Chapter 7

SCATTERING

This chapter (Chapter 11 in the textbook) discusses the scattering problem. In the presence of scatterers, the incident field is scattered. We aim to find the scattered electromagnetic fields. Techniques used to find scattered fields are: (1) Geometrical optics (GO); (2) Physical optics (PO); (3) Model techniques (MT). In this chapter, we will discuss some analytical methods for canonical problems and the PO approximation for PEC scattering problems.

7.1 Infinite Line Sources

In this section we study the radiation of uniform electric and magnetic line sources extending in z direction to infinity. The current densities on the line sources are assumed uniform. Thus this situation is different from the radiation of a point source—it becomes a two-dimensional problem in a homogeneous medium, or if the medium is also uniform along the z direction.

7.1.1 An Electric Line Source

The electric current density of a line source is given by

$$\mathbf{J}(\mathbf{r}) = \hat{z}\delta(\boldsymbol{\rho} - \boldsymbol{\rho}_0)I_e \tag{7.1}$$

Therefore the magnetic vector potential $\mathbf{A}$ satisfies

$$(\nabla^2 + k^2)\mathbf{A} = -\hat{z}\mu I_e \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_0)$$

Since both the source and the homogeneous medium are independent of z, we have $\partial/\partial z = 0$. In cylindrical coordinates, assuming that $\boldsymbol{\rho}_0 = 0$, we have

$$(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + k^2)A_z = -\mu\delta(\boldsymbol{\rho})I_e \tag{7.2}$$

The solution of this equation is

$$A_z = -\frac{j\mu}{4}I_e H_0^{(2)}(k\rho) \tag{7.3}$$

Note that to arrive at this solution, we have eliminated the other non-physical solution proportional to the zeroth order Hankel function of the first kind. From **A** we can obtain the electromagnetic fields as

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\phi_e = -j\omega\mathbf{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) = -\hat{z}I_e\frac{k^2}{4\omega\epsilon}H_0^{(2)}(k\rho) \tag{7.4}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} = -\hat{\phi}\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} = -\hat{\phi}jkI_e4H_1^{(2)}(k\rho) \tag{7.5}$$

If $\boldsymbol{\rho_0} \neq 0$, i.e., when the line source is located away from the origin, we have

$$\mathbf{E} = -\hat{z}I_e\frac{k^2}{4\omega\epsilon}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho_0}|) \tag{7.6}$$

$$\mathbf{H} = -\hat{\psi}j\frac{kI_e}{4}H_1^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho_0}|) \tag{7.7}$$

Here, ψ is the circumferential angle around the source.

We define the wave impedance as

$$Z_w^{+\rho} = -\frac{E_z}{H_\phi} = \frac{jk}{\omega\epsilon}\frac{H_0^{(2)}(k\rho)}{H_1^{(2)}(k\rho)} \overset{\rho\to\infty}{\longrightarrow} \eta \tag{7.8}$$

7.1.2 Magnetic Line Source

Similarly, for the magnetic line source located at $\boldsymbol{\rho}_0$, we have

$$\mathbf{E} = \hat{\psi}E_\psi = \hat{\psi}jI_m\frac{k}{4}H_1^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho_0}|) \tag{7.9}$$

$$\mathbf{H} = \hat{z}H_z = -\hat{z}I_m\frac{k^2}{4\omega\mu}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho_0}|) \tag{7.10}$$

Again, note that ψ is the circumferential angle around the source. In general $\hat{\psi} \neq \&\phi$ except for $\boldsymbol{\rho}_0 = 0$.

7.1.3 An Electric Line Source above an Infinite PEC Plane

Assuming that an infinite PEC plane is located at $y = 0$. If an electric line source with $\mathbf{J} = \hat{z}I_e\Delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0)$ is located at $\boldsymbol{\rho}_0 = \hat{y}h$, then its image will be $\mathbf{J}' = -\hat{z}I_e\Delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0')$ located at $\boldsymbol{\rho}_0' = -\hat{y}h$. The total electric field **E** will be the sum of the incident field $\mathbf{E}_i$ due to the original source and the reflected field $\mathbf{E}_r$ due to the image source:

