$$\mathbf{H}_F = -j\omega\mathbf{F} - \nabla\phi_m = -j\omega\mathbf{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F}) \tag{3.26}$$

or

$$\mathbf{H}_F = \frac{j}{\omega\mu}(\nabla \times \mathbf{E}_F + \mathbf{M}_i) \tag{3.27}$$

Again, Equation (3.26) is more convenient for far-field evaluation since the second term can be neglected as it decays faster than $1/r$; whereas equation (3.27) is more convenient for near-field evaluation since the the source singularity term $j\mathbf{M}_i/\omega\mu$ is explicitly given.

Because of the principle of superposition, the total field due to both electric and magnetic sources are

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F, \quad \mathbf{H} = \mathbf{H}_A + \mathbf{H}_F \tag{3.28}$$

3.4 Homogeneous Wave Equations

From the previous section, the total fields are given by (3.28), (3.13)–(3.16), and (3.24)–(3.27). More specifically,

$$\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = -j\omega\mathbf{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{A}) - \frac{1}{\epsilon}\nabla \times \mathbf{F} \tag{3.29}$$

$$\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla \times \mathbf{A} - j\omega\mathbf{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F}) \tag{3.30}$$

where for a source-free region,

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = 0 \tag{3.31a}$$

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = 0 \tag{3.31b}$$

The complex wave number is $k = \omega\sqrt{\mu\epsilon}$.

Equations (3.29) and (3.30) can be expanded in scalar forms. In Cartesian coordinates,

$$E_x = -j\omega A_x - \frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}\Big) - \frac{1}{\epsilon}\Big(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\Big) \tag{3.32a}$$

$$E_y = -j\omega A_y - \frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2 A_x}{\partial x\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\partial z}\Big) - \frac{1}{\epsilon}\Big(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\Big) \tag{3.32b}$$

$$E_z = -j\omega A_z - \frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2 A_x}{\partial x\partial z} + \frac{\partial^2 A_y}{\partial y\partial z} + \frac{\partial^2 A_z}{\partial z^2}\Big) - \frac{1}{\epsilon}\Big(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\Big) \tag{3.32c}$$

$$H_x = -j\omega F_x - \frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x\partial y} + \frac{\partial^2 F_z}{\partial x\partial z}\Big) + \frac{1}{\mu}\Big(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\Big) \tag{3.32d}$$

$$H_y = -j\omega F_y - \frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2 F_x}{\partial x\partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y\partial z}\Big) + \frac{1}{\mu}\Big(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\Big) \tag{3.32e}$$

$$H_z = -j\omega F_z - \frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2 F_x}{\partial x\partial z} + \frac{\partial^2 F_y}{\partial y\partial z} + \frac{\partial^2 F_z}{\partial z^2}\Big) + \frac{1}{\mu}\Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big) \tag{3.32f}$$

Similarly, in cylindrical coordinates, we have

$$E_\rho = -j\omega A_\rho - \frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial\rho}\Big[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\Big] - \frac{1}{\epsilon}\Big(\frac{1}{\rho}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\Big) \tag{3.33a}$$

$$E_\phi = -j\omega A_\phi - \frac{j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\Big] - \frac{1}{\epsilon}\Big(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\Big) \tag{3.33b}$$

$$E_z = -j\omega A_z - \frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial z}\Big[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}\Big] - \frac{1}{\epsilon}\frac{1}{\rho}\Big[\frac{\partial(\rho F_\phi)}{\partial\rho} - \frac{\partial F_\rho}{\partial\phi}\Big] \tag{3.33c}$$

$$H_\rho = -j\omega F_\rho - \frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial\rho}\Big[\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}\Big] + \frac{1}{\mu}\Big(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\Big) \tag{3.33d}$$

$$H_\phi = -j\omega F_\phi - \frac{j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big[\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}\Big] + \frac{1}{\mu}\Big(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\Big) \tag{3.33e}$$

$$H_z = -j\omega F_z - \frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial z}\Big[\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}\Big] + \frac{1}{\mu}\frac{1}{\rho}\Big[\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\Big] \tag{3.33f}$$

With respect to a given direction, we can define modes as the transverse electric (TE), transverse magnetic (TM), and transverse electromagnetic (TEM) modes, if the electric field, the magnetic field, or both are transverse to this direction, respectively. This direction is often chosen as the direction of propagation.

3.4.1 TEM, TM, and TE Modes in Cartesian Coordinates

A. TEM^z Modes

For TEM modes, both electric and magnetic fields are transverse to a given direction, say the z direction. In that case, $E_z = 0$ and $H_z = 0$ in (3.32c) and (3.32f). Let $A_x = A_y = 0$ and $F_x = F_y = 0$, then equations (3.32c) and (3.32f) reduce to

$$E_z = -\frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2}{\partial z^2} + k^2\Big)A_z = 0 \tag{3.34}$$

$$H_z = -\frac{j}{\omega\mu\epsilon}\Big(\frac{\partial^2}{\partial z^2} + k^2\Big)F_z = 0 \tag{3.35}$$

