Chapter 4

WAVE PROPAGATION AND POLARIZATION

In this chapter (Chapter 4 in textbook) we study the propagation of TEM waves in general lossy media and their polarizations.

In contrast to the textbook, we first consider the general case for lossy media by using the complex permittivity and complex wavenumber. The lossless case is just a special case where $\sigma = 0$, $\sigma_m = 0$ and ϵ, μ and k become real.

For a TEM wave, both $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to a certain direction, for example $\hat{a}$, where $\hat{a}$ can vary with space. At every point in space, the $\mathbf{E}$ and $\mathbf{H}$ vectors form an equiphase plane. These equiphase planes are in general not parallel to each other at different points in space.

If these equiphase planes are parallel to each other everywhere, then these TEM waves are plane waves. In addition, if these planes are also the equiamplitude planes, then the wave is called *uniform plane waves.* On such a plane, the field is not a function of position.

4.1 Uniform Plane Waves in a Homogeneous Medium

The homogeneous wave equation for $\mathbf{E}$ in a homogeneous medium is

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0, \quad k^2 = \omega^2 \tilde{\mu} \tilde{\epsilon} \tag{4.1}$$

where $\tilde{\epsilon}$ $\tilde{\mu}$ are the complex permittivity and permeability. In the rest of the class, this tilde will be omitted, keeping in mind that in general ϵ and μ can be complex.

The general solution of this equation is obtained in Chapter 3. In particular, for a uniform plane wave, we can write

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0^+ e^{-jk\hat{k}\cdot\mathbf{r}} + \mathbf{E}_0^- e^{jk\hat{k}\cdot\mathbf{r}} \equiv \mathbf{E}^+(\mathbf{r}) + \mathbf{E}^-(\mathbf{r}) \tag{4.2}$$

where the complex wavenumber $k = k' + jk'' = \beta - j\alpha = \omega\sqrt{\mu\epsilon}$ will take the following values

$$k' = \beta \geq 0, \quad -k'' = \alpha \geq 0. \tag{4.3}$$

As discussed in Chapter 3, the choice of (4.3) will ensure that $\mathbf{E}^+(\mathbf{r}) = \mathbf{E}_0^+ e^{-jk\hat{k}\cdot\mathbf{r}}$ is a wave propagating in $+\hat{k}$ direction, while $\mathbf{E}^-(\mathbf{r}) = \mathbf{E}_0^- e^{jk\hat{k}\cdot\mathbf{r}}$ is a wave propagating in $-\hat{k}$ direction. Alternatively, we can define the complex propagation constant

$$\gamma = \alpha + j\beta = jk = -k'' + jk', \quad \begin{cases} \alpha = -k'' \geq 0 \\ \beta = k' \geq 0 \end{cases} \tag{4.4}$$

where α is called the attenuation constant, and β the phase constant.

Given the plane wave expression (4.2), we can replace ∇ operator by $\mp j\mathbf{k} = \mp jk\hat{k}$ depending on whether its a $+\hat{k}$ or $-\hat{k}$ propagating wave (i.e., $\mathbf{E}^+$ or $\mathbf{E}^-$). Then from Gauss' law we have

$$\nabla \cdot \mathbf{E}^\pm(\mathbf{r}) = \mp j\mathbf{k} \cdot \mathbf{E}^\pm = 0 \tag{4.5}$$

that is, $\mathbf{E}$ is perpendicular to the propagation direction ($\pm\hat{k}$).