$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^r = \begin{cases} -\hat{z}\frac{k^2I_e}{4\omega\epsilon}[H_0^{(2)}(k\rho_i) - H_0^{(2)}(k\rho_r)] & \text{if } y \geq 0 \\ 0 & \text{if } y \leq 0 \end{cases} \tag{7.11}$$

$$\rho_i = |\boldsymbol{\rho}-\boldsymbol{\rho}_0| = [\rho^2 + h^2 - 2\rho h\sin\phi] \tag{7.12}$$

$$\rho_r = |\boldsymbol{\rho}-\boldsymbol{\rho}_0'| = [\rho^2 + h^2 + 2\rho h\sin\phi] \tag{7.13}$$

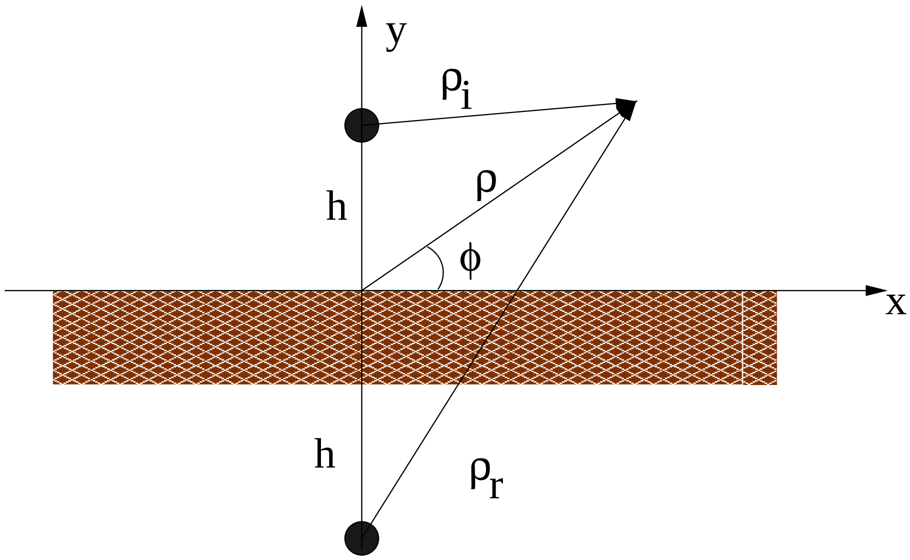

Figure 7.1: *An electric line source above a PEC plane.*

For the far field,

$$\rho_i = \rho - h\sin\phi, \;\; \rho_r = \rho + h\sin\phi$$

Therefore,

$$\mathbf{E} = \begin{cases} -\hat{z} j\eta I_e \sqrt{\frac{jk}{2\pi}} \sin\left(kh\sin\phi\right) \frac{e^{-jk\rho}}{\sqrt{\rho}} & \text{if } y \geq 0 \\ 0 & \text{if } y \leq 0 \end{cases} \tag{7.14}$$

7.2 Scattering by Planar Surfaces

Radar Cross Section (RCS) is the area intercepting the amount of incident power that, when scattered isotropically, produces at the receiver a density that equal to the density scattered by the actual target. Assuming the power density of the scattered field is S_s and that for the incident plane wave is S_i, the RCS $\sigma_{3\text{D}}$ is thus defined as

$$\lim_{r\to\infty}\left[4\pi r^2 S^s\right] \equiv S_i \sigma_{3\text{D}}$$

Thus, the 3-D RCS is given by the following equivalent expressions

$$\sigma_{3\text{D}} = \begin{cases} \lim_{r\to\infty} [4\pi r^2 \frac{S_s}{S_i}] \\ \lim_{r\to\infty} [4\pi r^2 \frac{|\mathbf{E}_s|^2}{|\mathbf{E}_i|^2}] \\ \lim_{r\to\infty} [4\pi r^2 \frac{|\mathbf{H}_s|^2}{|\mathbf{H}_i|^2}] \end{cases} \tag{7.15}$$