The solutions to (3.34) and (3.35) can be written in terms of waves propagating in $+z$ and $-z$ directions, i.e.,

$$A_z(x,y,z) = A_z^+(x,y)e^{-jkz} + A_z^-(x,y)e^{jkz} \tag{3.36}$$

$$F_z(x,y,z) = F_z^+(x,y)e^{-jkz} + F_z^-(x,y)e^{jkz} \tag{3.37}$$

Given $A_\pm$ and $F_\pm$, all other field components can be written by (3.32)

$$E_x = E_x^+e^{-jkz}+E_x^-e^{jkz} = (-\frac{1}{\sqrt{\mu\epsilon}}\frac{\partial A_z^+}{\partial x}-\frac{1}{\epsilon}\frac{\partial F_z^+}{\partial y})e^{-jkz}+(\frac{1}{\sqrt{\mu\epsilon}}\frac{\partial A_z^+}{\partial x}-\frac{1}{\epsilon}\frac{\partial F_z^-}{\partial y})e^{jkz} \tag{3.38}$$

$$E_y = E_y^+e^{-jkz}+E_y^-e^{jkz} = (-\frac{1}{\sqrt{\mu\epsilon}}\frac{\partial A_z^+}{\partial y}+\frac{1}{\epsilon}\frac{\partial F_z^+}{\partial x})e^{-jkz}+(\frac{1}{\sqrt{\mu\epsilon}}\frac{\partial A_z^+}{\partial y}+\frac{1}{\epsilon}\frac{\partial F_z^-}{\partial x})e^{jkz} \tag{3.39}$$

$$H_x = H_x^+e^{-jkz} + H_x^-e^{jkz} = -\sqrt{\frac{\epsilon}{\mu}}E_y^+e^{-jkz} + \sqrt{\frac{\epsilon}{\mu}}E_y^-e^{jkz} \tag{3.40}$$

$$H_y = H_y^+e^{-jkz} + H_y^-e^{jkz} = \sqrt{\frac{\epsilon}{\mu}}E_y^+e^{-jkz} - \sqrt{\frac{\epsilon}{\mu}}E_y^-e^{jkz} \tag{3.41}$$

The following ratios are called the wave impedance for the $+z$ and $-z$ propagating waves,

$$Z_w^+ = \frac{E_x^+}{H_y^+} = -\frac{E_y^+}{H_x^+} = \sqrt{\frac{\mu}{\epsilon}} \tag{3.42}$$

$$Z_w^- = -\frac{E_x^-}{H_y^-} = \frac{E_y^-}{H_x^-} = \sqrt{\frac{\mu}{\epsilon}} \tag{3.43}$$

Note that the signs of the ratio follow the right-hand rule. For example, for the propagation direction $(+\hat{z})$, the electric field component $(\hat{x}E_x)$ and magnetic field component $(\hat{y}H_y)$ form a right-hand rule.

Two other special cases (i) $A_z = 0$ but $F_z \neq 0$, and (ii) $A_z \neq 0$ but $F_z =$ can be obtained easily from the above.

B. TM^z Modes

For TM^z modes, we have $H_z = 0$ but $E_z \neq 0$. We let $\mathbf{F} = 0$ and $\mathbf{A} = \hat{z}A_z(x, y, z)$, then from (3.31a) we have

$$\nabla^2 A_z + k^2 A_z = 0 \tag{3.44}$$

Note that equation (3.44) is different from (3.34). It hence accepts a general solution of

$$A_z = [C_1 \cos(k_x x) + D_1 \sin(k_x x)][C_2 \cos(k_y y) + D_2 \sin(k_y y)](A_3 e^{-jk_z z} + B_3 e^{jk_z z}) \tag{3.45}$$

where

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{3.46}$$

Field components can be found from (3.32),

$$E_x = -\frac{j}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial x \partial z} \tag{3.47}$$

$$E_y = -\frac{j}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial y \partial z} \tag{3.48}$$

$$E_z = -\frac{j}{\omega\mu\epsilon}(\frac{\partial^2}{\partial z^2} + k^2)A_z \tag{3.49}$$

$$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} \tag{3.50}$$

$$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} \tag{3.51}$$

$$H_z = 0 \tag{3.52}$$

C. TE^z Modes

For TE^z modes, we have $E_z = 0$ but $H_z \neq 0$. We let $\mathbf{A} = 0$ and $\mathbf{F} = \hat{z}F_z(x, y, z)$, then from (3.31b) we have

$$\nabla^2 F_z + k^2 F_z = 0 \tag{3.53}$$

Note that equation (3.44) is different from (3.34). It hence accepts a general solution of

$$F_z = [C_1 \cos(k_x x) + D_1 \sin(k_x x)][C_2 \cos(k_y y) + D_2 \sin(k_y y)](A_3 e^{-jk_z z} + B_3 e^{jk_z z}) \tag{3.54}$$

where

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{3.55}$$

Field components can be found from (3.32),

$$E_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} \tag{3.56}$$

$$E_y = \frac{1}{\epsilon}\frac{\partial F_z}{\partial x} \tag{3.57}$$

$$E_z = 0 \tag{3.58}$$

$$H_x = -\frac{j}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial x \partial z} \tag{3.59}$$

$$H_y = -\frac{j}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial y \partial z} \tag{3.60}$$

$$H_z = -\frac{j}{\omega\mu\epsilon}(\frac{\partial^2}{\partial z^2} + k^2)F_z \tag{3.61}$$

In Cartesian coordinates, once the modes with respect to z direction are derived, those with respect to other directions are easily found.