The corresponding magnetic field can be obtained by Faraday' law:

$$\mathbf{H}^\pm(\mathbf{r}) = \frac{j}{\omega\mu}\nabla \times \mathbf{E}^\pm = \frac{\pm\mathbf{k}}{\omega\mu} \times \mathbf{E}^\pm = \frac{1}{\eta}(\pm\hat{\mathbf{k}}) \times \mathbf{E}^\pm \equiv \mathbf{H}_0^\pm e^{\mp j\mathbf{k}\cdot\mathbf{r}} \tag{4.6}$$

Noting that the form of the plane waves propagating in $\hat{k}$ and $-\hat{k}$ are the same, we can summarize the expressions for uniform plane waves in a homogeneous medium as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{H}(\mathbf{r}) = \frac{1}{\eta}\hat{k} \times \mathbf{E} \tag{4.7}$$

Conversely, if $\mathbf{H}$ is know, we have

$$\mathbf{E}(\mathbf{r}) = -\eta\hat{k} \times \mathbf{H} \tag{4.8}$$

4.1.1 Wave Impedance

The wave impedance of plane wave is

$$Z_w = \eta = \sqrt{\frac{\tilde{\mu}}{\tilde{\epsilon}}} = \sqrt{\frac{\mu - \frac{j\sigma_m}{\omega}}{\epsilon - \frac{j\sigma}{\omega}}} = |\eta|e^{j\theta_\eta} \tag{4.9}$$

where

$$|\eta| = \left(\frac{\mu^2 + \frac{\sigma_m^2}{\omega^2}}{\epsilon^2 + \frac{\sigma^2}{\omega^2}}\right)^{1/2}, \quad \theta_\eta = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\sigma_m}{\omega\mu}\right). \tag{4.10}$$

4.1.2 Phase and Group Velocities, Power and Energy Densities

The complex wavenumber is

$$k = \sqrt{(\omega\mu - j\sigma_m)(\omega\epsilon - j\sigma)} = k' + jk'' = \beta - j\alpha \tag{4.11}$$

where β and α can be obtained from the parameters μ, ϵ, σ and σ_m

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{\left(1-\frac{\sigma\sigma_m}{\omega^2\mu\epsilon}\right)^2 + \left(\frac{\sigma}{\omega\epsilon}+\frac{\sigma_m}{\omega\mu}\right)^2} - \left(1-\frac{\sigma\sigma_m}{\omega^2\mu\epsilon}\right)\right]^{1/2}, \tag{4.12a}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{\left(1-\frac{\sigma\sigma_m}{\omega^2\mu\epsilon}\right)^2 + \left(\frac{\sigma}{\omega\epsilon}+\frac{\sigma_m}{\omega\mu}\right)^2} + \left(1-\frac{\sigma\sigma_m}{\omega^2\mu\epsilon}\right)\right]^{1/2} \tag{4.12b}$$

For the special case where $\sigma_m = 0$, we have

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{1/2}, \tag{4.12c}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2} \tag{4.12d}$$

The phasor notations of $\mathbf{E}$ and $\mathbf{H}$ given in (4.7) are

$$\mathbf{E} = \mathbf{E}_0 e^{-\alpha\hat{k}\cdot\mathbf{r}} e^{-j\beta\hat{k}\cdot\mathbf{r}}, \quad \mathbf{H} = \frac{1}{|\eta|}\hat{k}\times\mathbf{E}_0 e^{-\alpha\hat{k}\cdot\mathbf{r}} e^{-j[\beta\hat{k}\cdot\mathbf{r}-\theta_\eta]} \tag{4.13}$$

As can be seen from (4.13), then is a phase difference between the $\mathbf{E}$ and $\mathbf{H}$ fields. Assuming $\mathbf{E}_0$ is real, the instantaneous forms of the fields are

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{-\alpha\hat{k}\cdot\mathbf{r}}\cos(\omega t - \beta\hat{k}\cdot\mathbf{r}), \quad \mathbf{H}(\mathbf{r},t) = \frac{\hat{k}\times\mathbf{E}_0}{|\eta|} e^{-\alpha\hat{k}\cdot\mathbf{r}}\cos(\omega t - \beta\hat{k}\cdot\mathbf{r} + \theta_\eta) \tag{4.14}$$

The phase velocity along $\hat{k}$ direction is then

$$v_p = \frac{d(\hat{k}\cdot\mathbf{r})}{dt} = \frac{\omega}{\beta}. \tag{4.15}$$

In general, as σ increases (assuming $\sigma_m = 0$), β increases, and hence v_p decreases. When the conduction current dominates over the displacement current (i.e., $\sigma/\omega\epsilon \gg 1$), $\beta \propto \sqrt{\sigma}$ and $v_p \propto 1/\sqrt{\sigma}$.