For a two-dimensional target, since we assume that the medium and fields are invariant in one (say z) direction, the target is actually infinite in size in that direction. Therefore the RCS defined above is finite. To avoid this difficulty, we define a **scattering width** to be the RCS per unit length in the invariant direction. RCS $\longleftrightarrow$ scattering width (SW)

$$\sigma_{2\mathrm{D}} = \begin{cases} \lim_{\rho\to\infty} [2\pi\rho \frac{S_s}{S_i}] \\ \lim_{\rho\to\infty} [2\pi\rho \frac{|\mathbf{E}_s|^2}{|\mathbf{E}_i|^2}] \\ \lim_{\rho\to\infty} [2\pi\rho \frac{|\mathbf{H}_s|^2}{|\mathbf{H}_i|^2}] \end{cases} \tag{7.16}$$

Note that here a plane wave incidence is assumed. For an arbitrary target, the scattered field in general cannot be found analytically. Therefore,either numerical methods or some approximation should be used. In this chapter, we will use the physical optics approximation. Full-wave numerical methods will be discussed in a subsequent course on computational electromagnetics.

The dimensions of the RCS and SW defined above is m^2 and m respectively. Most of the time, these are also shown in $\mathrm{dB}\cdot\mathrm{m}^2$ and $\mathrm{dB}\cdot\mathrm{m}$ respectively by defining

$$\mathrm{RCS}\,(\mathrm{dB}\cdot\mathrm{m}^2) = 10\log_{10}[\mathrm{RCS}\,(\mathrm{m}^2)]$$

$$\mathrm{SW}\,(\mathrm{dB}\cdot\mathrm{m}) = 10\log_{10}[\mathrm{SW}\,(\mathrm{m})]$$

7.2.1 TM$_z$ Plane Wave Scattering from a Strip: The PO Approximation

Consider a strip invariant in z direction. The PEC strip is assumed infinitely thin in y direction and located at $0 \le x \le w$. We assume a TM$_z$ plane wave incidence so that the incidence direction is

$$\hat{k}_i = -\hat{x}\cos\phi_i - \hat{y}\sin\phi_i$$

and the incident electric field is

$$\mathbf{E}_i = \hat{z}E_0 e^{-jk\hat{k}_i\cdot\mathbf{r}} = \hat{z}E_0 e^{jk(x\cos\phi_i + y\sin\phi_i)}$$

The incident magnetic field $\mathbf{H}_i$ can be obtained by

$$\mathbf{H}_i = \frac{1}{\eta}\hat{k}_i \times \mathbf{E}_i = \frac{E_0}{\eta}(-\hat{x}\sin\phi_i + \hat{y}\cos\phi_i)e^{jk(x\cos\phi_i + y\sin\phi_i)}$$

The scattered fields cannot be found in a closed form. In this case, the rigorous method requires numerically solving for the physical optics equivalent current, from which the scattered fields can be evaluated.

However, when the width w of the strip is much larger than the wavelength, we can use the **physical optics approximation**

$$\mathbf{J}_p \approx 2\hat{n}\times\mathbf{H}_i, \quad \mathbf{M}_p = 0 \tag{6.77}$$

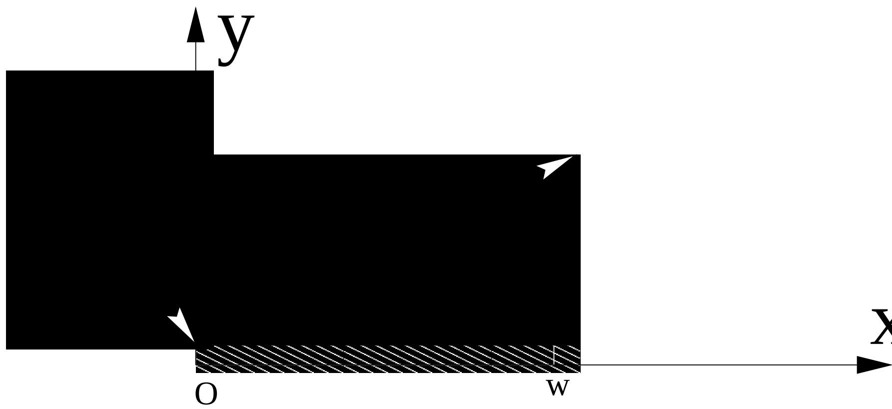

Figure 7.2: *Scattering by a PEC strip along the z direction. For TM_z incidence, $\mathbf{E}_i = \hat{z}E_i$. For TE_z incidence, $\mathbf{H}_i = \hat{z}H_i$.*

From this physical optics approximated current, we can find the scattered field.