3.4.2 TEM, TM, and TE Modes in Cylindrical Coordinates

A. TEM Modes

In cylindrical coordinates, TEM modes have different forms for ρ, ϕ, and z directions, in contrast to Cartesian coordinates. Therefore we will derive for different TEM modes. One equation useful is

$$\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial(\rho f)}{\partial\rho} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f}{\partial\rho} - \frac{f}{\rho^2} \tag{3.62}$$

a. TEM$^\rho$ Modes

For TEM$^\rho$ modes, $E_\rho = 0$ and $H_\rho = 0$. We choose $A_\phi = A_z = 0$ and $F_\phi = F_z = 0$. Therefore, from (3.33), we have

$$E_\rho = -j\omega A_\rho - \frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial\rho}\Big[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}\Big] = -\frac{j}{\omega\mu\epsilon}\Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + (k^2 - \frac{1}{\rho^2})\Big]A_\rho = 0 \tag{3.63}$$

$$H_\rho = -j\omega F_\rho - \frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial\rho}\Big[\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho}\Big] = -\frac{j}{\omega\mu\epsilon}\Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + (k^2 - \frac{1}{\rho^2})\Big]F_\rho = 0 \tag{3.64}$$

Note that equations (3.63) and (3.64) are Bessel differential equations with $n = 1$, and hence have solutions of the forms

$$A_\rho = A_\rho^+(\phi, z)H_1^{(2)}(k\rho) + A_\rho^-(\phi, z)H_1^{(1)}(k\rho) \tag{3.65}$$

$$F_\rho = F_\rho^+(\phi, z)H_1^{(2)}(k\rho) + F_\rho^-(\phi, z)H_1^{(1)}(k\rho) \tag{3.66}$$

The other field components are given by (3.33) as

$$E_\phi = -\frac{j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}\Big] - \frac{1}{\epsilon}\frac{\partial F_\rho}{\partial z} \tag{3.67a}$$

$$E_z = -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial z}\Big[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}\Big] + \frac{1}{\epsilon}\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi} \qquad (3.67b)$$

$$H_\phi = -\frac{j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big[\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho}\Big] + \frac{1}{\mu}\frac{\partial A_\rho}{\partial z} \qquad (3.67c)$$

$$H_z = -\frac{j}{\omega\mu\epsilon}\frac{\partial}{\partial z}\Big[\frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho}\Big] - \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi} \qquad (3.67d)$$

One important thing to note is that the wave impedances, if defined as the ratio between E_ϕ and H_z, or between $-E_z$ and H_ϕ, are functions of space; moreover, these two definitions yield different values. Only for $\rho \to \infty$ do these definitions converge to the same constant value.

b. TEM$^\phi$ Modes

For TEM$^\phi$ modes, $E_\phi = 0$ and $H_\phi = 0$. We choose $A_\rho = A_z = 0$ and $F_\rho = F_z = 0$. Therefore, from (3.33), we have

$$E_\phi = -\frac{j}{\omega\mu\epsilon}\Big[\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2\Big]A_\phi = 0 \qquad (3.68a)$$

$$H_\phi = -\frac{j}{\omega\mu\epsilon}\Big[\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2\Big]F_\phi = 0 \qquad (3.68b)$$

Unfortunately, one cannot find a solution to (3.68) which satisfies the periodic condition $A_\phi(\phi+2\pi) = A_\phi(\phi)$. This seems to suggest that TEM$^\phi$ Modes cannot exist in a homogeneous medium, unlike what the textbook has implied.

c. TEM^z Modes

The derivation of TEM^z modes is similar to that for Cartesian coordinates, and will not be repeated here. Note that for these modes, the wave impedance can be uniquely defined.

B. TM Modes

a. TM$^\rho$ Modes

For TM$^\rho$ modes, $H_\rho = 0$, so we choose $\mathbf{F} = 0$ and $\mathbf{A} = \hat{\rho}A_\rho$. The vector wave equation is $\nabla^2\mathbf{A} + k^2\mathbf{A} = 0$. However, in cylindrical coordinates,

$$\begin{aligned}\nabla^2\hat{\rho}A_\rho &= \Big(\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\rho\tfrac{\partial}{\partial\rho} + \tfrac{1}{\rho^2}\tfrac{\partial^2}{\partial\phi^2} + \tfrac{\partial^2}{\partial z^2}\Big)\hat{\rho}A_\rho \\ &= \hat{\rho}\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\rho\tfrac{\partial A_\rho}{\partial\rho} + \tfrac{1}{\rho^2}\tfrac{\partial}{\partial\phi}(\hat{\rho}\tfrac{\partial A_\rho}{\partial\phi} + \hat{\phi}A_\rho) + \hat{\rho}\tfrac{\partial^2 A_\rho}{\partial z^2} \\ &= \hat{\rho}\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\rho\tfrac{\partial A_\rho}{\partial\rho} + \tfrac{1}{\rho^2}\Big[\hat{\rho}\tfrac{\partial^2 A_\rho}{\partial\phi^2} + \hat{\phi}2\tfrac{\partial A_\rho}{\partial\phi} - \hat{\rho}A_\rho\Big] + \hat{\rho}\tfrac{\partial^2 A_\rho}{\partial z^2} \\ &= \hat{\rho}\Big[\tfrac{1}{\rho}\tfrac{\partial}{\partial\rho}\rho\tfrac{\partial A_\rho}{\partial\rho} - \tfrac{A_\rho}{\rho^2} + \tfrac{1}{\rho^2}\tfrac{\partial^2 A_\rho}{\partial\phi^2} + \tfrac{\partial^2 A_\rho}{\partial z^2}\Big] + \hat{\phi}\tfrac{2}{\rho^2}\tfrac{\partial A_\rho}{\partial\phi}\end{aligned} \qquad (3.68)$$