The instantaneous energy densities are

$$w_e(\mathbf{r},t) = \frac{1}{2}\epsilon\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\epsilon E_0^2 e^{-2\alpha\hat{k}\cdot\mathbf{r}}\cos^2(\omega t - \beta\hat{k}\cdot\mathbf{r}) \tag{4.16}$$

$$w_m(\mathbf{r},t) = \tfrac{1}{2}\mu\mathbf{H}(\mathbf{r},t)\cdot\mathbf{H}(\mathbf{r},t) = \tfrac{1}{2}\mu H_0^2 e^{-2\alpha\hat{k}\cdot\mathbf{r}}\cos^2(\omega t - \beta\hat{k}\cdot\mathbf{r} + \theta_\eta)$$

$$= \tfrac{\mu}{2|\eta|^2} E_0^2 e^{-2\alpha\hat{k}\cdot\mathbf{r}} \cos^2(\omega t - \beta\hat{k}\cdot\mathbf{r} + \theta_\eta) \tag{4.17}$$

The instantaneous power density is

$$\mathbf{p}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t) = \hat{k}\frac{E_0^2}{|\eta|}e^{-2\alpha\hat{k}\cdot\mathbf{r}}\cos(\omega t-\beta\hat{k}\cdot\mathbf{r}+\theta_\eta)\cos(\omega t-\beta\hat{k}\cdot\mathbf{r}) \tag{4.18}$$

The instantaneous group (energy) velocity is defined by the ratio between the total power and the total energy density

$$\mathbf{v}_g = \frac{\mathbf{p}(\mathbf{r},t)}{w_e + w_m} \tag{4.19}$$

Obviously from (4.16)–(4.18), for a lossy medium ($\sigma \neq 0$) this group velocity depends on space and time. This indicates the problem of the definition of group velocity for a lossy medium using (4.19). Perhaps a more appropriate definition for lossy media can be written in terms of the time-averaged quantities

$$\mathbf{v}_g = \frac{\frac{1}{T}\int_0^T \mathbf{p}(\mathbf{r},t)dt}{\frac{1}{T}\int_0^T (w_e+w_m)dt} = \hat{k}\frac{\cos\theta_\eta/2|\eta|}{\frac{1}{4}(\epsilon + \frac{\mu}{|\eta|^2})} = \hat{k}\frac{2|\eta|\cos\theta_\eta}{\epsilon|\eta|^2+\mu} \tag{4.20}$$

For a lossless medium, $\sigma = 0$ and $\theta_\eta = 0$, $\eta = \sqrt{\frac{\mu}{\epsilon}}$, $\alpha = 0$. Then both definitions (4.19) and (4.20) give

$$\mathbf{v}_g = \frac{\hat{k}}{\sqrt{\mu\epsilon}} \tag{4.21}$$

Only for lossless media, we have $v_g = v_p = 1/\sqrt{\mu\epsilon}$.

The time-average power density (Poynting vector) is

$$\mathbf{p}_{av}(\mathbf{r}) = \frac{1}{2}\Re e[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})] = \hat{k}\frac{|E_0|^2}{2|\eta|}e^{-2\alpha\hat{k}\cdot\mathbf{r}}\cos\theta_\eta \tag{4.22}$$

4.1.3 Skin Depth

In a conductive medium, a uniform plane wave is

$$\mathbf{E} = \mathbf{E}_0 e^{-\alpha\hat{k}\cdot\mathbf{r}}e^{-j\beta\hat{k}\cdot\mathbf{r}} \tag{4.23}$$