The physical optics equivalent magnetic current density on the PEC surface is zero. In the PO approximation, the physical optics equivalent electric current density on the strip is

$$\mathbf{J_s}(x) = \hat{n}\times\mathbf{H}_t|_{y=0} \approx 2\hat{n}\times\mathbf{H}_i|_{y=0} = \hat{z}\frac{2E_0}{\eta}\sin(\phi_i)e^{jkx\cos\phi_i}, \quad x\in[0,w] \quad (7.17)$$

From this electric current density, we can calculate the scattered electromagnetic field through the magnetic vector potential

$$\mathbf{A} = \mu\int_S \mathbf{J}_p(x',y',z')g(|\mathbf{r}-\mathbf{r}'|)dx'dz' \quad (7.18)$$

where $g(|\mathbf{r}-\mathbf{r}'|) = e^{-jk|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r}-\mathbf{r}'|$ is the 3-D Green's function. Note that in the above equation, the integration is over (x', z'), i.e., both in x direction and the invariant direction z. However, in our case the current density is only a function of x'. Therefore, the integration over z' can be carried out on the Green's function, resulting in

$$\int_{-\infty}^{\infty} \frac{\exp\left[-jk\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}\right]}{4\pi\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}} = \frac{-j}{4}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)$$

This result is obtained by noting that

$$\int_{-\infty}^{\infty} \frac{\exp[-j\alpha\sqrt{x^2+t^2}]}{\sqrt{x^2+t^2}}dt = -j\pi H_0^{(2)}(\alpha x) \quad (7.19)$$

Finally, equation (7.18) becomes

$$\mathbf{A} = \mu\int_C \mathbf{J}_p(x')g_{2D}(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)dx' \quad (7.20)$$

where C is the curve on which the surface current exists, in this case over the strip with $x \in [0, w]$;

$$g_{2D}(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = \frac{-j}{4} H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$$

is the 2-D Green's function.

The far-zone scattered field can be obtained by using the approximation

$$|\boldsymbol{\rho} - \boldsymbol{\rho}'| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi_s - \phi')} \approx \rho - \rho' \cos(\phi - \phi') \tag{7.21}$$

and the asymptotic expression for the Hankel function

$$H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) \approx \sqrt{\frac{2j}{\pi k \rho}} e^{-jk(\rho - \rho'' \cos(\phi - \phi'))} = \sqrt{\frac{2j}{\pi k}} \frac{e^{-jk\rho}}{\sqrt{\rho}} e^{jk\rho' \cos(\phi - \phi')} \tag{7.22}$$

Using these far-field approximations we have the approximate magnetic vector potential $\mathbf{A}$

$$\mathbf{A} \approx -\hat{z} A_0 e^{j\frac{kw}{2}(\cos\phi + \phi_i)} \sin\phi_i \left[\frac{\sin X}{X}\right] \frac{e^{-jk\rho}}{\sqrt{\rho}} \equiv \hat{z} A_z \tag{7.23}$$

where

$$A_0 = \frac{j\omega\mu E_0}{\eta} \sqrt{\frac{j}{2\pi k}}, \qquad X = \frac{kw}{2}(\cos\phi + \cos\phi_i)$$

Noting that $\nabla \cdot \mathbf{A} = 0$, we arrive at the far-zone electric field

$$\mathbf{E}_s = -j\omega\mathbf{A} - \frac{j}{\omega\mu\epsilon} \nabla(\nabla \cdot \mathbf{A}) = -j\omega\mathbf{A} = -\hat{z} j\omega A_z \tag{7.24}$$

and the far-zone magnetic field

$$\mathbf{H}_s = \frac{1}{\eta} \hat{\rho} \times \mathbf{E}_s = \hat{\phi} \frac{j\omega}{\eta} A_z \tag{7.25}$$

(**Note the textbook has an important error here: it should be E_{sz} rather than $\mathbf{E}_{s\theta}$. This also affects the SW below.**)