Therefore, we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_\rho}{\partial\rho} + (k^2 - \frac{1}{\rho^2})A_\rho + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial\phi^2} + \frac{\partial^2 A_\rho}{\partial z^2} = 0 \qquad (3.69a)$$

$$\frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi} = 0 \tag{3.69b}$$

From (3.69b), it is seen that A_ρ is not a function of ϕ. The solution to (3.69a) is

$$A_\rho = [A_1 H_1^{(2)}(k_\rho \rho) + B_1 H_1^{(1)}(k_\rho \rho)][A_3 e^{-jk_z z} + B_3 e^{jk_z z}] \tag{3.70}$$

where $k_z^2 + k_\rho^2 = k^2$. Other field components are

$$E_\rho = -\frac{j}{\omega\mu\epsilon}[\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial A_\rho}{\partial \rho} - \frac{A_\rho}{\rho^2} + k^2 A_\rho] \tag{3.71a}$$

$$E_\phi = 0 \tag{3.71b}$$

$$E_z = -\frac{j}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial(\rho A_\rho)}{\partial \rho} \tag{3.71c}$$

$$H_\rho = 0 \tag{3.71d}$$

$$H_\phi = \frac{1}{\mu}\frac{\partial A_\rho}{\partial z} \tag{3.71e}$$

$$H_z = 0 \tag{3.71f}$$

b. TM$^\phi$ Modes

TM$^\phi$ modes can be derived in the same way as for equations (3.69)–(3.71).

c. TM^z Modes

TM^z modes are derived in the textbook and are similar to those in Cartesian coordinates.

$$A_z = [A_1 H_m^{(2)}(k_\rho \rho) + B_1 H_m^{(1)}(k_\rho \rho)][A_2 \cos m\phi + B_2 \sin m\phi][A_3 e^{-jk_z z} + B_3 e^{jk_z z}] \tag{3.72}$$

$$E_\rho = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial \rho \partial z} \tag{3.73a}$$

$$E_\phi = \frac{-j}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial \phi \partial z} \tag{3.73b}$$

$$E_z = \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z \tag{3.73c}$$

$$H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} \tag{3.73d}$$

$$H_\phi = -\frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \rho} \tag{3.73e}$$

$$H_z = 0 \tag{3.73f}$$

C. TE Modes

a. TE$^\rho$ Modes
The derivation for TE$^\rho$ modes is similar to (3.69)–(3.71) and will not be repeated here.

b. TE$^\phi$ Modes
For TE$^\phi$ modes, we choose $\mathbf{A} = 0$ and $\mathbf{F} = \hat{\phi} F_\phi$. Then equation $\nabla^2 \mathbf{F} + k^2 \mathbf{F} = 0$ becomes

$$\hat{\phi}\Big[\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial F_\phi}{\partial \rho} + (k^2 - \frac{1}{\rho^2})F_\phi + \frac{1}{\rho^2}\frac{\partial^2 F_\phi}{\partial \phi^2} + \frac{\partial^2 F_\phi}{\partial z^2}\Big] - \hat{\rho}\frac{2}{\rho}\frac{\partial F_\phi}{\partial \phi} = 0 \qquad (3.74)$$

The solution of (3.74) is

$$F_\phi = [A_1 H_1^{(2)}(k_\rho \rho) + B_1 H_1^{(1)}(k_\rho \rho)][A_3 e^{-jk_z z} + B_3 e^{jk_z z}] \qquad (3.75)$$

where $k_z^2 + k_\rho^2 = k^2$. Other components are derivable through (3.33).

c. TE^z Modes
TE^z modes are dual to equations (3.72) and (3.73).

3.5 Inhomogeneous Wave Equations

In this section we aim at solving equations

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}(\mathbf{r}) \qquad (3.76)$$

$$\nabla^2 \mathbf{F} + k^2 \mathbf{F} = -\epsilon \mathbf{M}(\mathbf{r}) \qquad (3.77)$$

where the sources are arbitrary functions of space. Note for convenience, we have dropped the subscript i from the impressed sources. We will first solve for these vector functions due to a point source, and then extend the results to arbitrary sources.

3.5.1 A Point Electric Dipole Source

First we look at a z directed electric dipole source $\mathbf{J} = \hat{z} J_0 \delta(\mathbf{r})$. Then (3.76) becomes

$$\nabla^2 A_z + k^2 A_z = -\mu J_0 \delta(\mathbf{r}) \qquad (3.78)$$

The solution of $A_z(x, y, z)$ should depend on $r = \sqrt{x^2 + y^2 + z^2}$ only. Further, if we write

$$A_z = \frac{f(r)}{r} \qquad (3.79)$$

then (3.78) becomes

$$\frac{d^2 f}{dr^2} + k^2 f = -\mu J_0 r \delta(\mathbf{r}) \qquad (7.80)$$

The general solution to (7.80) is

$$f(r) = C_1 e^{-jkr} + C_2 e^{jkr} \tag{3.81}$$

where C_1 and C_2 still have to be determined. For a out-going wave, we choose $C_2 = 0$. Then

$$A_z(r) = C_1 \frac{e^{-jkr}}{r} \tag{3.82}$$

Integrating (3.78) over a sphere of radius $a \to 0$, and using

$$\nabla A_z = \hat{r}\frac{\partial A_z}{\partial r} = \hat{r}(-\frac{1}{r^2} + \frac{-jk}{r})C_1 e^{-jkr}$$

and Gauss theorem, we have

$$\oint_S \nabla A_z \cdot \hat{r} ds + k^2 \int_V A_z dv = -\mu J_0.$$

That is,

$$4\pi a^2(\frac{-1}{a^2} - \frac{jk}{a})C_1 e^{-jkr} + k^2 \int_0^a 4\pi r^2 C_1 \frac{e^{-jkr}}{r} dr = -\mu J_0$$

As $a \to 0$, the second term approaches zero. So

$$-4\pi C_1 = -\mu J_0, \qquad C_1 = \frac{\mu J_0}{4\pi}$$

Hence, solution (3.82) is

$$A_z = \frac{\mu J_0}{4\pi r} e^{-jkr} \tag{3.83}$$

For a point electric dipole in an arbitrary direction at origin, $\mathbf{J} = \hat{a} J_0 \delta(\mathbf{r})$, we then have

$$\mathbf{A}(\mathbf{r}) = \hat{a}\frac{\mu J_0}{4\pi r} e^{-jkr} \tag{3.84}$$

For a point dipole source away from the origin, $\mathbf{J} = \hat{a} J_0 \delta(\mathbf{r} - \mathbf{r}')$,

$$\mathbf{A}(\mathbf{r}) = \hat{a}\frac{\mu J_0}{4\pi R} e^{-jkR}, \qquad R = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{3.85}$$

where R is the distance between the source and the observation point.