The skin depth is defined as $\delta = (\hat{k}\cdot\mathbf{r}_2 - \hat{k}\cdot\mathbf{r}_1)$ such that $|E(\mathbf{r}_2)|/|E(\mathbf{r}_1)| = 1/e$. Hence

$$\delta = \frac{1}{\alpha} = 1/\omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{\left(1-\frac{\sigma\sigma_m}{\omega^2\mu\epsilon}\right)^2 + \left(\frac{\sigma}{\omega\epsilon}+\frac{\sigma_m}{\omega\mu}\right)^2} - \left(1-\frac{\sigma\sigma_m}{\omega^2\mu\epsilon}\right)\right]^{1/2},$$

For $\sigma_m = 0$, we have

$$\delta = \frac{1}{\alpha} = 1/\omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2}. \tag{4.24}$$

For a **good dielectric**, the displacement current dominates the conduction current, i.e., $(\sigma/\omega\epsilon) << 1$, then

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}, \quad \beta \approx \omega\sqrt{\mu\epsilon}, \quad \delta \approx \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}, \quad \eta \approx \sqrt{\frac{\mu}{\epsilon}} \tag{4.25}$$

For a **good conductor**, the conduction current dominates, and $(\sigma/\omega\epsilon) >> 1$. Then

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \delta \approx \sqrt{\frac{2}{\omega\mu\sigma}}, \quad \eta \approx \sqrt{\frac{\omega\mu}{2\sigma}}(1+j) \tag{4.26}$$

In these approximations, we have assumed that $\sigma_m = 0$.

4.1.4 Decomposition of a TEM^k Mode into TE^y and TM^y Modes

Assuming the propagation direction $\hat{k}$ is on the (x, z) plane and makes an angle of θ_i with the $\hat{z}$ direction, so that

$$\hat{k} = \hat{z}\cos\theta_i + \hat{x}\sin\theta_i, \quad \hat{k}\cdot\mathbf{r} = x\sin\theta_i + z\cos\theta_i \tag{4.27}$$

A uniform plane wave $\mathbf{E} = \mathbf{E}_0 e^{-jk\hat{k}\cdot\mathbf{r}}$ is a TEM^k mode as both $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to $\hat{k}$, that is $\mathbf{E}_0\cdot\hat{k} = 0$ and $\mathbf{H}_0\cdot\hat{k} = 0$. If we can always write

$$\mathbf{E} = E_0[\hat{y}\cos\theta_y + (\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)\sin\theta_y]e^{-jk\hat{k}\cdot\mathbf{r}} \equiv \mathbf{E}_1 + \mathbf{E}_2, \quad \mathbf{E}_1 = \hat{y}\cos\theta_y \tag{4.28}$$

where θ_y is the angle between $\mathbf{E}_0$ and y axis. Then $\mathbf{E}_1$ and $\mathbf{E}_2$ are respectively the TM^y and TE^y modes.

A. TM^y wave

For the TM^y wave

$$\mathbf{E}_1 = \hat{y}E_0\cos\theta_y e^{-jk\hat{k}\cdot\mathbf{r}} = \hat{y}E_0\cos\theta_y e^{-jk(x\sin\theta_i + z\cos\theta_i)} \tag{4.29}$$

$$\mathbf{H}_1 = \frac{1}{\eta}\hat{k}\times\mathbf{E}_1 = \frac{E_0\cos\theta_y}{\eta}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-jk(x\sin\theta_i + z\cos\theta_i)} \tag{4.30}$$

The directional impedance in x and z directions are, by definition,

$$(Z_x)_{TM_y} = \frac{E_{1y}}{H_{1z}} = \frac{\eta}{\sin\theta_i} \tag{4.31}$$

$$(Z_z)_{TM_y} = -\frac{E_{1y}}{H_{1x}} = \frac{\eta}{\cos\theta_i}. \tag{4.32}$$

Note that in general for lossy media η, $(Z_x)_{\mathrm{TM_y}}$, and $(Z_z)_{\mathrm{TM_y}}$ are complex.