The bistatic scattering width at (θ, ϕ) direction is then

$$\sigma_{2D}\,(\text{bistatic}) = \lim_{\rho \to \infty} \left[2\pi\rho \frac{|\mathbf{E}_s|^2}{|\mathbf{E}_i|^2}\right] = \frac{2\pi w^2}{\lambda} \sin^2\phi_i \left[\frac{\sin^2 \frac{kw}{2}(\cos\phi + \cos\phi_i)}{\left(\frac{kw}{2}\right)^2 (\cos\phi + \cos\phi_i)^2}\right] \tag{7.26}$$

And the monostatic SW ($\phi_s = \phi_i$) is

$$\sigma_{2D}\,(\text{monostatic}) = \frac{2\pi w^2}{\lambda} \sin^2\phi_i \frac{\sin^2(kw\cos\phi_i)}{k^2 w^2 \cos^2\phi_i} \tag{7.27}$$

Both are of the form $\frac{\sin^2 X}{X^2}$, which has a maximum at $\phi = \pi - \phi_i$ for bistatic and $\frac{\phi_i = \pi}{2}$ for monostatic case.

TE$_z$ Incidence on A PEC Strip

Similarly, for TE$_z$ incidence,

$$\mathbf{A} \cong -\hat{x}j\mu\omega H_0\sqrt{\frac{j}{2\pi k}}e^{j(kw/2)(\cos\phi+\cos\phi_i)}\left[\frac{\sin X}{X}\right]\frac{e^{-jk\rho}}{\sqrt{\rho}} \equiv \hat{x}A_x \tag{7.28}$$

The far-zone fields are

$$\mathbf{E}_s \approx -j\omega[\hat{z}(\hat{z}\cdot\mathbf{A}) + \hat{\phi}(\hat{\phi}\cdot\mathbf{A})] = -\hat{\phi}j\omega A_x \sin\phi \tag{7.29}$$

$$\mathbf{H}_s \approx \frac{1}{\eta}\hat{\rho}\times\mathbf{E}_s = \hat{z}\frac{E_{s\phi}}{\eta} = -\hat{z}\frac{j\omega}{\eta}A_x\sin\phi \tag{7.30}$$

The scattering widths are

$$\sigma_{2\mathrm{D}}\,(\text{bistatic}) = \frac{2\pi w^2}{\lambda}\sin^2\phi\frac{\sin^2 X}{X^2} \tag{7.31}$$

$$\sigma_{2\mathrm{D}}\,(\text{monostatic}) = \frac{2\pi w^2}{\lambda}\sin^2\phi_i\frac{\sin^2(kw\cos\phi_i)}{k^2w^2\cos^2\phi_i} \tag{7.32}$$

Note maximum for σ_{2D} (bistatic) does not occur at $\phi = \pi - \phi_i$. This is approximately true for $\omega \gg \lambda$, though.

7.2.2 Plane Wave Scattering from a Flat Rectangular Plate

In Chapter 7 we already studied this case using both the induction equivalent and physical equivalent approximations, except that we used a different coordinate system where the plate was on the yz plane. In this subsection, we have a similar case except that the plate is on the xy plane located at $z = 0$. The procedures are identical to those in the last example of Chapter 7, and will not be repeated here. We only present a summary below.

Here we assume that the PEC plate is located at $z = 0$ with $-a/2 \leq x \leq a/2$ and $-b/2 \leq y \leq b/2$. The plane of incidence of the plane wave is on yz plane, with the incidence direction

$$\hat{k}_i = (\hat{y}\sin\theta_i - \hat{z}\cos\theta_i).$$

A. TE$_x$ Scattering

For the incident TE$_x$ plane wave, we have

$$\mathbf{E_i} = \eta H_0(\hat{y}\cos\theta_i + \hat{z}\sin\theta_i)e^{-jk(y\sin\theta_i - z\cos\theta_i)} \tag{7.33}$$

$$\mathbf{H_i} = \hat{x}H_0 e^{-jk(y\sin\theta_i - z\cos\theta_i)} \tag{7.34}$$

Using the physical optics approximation, the induced current on the plate are $\mathbf{M}_p = 0$ and

$$\begin{aligned}\mathbf{J_p} &= \hat{n}\times\mathbf{H}|_{z=0} = \hat{z}\times\hat{x}(H_i + H_r)|_{z=0} \\ &\cong 2\hat{z}\times\hat{x}H_i|_{z=0} = \hat{y}2H_0e^{-jky\sin\theta_i}\end{aligned} \tag{7.35}$$