3.5.2 Arbitrary Volume Sources

By the principle of superposition and equation (3.84), if the electric current density is $\mathbf{J}(\mathbf{r})$, the magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dv' \tag{3.86}$$

Similarly, the electric vector potential due to an arbitrary volume magnetic current source $\mathbf{M}(\mathbf{r})$ is given by

$$\mathbf{F}(\mathbf{r}) = \frac{\epsilon}{4\pi} \int_V \mathbf{M}(\mathbf{r}') \frac{e^{-jkR}}{R} dv' \tag{3.87}$$

which is the solution to equation (3.77).

3.5.3 Arbitrary Surface Sources

In many applications, surface currents $\mathbf{J}_s$ and $\mathbf{M}_s$ exist on a surface S described by equation $S(\mathbf{r}) = 0$. Recall that the surface current densities are defined as

$$\mathbf{J}_s(\mathbf{r}) = \lim_{\Delta h \to 0} \mathbf{J}(\mathbf{r})\Delta h, \qquad \mathbf{M}_s(\mathbf{r}) = \lim_{\Delta h \to 0} \mathbf{M}(\mathbf{r})\Delta h,$$

where Δh is the thickness of a thin surface layers parallel to the surface S. Hence, in terms of general functions, the current densities are

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}_s(\mathbf{r})\delta[S(\mathbf{r})], \quad \mathbf{M}(\mathbf{r}) = \mathbf{M}_s(\mathbf{r})\delta[S(\mathbf{r})] \tag{3.88}$$

where $S(\mathbf{r})$ is the equation describing the surface S. The magnetic and electric vector potentials due to these sources are obtained from (3.86) and (3.87), and reduced to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_S \mathbf{J}_s(\mathbf{r}') \frac{e^{-jkR}}{R} ds' \tag{3.88}$$

$$\mathbf{F}(\mathbf{r}) = \frac{\epsilon}{4\pi} \int_S \mathbf{M}_s(\mathbf{r}') \frac{e^{-jkR}}{R} ds' \tag{3.89}$$

3.5.4 Arbitrary Line Sources

Similarly, sometimes line currents $\mathbf{I}_e$ and $\mathbf{I}_m$ exist on a curve C described by the intersection of two surfaces $S_1(\mathbf{r}) = 0$ and $S_2(\mathbf{r}) = 0$. In terms of general functions, the current densities are

$$\mathbf{J}(\mathbf{r}) = \mathbf{I}_e(\mathbf{r})\delta[S_1(\mathbf{r})]\delta[S_2(\mathbf{r})], \quad \mathbf{M}(\mathbf{r}) = \mathbf{I}_m(\mathbf{r})\delta[S_1(\mathbf{r})]\delta[S_2(\mathbf{r})] \tag{3.90}$$

The magnetic and electric vector potentials due to these sources are obtained from (3.86) and (3.87), and reduced to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int_C \mathbf{I}_e(\mathbf{r}') \frac{e^{-jkR}}{R} dl' \tag{3.91}$$

$$\mathbf{F}(\mathbf{r}) = \frac{\epsilon}{4\pi} \int_C \mathbf{I}_m(\mathbf{r}') \frac{e^{-jkR}}{R} dl' \tag{3.92}$$

3.5.5 An Infinitesimal Electric Dipole

As infinitesimal electric dipole can be either regarded as a special case of (3.90) with $\mathbf{I}_e(z) = \hat{z} I_0 \ell \delta(z)$, or as a special case of (3.84) with $J_0 = I_0 \ell$. The result is

$$\mathbf{A} = \hat{z} A_z = \hat{z} \frac{\mu I_0 \ell}{4\pi r} e^{-jkr}. \tag{3.93}$$

To obtain the expressions for spherical coordinates, we use the transformation

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = T \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}, \qquad T = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \tag{3.94}$$

which gives

$$A_r = A_z \cos\theta = \frac{\mu I_0 \ell}{4\pi r} \cos\theta e^{-jkr}, \quad A_\theta = -A_z \sin\theta = -\frac{\mu I_0 \ell}{4\pi r} \sin\theta e^{-jkr}, \quad A_\phi = 0 \tag{3.96}$$

The electric and magnetic fields are

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \hat{\phi} j \frac{k I_0 \ell \sin\theta}{4\pi r} (1 - \frac{j}{kr}) e^{-jkr} \tag{3.97}$$

$$\mathbf{E} = \hat{r} \eta \frac{I_0 \ell \cos\theta}{2\pi r^2} (1 - \frac{j}{kr}) e^{-jkr} + \hat{\theta} j \eta \frac{k I_0 \ell \sin\theta}{4\pi r} [1 - \frac{j}{kr} - \frac{1}{k^2 r^2}] e^{-jkr} \tag{3.98a}$$

where $\eta = \sqrt{\mu/\epsilon}$. However, if we use equation (3.16), i.e., $\mathbf{E}_A = -\frac{j}{\omega\epsilon}(\nabla \times \mathbf{H}_A - \mathbf{J}_i)$, we obtain

$$\mathbf{E} = \hat{r} \eta \frac{I_0 \ell \cos\theta}{2\pi r^2} (1 - \frac{j}{kr}) e^{-jkr} + \hat{\theta} j \eta \frac{k I_0 \ell \sin\theta}{4\pi r} [1 - \frac{j}{kr} - \frac{1}{k^2 r^2}] e^{-jkr} + \hat{z} \frac{j I_0 \ell}{\omega\epsilon} \delta(\mathbf{r}). \tag{3.98}$$

Note that (3.98) and (3.98a) differ by the last term of (3.98), which corresponds to the singularity of the electric field. This term has been missed in the differentiation of the vector potential leading to (3.97) and (3.98a), showing the potential pitfall of the approach using vector potentials. Of course, such a difficulty leading to (3.96a) can be overcome if one does the differentiation of the vector potentials *properly*, by making use of the PDE (3.78) for A_z.