Recall that $jk = \alpha + j\beta$ is complex, we see that the phase of $\mathbf{E}_1$ is

$$\theta_E = \omega t - \beta\hat{k}\cdot\mathbf{r} = \omega t - \beta(z\cos\theta_i + x\sin\theta_i) \tag{4.33}$$

Letting $\theta_E =$ constant, (4.33) gives the directional phase velocity in z direction as

$$v_{pz} = \frac{dz}{dt} = \frac{\omega}{\beta \cos\theta_i} \geq v_p \tag{4.34}$$

The directional group velocity in z direction can be found in the same way as (4.20), that is

$$v_{gz} = \frac{\frac{1}{T}\int_0^T \mathbf{p}(\mathbf{r},t)\cdot\hat{z}dt}{\frac{1}{T}\int_0^T (w_e + w_m)dt} = v_{gk}\hat{k}\cdot\hat{z} = v_{gk}\cos\theta_i = \frac{2|\eta|\cos\theta_\eta}{\epsilon|\eta|^2+\mu}\cos\theta_i \leq v_{gk} \tag{4.35}$$

where v_{gk} is the group velocity in $\hat{k}$ direction.

B. TE^y wave

Similarly, for the TE^y component $\mathbf{E}_2$, we have

$$\mathbf{E}_2 = E_0 \sin\theta_y(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-jk(x\sin\theta_i + z\cos\theta_i)} \tag{4.36}$$

$$\mathbf{H}_2 = \hat{y}\frac{E_0\sin\theta_y}{\eta}e^{-jk(x\sin\theta_i+z\cos\theta_i)} \tag{4.37}$$

The directional impedance in x and z directions are, by definition,

$$(Z_x)_{TE_y} = -\frac{E_{1z}}{H_{1y}} = \eta\sin\theta_i \tag{4.38}$$

$$(Z_z)_{TE_y} = \frac{E_{1x}}{H_{1y}} = \eta\cos\theta_i. \tag{4.39}$$

Note that in general for lossy media η, $(Z_x)_{\mathrm{TM_y}}$, and $(Z_z)_{\mathrm{TM_y}}$ are complex. The directional phase and group velocities are the same as in (4.34) and (4.35).

4.1.5 Standing Waves

Let's consider the superposition of two plane waves propagating in $\hat{k} = \pm\hat{z}$ directions with $\mathbf{E} = \hat{x}E_x$ and

$$\begin{aligned} E_x(z) &= E_0^+ e^{-\alpha z}e^{-j\beta z} + E_0^- e^{\alpha z}e^{j\beta z} \\ &= \sqrt{(E_0^+)^2e^{-2\alpha z} + (E_0^-)^2 e^{2\alpha z} + 2E_0^+E_0^-\cos 2\beta z} \\ &\quad \times \exp\left\{-j\tan^{-1}\left[\frac{E_0^+e^{-\alpha z}-E_0^-e^{\alpha z}}{E_0^+e^{-\alpha z}+E_0^-e^{\alpha z}}\tan\beta z\right]\right\} \\ &= S(z)\exp[-j\phi(z)] \end{aligned} \tag{4.40}$$

Then the instantaneous form of the field is

$$E_x(z,t) = S(z)\cos[\omega t - \phi(z)]. \tag{4.41}$$

It is clear that

$$S(z) = \sqrt{(E_0^+)^2 e^{-2\alpha z} + (E_0^-)^2 e^{2\alpha z} + 2E_0^+ E_0^- \cos 2\beta z} \tag{4.42}$$

is the envelop of the maximum values the instantaneous field (4.41) will ever achieves as a function of time at the given position z. This envelop does not move in position as a function of time, and is called the *standing wave pattern*. The associated field (4.40) or (4.41) is called the standing wave.

Note that the envelop is in general a complicated function depending on α and β. Clearly in the range of $-\infty < z < \infty$ the global maximum is at infinity if $\alpha \neq 0$. Moreover, the pattern is not periodic.