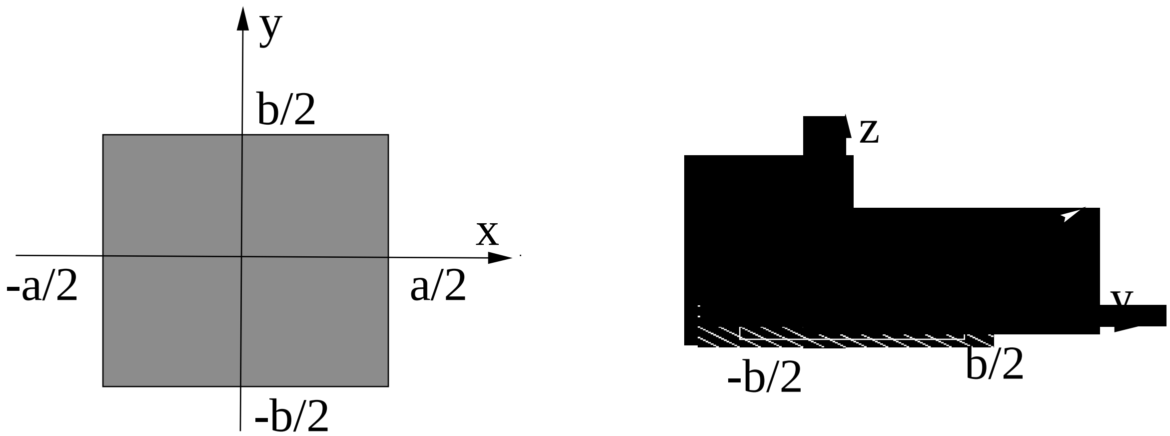

Figure 7.3: *Scattering by a rectangular PEC plate on xy plane. For TE$_x$ incidence,* $\mathbf{H}_i = \hat{x}H_i$. *For TM$_x$ incidence,* $\mathbf{E}_i = \hat{x}E_i$.

For far field zone,

$$E_r \approx 0 \tag{7.36}$$

$$H_r \approx 0 \tag{7.37}$$

$$E_\theta \approx -\frac{jk}{4\pi r e^{-jkr}}(L_\phi + \eta N_\theta) \tag{7.38}$$

$$E_\phi \approx \frac{jk}{4\pi r e^{-jkr}}(L_\theta - \eta N_\phi) \tag{7.39}$$

$$H_\theta \approx \frac{-E_\phi}{\eta} \tag{7.40}$$

$$H_\phi \approx \frac{E_\theta}{\eta} \tag{7.41}$$

$$N_\theta = \int_S (J_x \cos\theta\cos\phi + J_y \cos\theta\sin\phi - J_z\sin\theta) e^{jkr'\cos\phi} ds' \tag{7.42}$$

$$N_\phi = \int_S (-J_x \sin\phi + J_y\cos\phi) e^{jkr'\cos\phi'} ds' \tag{7.43}$$

$$\begin{aligned} r'\cos\phi' = \mathbf{r}'\cdot\hat{r} &= (\hat{x}x' + \hat{y}y')(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta) \\ &= x'\sin\theta\cos\phi + y'\sin\theta\sin\phi \end{aligned} \tag{7.44}$$

Noting that

$$\begin{aligned} &\int_{-b/2}^{b/2} e^{jky'(\sin\theta_s\sin\phi_s - \sin\theta_i)} dy' \int_{-a/2}^{a/2} e^{jkx'\sin\theta_s\cos\theta_s} dx' \\ &= 2abH_0 \Big\{ \cos\theta_s \sin\phi_s \Big[\frac{\sin X}{X}\Big]\Big[\frac{\sin Y}{Y}\Big]\Big\} \end{aligned} \tag{7.45}$$

we obtain

$$N_\phi = 2abH_0 \left\{ \cos\theta_s \left[\frac{\sin X}{X}\right] \left[\frac{\sin Y}{Y}\right] \right\} \tag{7.46}$$

where $X = \frac{ka}{2}\sin\theta_s \cos\phi_s$, and $Y = \frac{kb}{2}(\sin\theta_s \sin\phi_s - \sin\theta_i)$

$$\sigma_{3D} = 4\pi \left(\frac{ab}{\lambda}\right)^2 \left(\cos^2\theta_s \sin^2\phi_s + \cos^2\phi_s\right) \left[\frac{\sin X}{X}\right]^2 \left[\frac{\sin Y}{Y}\right]^2 \tag{7.47}$$