3.6 Radiation and Scattering Equations

3.6.1 Near Field Equations

First we consider fields due to an electric current source $\mathbf{J}$. The magnetic vector potential is given by (3.86). Then the magnetic field is

$$\mathbf{H}_A = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{-1}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \times \nabla \Big(\frac{e^{-jkR}}{R} \Big) dv' \tag{3.99}$$

where identity $\nabla \times (g\mathbf{J}) = (\nabla g) \times \mathbf{J} + g\nabla \times \mathbf{J}$ has been used. Therefore,

$$\mathbf{H}_A = \frac{1}{4\pi} \int_V [\mathbf{J}(\mathbf{r}') \times \mathbf{R}] \frac{1 + jkR}{R^3} e^{-jkR} dv' \tag{3.100}$$

Noting that

$$\nabla \times (\mathbf{J} \times \mathbf{R}) = -(\mathbf{J} \cdot \nabla)\mathbf{R} + \mathbf{J}\nabla \cdot \mathbf{R} + (\mathbf{R} \cdot \nabla)\mathbf{J} - \mathbf{R}\nabla \cdot \mathbf{J}$$

$$\nabla f(\mathbf{R}) = f'(R)\frac{\mathbf{R}}{R}$$

we can find the electric field

$$\begin{aligned} \mathbf{E}_A = & \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H}_A - \frac{1}{j\omega\epsilon}\mathbf{J}(\mathbf{r}) \\ = & -\frac{j}{4\pi\omega\epsilon} \int_V \Big\{ -(\mathbf{J} \times \mathbf{R}) \times \nabla\Big[\frac{1+jkR}{R^3} e^{-jkR}\Big] + \frac{1+jkR}{R^3} e^{-jkR} \nabla \times (\mathbf{J} \times \mathbf{R}) \Big\} dv' \\ & -\frac{1}{j\omega\epsilon}\mathbf{J}(\mathbf{r}) \\ = & -\frac{j\eta}{4\pi k} \int_V \Big\{ G_1 \mathbf{J}(\mathbf{r}') + G_2 \mathbf{R}[\mathbf{R} \cdot \mathbf{J}(\mathbf{r}')] \Big\} e^{-jkR} dv' - \frac{1}{j\omega\epsilon}\mathbf{J}(\mathbf{r}) \end{aligned} \tag{3.101}$$

where

$$G_1 = \frac{-1 - jkR + k^2R^2}{R^3}, \qquad G_2 = \frac{3 + j3kR - k^2R^2}{R^5} \tag{3.102}$$

Similar expressions can be obtained for the fields due to the magnetic current source $\mathbf{M}$:

$$\mathbf{E}_F = -\frac{1}{4\pi} \int_V [\mathbf{M}(\mathbf{r}') \times \mathbf{R}] \frac{1 + jkR}{R^3} e^{-jkR} dv' \tag{3.103}$$

$$\mathbf{H}_F = -\frac{j}{4\pi k\eta} \int_V \Big\{ G_1 \mathbf{M}(\mathbf{r}') + G_2 \mathbf{R}[\mathbf{R} \cdot \mathbf{M}(\mathbf{r}')] \Big\} e^{-jkR} dv' - \frac{1}{j\omega\mu}\mathbf{M}(\mathbf{r}) \tag{3.104}$$

These equations are valid everywhere in arbitrary coordinates. The textbook shows the expressions for Cartesian coordinates. Note that the expressions given for $\mathbf{E}_A$ and $\mathbf{H}_F$ are only valid *away* from the source since they do not include the source singularity terms as in (3.101) and (3.104).

3.6.2 Far Field

For the far field zone, we can approximate

$$R = [r^2 + r'^2 - 2rr' \cos\psi]^{1/2} \begin{cases} r - r' \cos\psi, & \text{for phase variations} \\ r, & \text{for amplitude variations} \end{cases} \tag{3.105}$$

if $r >> \lambda$ and $r \geq 2D^2/\lambda$ where D is the largest dimension of the radiator or scatterer, and $\cos\psi = \hat{r} \cdot \hat{r}'$ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Under this approximation,

$$\mathbf{A} \approx \frac{\mu}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \frac{e^{-jk(r - r' \cos\psi)}}{r} dv' = \frac{\mu e^{-jkr}}{4\pi r}\mathbf{N} \tag{3.106}$$

$$\mathbf{F} \approx \frac{\epsilon}{4\pi} \int_V \mathbf{M}(\mathbf{r}') \frac{e^{-jk(r-r'\cos\psi)}}{r} dv' = \frac{\epsilon e^{-jkr}}{4\pi r}\mathbf{L} \tag{3.107}$$

where

$$\mathbf{N} = \int_V \mathbf{J}(\mathbf{r}') e^{jkr'\cos\psi} dv', \quad \mathbf{L} = \int_V \mathbf{M}(\mathbf{r}') e^{jkr'\cos\psi} dv' \tag{3.108}$$