In the special case where the medium is lossless, $\alpha = 0$, and

$$S(z) = \sqrt{(E_0^+)^2 + (E_0^-)^2 + 2E_0^+ E_0^- \cos 2\beta z} \tag{4.43}$$

becomes periodic. The standing wave ratio is defined as the ratio of maximum/minimum values of $S(z)$, or of $\mathbf{E}(z)$,

$$\text{SWR} = \frac{|E_x(z)|_{\max}}{|E_x(z)|_{\min}} = \frac{1 + \frac{|E_0^-|}{|E_0^+|}}{1 - \frac{|E_0^-|}{|E_0^+|}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{4.44}$$

where $\Gamma = E_0^-/E_0^+$ is the reflection coefficient. In general $1 \leq \text{SWR} \leq \infty$, with SWR $= 1$ for $\Gamma = 0$, and SWR $= \infty$ for $|\Gamma| = 1$.

Assuming $E_0^\pm$ are positive and $E_0^+ \geq E_0^-$, we have

$$|E_x(z)|_{\max} = |E_0^+| + |E_0^-|, \quad \text{for } \beta z_{max} = m\pi, \quad m = \text{integer} \tag{4.45}$$

and

$$|E_x(z)|_{\min} = |E_0^+| - |E_0^-|, \quad \text{for } \beta z_{min} = (m + 1/2)\pi, \quad m = \text{integer} \tag{4.46}$$

The distance between maximum and minimum location is

$$|(z_{max})_m - (z_{min})_m| = \frac{\lambda}{4} \tag{4.47}$$

and that between adjacent maximums or minimums is

$$|(z_{max})_{m+1} - (z_{max})_m| = |(z_{min})_{m+1} - (z_{min})_m| = \frac{\lambda}{2} \tag{4.48}$$

4.2 Polarization

Polarization is a property describing the time-varying direction and relative magnitude of $\mathbf{E}(\mathbf{r}, t)$ at a fixed location. It consists of the trace of $\mathbf{E}$ as a function of time at a fixed location, and the sense of rotation of the trace with respect to the direction of propagation. For example, a single plane wave $\mathbf{E} = \mathbf{E}_0 e^{-jk\hat{k}\cdot\mathbf{r}}$ or

$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\omega t - k\hat{k}\cdot\mathbf{r})$ is called linearly polarized because $\mathbf{E}(\mathbf{r},t)$ is always in a direction parallel to $\mathbf{E}_0$.

Consider two linearly polarized waves given by

$$\mathbf{E}_1 = \hat{x}\mathbf{E}_{x0}e^{-jkz} = \hat{x}|E_{x0}|e^{-jkz+j\phi_x}, \quad \mathbf{E}_2 = \hat{y}\mathbf{E}_{y0}e^{-jkz} = \hat{y}|E_{y0}|e^{-jkz+j\phi_y} \tag{4.49}$$

where $\phi = \phi_y - \phi_x$ is the time phase lag of $\mathbf{E}_2$ with respect to $\mathbf{E}_1$.

The sum of these two fields gives

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_1(\mathbf{r},t)+\mathbf{E}_2(\mathbf{r},t) = \hat{x}|E_{x0}|\cos(\omega t-kz+\phi_x)+\hat{y}|E_{y0}|\cos(\omega t-kz+\phi_y) \equiv \hat{x}E_x+\hat{y}E_y \tag{4.50}$$

From (4.50) we can obtain that

$$\frac{E_x}{|E_{x0}|} = \cos(\omega t - kz + \phi_x) \tag{4.51}$$

$$\frac{E_y}{|E_{y0}|} = \cos(\omega t - kz + \phi_y) = \cos(\omega t - kz + \phi_x)\cos\phi - \sin(\omega t - kz + \phi_x)\sin\phi. \tag{4.52}$$

From (4.51) and (4.52) we have

$$\sin^2(\omega t - kz + \phi_x)\sin^2\phi = \left[\frac{E_y}{|E_{y0}|} - \frac{E_x}{|E_{x0}|}\cos\phi\right]^2 \tag{4.53}$$