Maximum scattering always lies in a plane parallel to the incidence plane ($\phi_s = \pi/2$). So for this plane, the principal bistatic ($\phi_s = \pi/2, 3\pi/2$)

$$\sigma_{3D}(\phi_s = \pi/2, 3\pi/2) = 4\pi \left(\frac{ab}{\lambda}\right)^2 \cos^2\theta_s \left\{ \frac{\sin[\frac{kb}{2}(\sin\theta_s \mp \sin\theta_i)]}{\frac{kb}{2}(\sin\theta_s \mp \sin\theta_i)} \right\}^2 \quad \textit{for } 0 \le \theta_s \le \pi/2 \tag{7.48}$$

Back scattering direction ($\phi_s = \phi_i = 3\pi/2$, $\theta_s = \theta_i$),

$$\sigma_{3D} = 4\pi \left(\frac{ab}{\lambda}\right)^2 \cos^2\theta_i \left[\frac{\sin(kb\sin\theta_i)}{kb\sin\theta_i)}\right]^2 \tag{7.49}$$

For large $b \gg \lambda$, ($\phi_s = \pi/2, 3\pi/2$) has a maximum at $\phi_s \approx \phi_i$, similar to the TE^z scattering in strip.

B. TM$_x$ Scattering

Similarly, for a TM$_x$ Plane wave incidence,

$$\sigma_{3D} = 4\pi \left(\frac{ab}{\lambda}\right)^2 \left[\cos^2\theta_i \left(\cos^2\theta_s \sin^2\phi_s + \sin^2\phi_s\right)\right] \left[\frac{\sin X}{X}\right]^2 \left[\frac{\sin Y}{Y}\right]^2 \tag{7.50}$$

The above solutions use the physical optics (PO) approximation to predict RCS. This approximation is most accurate at the specular direction.

The monostatic RCS σ_{3D} and σ_{2D} are independent of polarization, which is only true in PO approximation.

7.3 Cylindrical Wave Transformations and Theorems

In this section we will study the scattering by circular cylinders. To facilitate the formulation of the scattering of plane waves, we will first introduce the transformation of plane waves into cylindrical waves. For scattering of line sources, we will introduce the addition theorem.

7.3.1 Plane Waves in Terms of Cylindrical Wave Functions

First we will prove the important formula

$$\boxed{e^{\pm jkx} = e^{\pm jk\rho\cos\phi} = \sum_{n=-\infty}^{\infty} j^{\pm n} J_n(k\rho) e^{jn\phi}} \tag{7.51}$$

To prove this formula, we only need to expand the left-hand side as the summation of cylindrical functions, for example,

$$e^{-jkx} = e^{-jk\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho)e^{jn\phi} \tag{7.52}$$

Here we have used $J_n(k\rho)$ rather than the other three cylindrical harmonics (Y_n, $H_n^{(1)}$ and $H_n^{(2)}$) because only this function is regular at the origin. The other harmonics are singular at the origin and do not satisfy the regularity of the left-hand side. Multiplying both sides by $e^{-jm\phi}$ and integrating over ϕ, we arrive

$$\int_0^{2\pi} e^{-jk\rho\cos\phi}e^{-jm\phi}d\phi = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho)\int_0^{2\pi} e^{j(n-m)}d\phi \tag{7.53}$$

Noting that

$$\int_0^{2\pi} e^{j(n-m)}d\phi = 2\pi\delta_{mn} \tag{7.54}$$

$$\int_0^{2\pi} e^{-j(k\rho\cos\phi+m\phi)}d\phi = 2\pi j^{-m}J_{-m}(-k\rho) = 2\pi j^{-m}(-1)^{2m}J_m(k\rho) \tag{7.55}$$

we have,

$$a_m = j^{-m} \tag{7.56}$$

which proves the first part of equation (7.51). The other part of (7.51) following the same proof.

Note that (7.51) is the cylindrical harmonic expansion of a plane wave propagating in $+x$ or $-x$ direction.