For the surface current densities, we have

$$\mathbf{N} = \int_S \mathbf{J}_s(\mathbf{r}') e^{jkr'\cos\psi} ds', \quad \mathbf{L} = \int_S \mathbf{M}_s(\mathbf{r}') e^{jkr'\cos\psi} ds' \tag{3.109}$$

And for line current sources

$$\mathbf{N} = \int_C \mathbf{I}_e(\mathbf{r}') e^{jkr'\cos\psi} d\ell', \quad \mathbf{L} = \int_C \mathbf{I}_m(\mathbf{r}') e^{jkr'\cos\psi} d\ell' \tag{3.110}$$

Now let us write these vectors **N** and **L** in component forms

$$\mathbf{N} = \hat{r}N_r(\theta,\phi) + \hat{\theta}N_\theta(\theta,\phi) + \hat{\phi}N_\phi(\theta,\phi) \tag{3.111}$$

$$\mathbf{L} = \hat{r}L_r(\theta,\phi) + \hat{\theta}L_\theta(\theta,\phi) + \hat{\phi}L_\phi(\theta,\phi) \tag{3.112}$$

Using $E_A = -\frac{j}{\omega\mu\epsilon}[\nabla(\nabla\cdot\mathbf{A}) + k^2\mathbf{A}]$, we have

$$\begin{aligned} \mathbf{E}_A = \quad & \hat{r}\left[\tfrac{0}{r} + \sum_{n=2,3,\cdots} \tfrac{C_{rn}}{r^n}\right] e^{-jkr} \\ & +\hat{\theta}\left[\tfrac{-j\omega\mu}{4\pi r}N_\theta + \sum_{n=2,3,\cdots} \tfrac{C_{\theta n}}{r^n}\right] e^{-jkr} \\ & +\hat{\phi}\left[\tfrac{-j\omega\mu}{4\pi r} + \sum_{n=2,3,\cdots} \tfrac{C_{\phi n}}{r^n}\right] e^{-jkr} \end{aligned} \tag{3.113}$$

Keeping only the $(1/r)$ terms, we obtain

$$\mathbf{E}_A \approx -\hat{\theta}j\omega A_\theta - \hat{\phi}j\omega A_\phi = -j\omega[\mathbf{A} - \hat{r}(\mathbf{A}\cdot\hat{r})] \tag{3.114a}$$

Note to make sure that only the θ and ϕ components are calculated using the about equation, we can write (3.114a) as

$$\mathbf{E}_A \approx -\hat{\theta}j\omega A_\theta - \hat{\phi}j\omega A_\phi = -j\omega[\hat{\theta}(\hat{\theta}\cdot\mathbf{A}) + \hat{\phi}(\hat{\phi}\cdot\mathbf{A})] \tag{3.114}$$

Similarly, for far fields due to **M**,

$$\mathbf{H}_F \approx -\hat{\theta}j\omega F_\theta - \hat{\phi}j\omega F_\phi = -j\omega[\mathbf{F} - \hat{r}(\mathbf{F}\cdot\hat{r})] \tag{3.115a}$$

or

$$\mathbf{H}_F \approx -\hat{\theta}j\omega F_\theta - \hat{\phi}j\omega F_\phi = -j\omega[\hat{\theta}(\hat{\theta}\cdot\mathbf{A}) + \hat{\phi}(\hat{\phi}\cdot\mathbf{A})] \tag{3.115}$$

Once $\mathbf{E}_A$ and $\mathbf{H}_F$ are obtained by above equations, $\mathbf{H}_A$ and $\mathbf{E}_F$ can be obtained by Maxwell's equations. Keeping only the $(1/r)$ terms, we have

$$H_{Ar} = \frac{j}{\omega\mu}\frac{1}{r\sin\theta}[\frac{\partial}{\partial\theta}\sin\theta E_{A\phi} - \frac{\partial E_A\theta}{\partial\phi}] \approx 0 \tag{3.116}$$

$$H_{A\theta} = \frac{j}{\omega\mu}\frac{1}{r}[\frac{1}{\sin\theta}\frac{\partial E_{Ar}}{\partial\phi} - \frac{\partial(rE_A\phi)}{\partial r}] \approx -\hat{\theta}\frac{j}{\omega\mu}\frac{\partial E_{A\phi}}{\partial r} = -\hat{\theta}\frac{E_{A\phi}}{\eta} \tag{3.117}$$

$$H_{A\phi} = \frac{j}{\omega\mu}\frac{1}{r}[\frac{\partial(rE_A\theta)}{\partial r} - \frac{\partial E_{Ar}}{\partial\theta} \approx \hat{\phi}\frac{j}{\omega\mu}\frac{\partial E_{A\theta}}{\partial r} = \hat{\theta}\frac{E_{A\theta}}{\eta} \tag{3.118}$$

or

$$\mathbf{H}_A = \frac{1}{\eta}\hat{r}\times\mathbf{E}_A = -\frac{j\omega}{\eta}\hat{r}\times\mathbf{A}. \tag{3.119}$$

Similarly,

$$\mathbf{E}_F = -\eta\hat{r}\times\mathbf{H}_F = -\eta\hat{r}\times\mathbf{F}. \tag{3.120}$$