Using (4.51) and (4.53), we then obtain

$$1 - \left[\frac{E_x}{|E_{x0}|}\right]^2 = \left[\frac{E_y}{|E_{y0}|\sin\phi} - \frac{E_x\cos\phi}{|E_{x0}|\sin\phi}\right]^2$$

or

$$\left[\frac{E_x}{|E_{x0}|\sin\phi}\right]^2 + \left[\frac{E_y}{|E_{y0}|\sin\phi}\right]^2 - 2\frac{\cos\phi}{\sin^2\phi}\frac{E_xE_y}{|E_{x0}||E_{y0}|} = 1 \tag{4.54}$$

which is an equation of ellipse in general. Therefore, the polarization of the sum of two perpendicularly polarized plane waves is in general elliptical.

The sense of rotation of this elliptically polarized wave can be described by the instantaneous angle $\psi(t)$ that $\mathbf{E}(\mathbf{r},t)$ makes with x-axis:

$$\psi(\mathbf{r},t) = \tan^{-1}\frac{E_y(\mathbf{r},t)}{E_x(\mathbf{r},t)} = \tan^{-1}\left\{\frac{|E_{y0}|}{|E_{x0}|}[\cos\phi - \tan(\omega t-kz+\phi_x)\sin\phi]\right\}. \tag{4.55}$$

Depending on the sign of $d\psi/dt$, there are three different polarization senses

- **RHEP—Right-hand elliptical polarization.** If this angle increases with time (i.e., $d\psi/dt > 0$) this wave is said to have a clockwise sense of rotation (right-hand polarization). Obviously, for $-\pi < \phi < 0$, it has a right-hand polarization.

- **LHEP—Left-hand elliptical polarization.** If this angle decreases with time (i.e., $d\psi/dt < 0$ or $0 < \phi < \pi/2$) this waves is said to have a counterclockwise sense of rotation (left-hand polarization).

- **Linear polarization.** If $\phi = 0$, ψ does not change with time ($d\psi/dt = 0$) then it is linearly polarized.

Three special cases can be obtained from equations (4.54) and (4.55).

4.2.1 Elliptical polarization with principal axes on x and y axes

In (4.54) and (4.55), if $\phi = \pm\frac{\pi}{2}$, then

$$\left[\frac{E_x}{|E_{x0}|}\right]^2 + \left[\frac{E_y}{|E_{y0}|}\right]^2 = 1 \tag{4.56}$$

$$\psi(\mathbf{r}, t) = \tan^{-1}\left\{\mp\frac{|E_{y0}|}{|E_{x0}|}\tan(\omega t - kz + \phi_x)\right\} \tag{4.57}$$

Obviously if $\phi = \pi/2$, it has a left-hand polarization; if $\phi = -\pi/2$, it has a right-hand polarization.

4.2.2 Circular polarization

If $\phi = \pm\frac{\pi}{2}$ and $|E_{x0}| = |E_{y0}|$, then (4.54) and (4.55) become

$$E_x^2 + E_y^2 = |E_{x0}|^2 \tag{4.58}$$

$$\psi(\mathbf{r}, t) = \mp(\omega t - kz + \phi_x) \tag{4.59}$$

which is a left-hand circularly polarized (**LHCP**) wave if $\phi = \pi/2$, and a right-hand circularly polarized (**LHCP**) wave if $\phi = -\pi/2$.

4.2.3 Linear polarization

If E_x and E_y are in phase or out of phase, that is $\phi = 0$ or π, then

$$\frac{E_x}{|E_{x0}|} = \pm\frac{E_y}{|E_{y0}|} \tag{4.60}$$

$$\psi(\mathbf{r}, t) = \pm\tan^{-1}\frac{|E_{y0}|}{|E_{x0}|} = \text{constant} \tag{4.61}$$

which is a linearly polarized wave with the **E** field making a constant angle of $\psi = \pm\tan^{-1}\frac{|E_{y0}|}{|E_{x0}|}$ with respect to x axis.