Therefore, the total far fields in terms of $\mathbf{N}$ and $\mathbf{L}$ vectors are

$$\begin{cases} E_r & \approx 0 \\ E_\theta & \approx -\frac{jke^{-jkr}}{4\pi r}(L_\phi + \eta N_\theta) \\ E_\phi & \approx \frac{jke^{-jkr}}{4\pi r}(L_\theta - \eta N_\phi) \end{cases} \tag{3.121}$$

$$\begin{cases} H_r & \approx 0 \\ H_\theta & \approx \frac{jke^{-jkr}}{4\pi r}(N_\phi - \frac{L_\theta}{\eta}) \\ H_\phi & \approx -\frac{jke^{-jkr}}{4\pi r}(N_\theta + \frac{L_\phi}{\eta}) \end{cases} \tag{3.122}$$

A. Evaluation of N and L in Cartesian Coordinates

For radiators or scatterers with rectangular geometries, it is more convenient to perform the integration in Cartesian coordinates. But the observation point (components) should be represented in spherical coordinates. Note that

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = T_{sr}\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}, \quad T_{sr} = \begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix} \tag{3.123}$$

We then have in spherical coordinates

$$\begin{aligned}\mathbf{N} &= \int_V dv' e^{jkr'\cos\psi}\,[J_x \quad J_y \quad J_z]\,T_{sr}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} \\ &= \int_V dv' e^{jkr'\cos\psi}\begin{bmatrix}J_x\sin\theta\cos\phi + J_y\sin\theta\sin\phi + J_z\cos\theta\\ J_x\cos\theta\cos\phi + J_y\cos\theta\sin\phi - J_z\sin\theta\\ -J_x\sin\phi + J_y\cos\phi\end{bmatrix}^t\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}\end{aligned} \tag{3.124}$$

Similarly

$$\mathbf{L} = \int_V dv' e^{jkr'\cos\psi} [\, M_x \quad M_y \quad M_z \,] T_{sr} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

$$= \int_V dv' e^{jkr'\cos\psi} \begin{bmatrix} M_x \sin\theta\cos\phi + M_y \sin\theta\sin\phi + M_z\cos\theta \\ M_x \cos\theta\cos\phi + M_y\cos\theta\sin\phi - M_z\sin\theta \\ -M_x\sin\phi + M_y\cos\phi \end{bmatrix}^t \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \quad (3.125)$$

In the above,

$$dv' = dx'dy'dz' \quad (3.126)$$

$$r'\cos\psi = \mathbf{r}' \cdot \hat{r} = x'\sin\theta\cos\phi + y'\sin\theta\sin\phi + z'\cos\theta \quad (3.127)$$

In general, for the far field we only need the θ and ϕ components of $\mathbf{N}$ and $\mathbf{L}$.

If the surface current densities are present, then the above volume integration need to be changed to surface integration. Those can be easily derived from above.

B. Evaluation of N and L in Cylindrical Coordinates

In cylindrical coordinates, we first note

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = T_{rc} \begin{bmatrix} J_\rho \\ J_\phi \\ J_z \end{bmatrix}, \quad T_{rc} = \begin{bmatrix} \cos\phi' & -\sin\phi' & 0 \\ \sin\phi' & \cos\phi' & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.128)$$

Then

$$\begin{aligned} \mathbf{J} &= [\, J_x \quad J_y \quad J_z \,] \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\ &= [\, J_\rho \quad J_\phi \quad J_z \,] T_{rc}^t T_{sr} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \\ &= [\, J_\rho \quad J_\phi \quad J_z \,] \begin{bmatrix} \sin\theta\cos(\phi-\phi') & \cos\theta\cos(\phi-\phi') & -\sin(\phi-\phi') \\ \sin\theta\sin(\phi-\phi') & \cos\theta\sin(\phi-\phi') & \cos(\phi-\phi') \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \end{aligned}$$

Thus,

$$\mathbf{N} = \int_V dv' e^{jkr'\cos\psi}$$

$$\cdot \begin{bmatrix} J_\rho \sin\theta\cos(\phi-\phi') + J_\phi\sin\theta\sin(\phi-\phi') + J_z\cos\theta \\ J_\rho\cos\theta\cos(\phi-\phi') + J_\phi\cos\theta\sin(\phi-\phi') - J_z\sin\theta \\ -J_\rho\sin(\phi-\phi') + J_\phi\cos(\phi-\phi') \end{bmatrix}^t \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \quad (3.129)$$

where

$$dv' = \rho' d\rho' d\phi' dz' \tag{3.130}$$

$$r' \cos\psi = x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta = \rho' \sin\theta \cos(\phi - \phi') + z' \cos\theta \tag{3.131}$$

Again, the expression in (3.129) is a volume integral. For a surface current, it should be changed to a surface integral. For vector **L**, one needs only to replace **J** by **M** in (3.129):

$$\mathbf{L} = \int_V dv' e^{jkr'\cos\psi} \cdot \begin{bmatrix} M_\rho \sin\theta \cos(\phi - \phi') + M_\phi \sin\theta \sin(\phi - \phi') + M_z \cos\theta \\ M_\rho \cos\theta \cos(\phi - \phi') + M_\phi \cos\theta \sin(\phi - \phi') - M_z \sin\theta \\ -M_\rho \sin(\phi - \phi') + M_\phi \cos(\phi - \phi') \end{bmatrix}^t \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{3.132}$$

Some useful integrals are listed below:

$$\int_{-c/2}^{c/2} e^{j\alpha z} dz = c \cdot \frac{\sin(\alpha c/2)}{\alpha c/2} \tag{3.133}$$

$$\int_{\alpha}^{2\pi+\alpha} e^{jz\cos(\phi-\phi')} d\phi' = 2\pi J_0(z) \tag{3.134}$$

$$\int_0^c z J_0(z) dz = z J_1(z)|_0^c = c J_1(c) \tag{3.135}$$

$$\int x^{1-p} J_p(\alpha x) dx = -\frac{1}{\alpha} x^{1-p} J_{p-1}(\alpha x) \tag{3.136}$$