7.3.2 Addition Theorem of Cylindrical Wave Functions

The addition theorem for cylindrical wave functions are

$$\boxed{H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = \sum_{n=-\infty}^{\infty} J_n(k\rho_<)H_n^{(2)}(k\rho_>)e^{jn(\phi-\phi')}} \tag{7.57}$$

$$\boxed{H_0^{(1)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = \sum_{n=-\infty}^{\infty} J_n(k\rho_<)H_n^{(1)}(k\rho_>)e^{jn(\phi-\phi')}} \tag{7.58}$$

$$\boxed{J_0(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = \sum_{n=-\infty}^{\infty} J_n(k\rho)J_n(k\rho')e^{jn(\phi-\phi')}} \tag{7.59}$$

$$Y_0(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = \sum_{n=-\infty}^{\infty} J_n(k\rho_<)Y_n(k\rho_>)e^{jn(\phi-\phi')} \tag{7.60}$$

where the short-hand notations of $\rho_>$ and $\rho_<$ are defined as

$$\rho_> = \max(\rho, \rho'), \qquad \rho_< = \min(\rho, \rho')$$

These notations give a more compact formula than the fully expanded one, for example for (7.56)

$$H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = \begin{cases} \sum_{n=-\infty}^{\infty} J_n(k\rho)H_n^{(2)}(k\rho')e^{jn(\phi-\phi')} & \text{for } \rho \le \rho' \\ \sum_{n=-\infty}^{\infty} J_n(k\rho')H_n^{(2)}(k\rho)e^{jn(\phi-\phi')} & \text{for } \rho \ge \rho' \end{cases} \tag{7.61}$$

To show these addition theorems we expand the left-hand sides with the cylindrical wave functions centered at the origin. The unknown expansion coefficients can be solved in two ways: The first is that in the textbook, which uses the continuity at the source radius $\rho = \rho'$, and the asymptotic behaviors the the functions at $\rho \to \infty$; the second way is to use the continuity at the source radius $\rho = \rho'$, and the jump condition of the derivative of the left-hand side. This second way need to use the **Wronskian** defined as $W[X, Y] = XY' - YX'$ for the cylindrical wave functions. In particular, these **Wronskians** are

$$W[J_n, Y_n] = J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = \frac{2}{\pi x} \tag{7.62}$$

$$W[J_n, H_n^{(2)}] = J_n(x)H_n^{(2)'}(x) - J_n'(x)H_n^{(2)}(x) = -\frac{2j}{\pi x} \tag{7.63}$$

7.4 Scattering by Circular PEC Cylinders

7.4.1 Normal Incident TM$_z$ Plane Wave Scattering by Conductive Circular Cylinder

Using the cylindrical wave transformation, the incident TM$_z$ electric field is given by

$$\mathbf{E}_i = \hat{z}E_0 e^{-jk\rho\cos\phi} = \hat{z}E_0 \sum_{n=-\infty}^{\infty} (-j)^n J_n(k\rho)e^{jn\phi} \tag{7.64}$$

Similarly, we can write the scattered electric field as

$$\mathbf{E}^s = \hat{z}E_z^s = \hat{z}E_0 \sum_{n=-\infty}^{\infty} c_n H_n^{(2)}(k\rho) \tag{7.65}$$

where c_n is an unknown coefficient to be solved by applying the boundary conditions. Since this is a PEC circular cylinder, on the PEC surface at $r = a$, the total tangential electric field components should be zero, or

$$c_n = -j^{-n}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{jn\phi} \tag{7.66}$$

Therefore, the scattered field is

$$\begin{aligned} E_{s,z} &= -E_0\sum_{n=-\infty}^{\infty} j^{-n}\frac{J_n(ka)}{H_n^{(2)}(ka)}H_n^{(2)}(k\rho)e^{jn\phi} \\ &= -E_0\sum_{n=-\infty}^{\infty}(-j)^n(2-\delta_{n0})\frac{J_n(ka)}{H_n^{(2)}(ka)}H_n^{(2)}(k\rho)\cos n\phi \end{aligned} \tag{7.67}$$

The total electric field is $\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s$, while the total magnetic field can be obtained by

$$\mathbf{H} = -\hat{\rho}\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} + \hat{\phi}\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial\rho} \tag{7.68}$$

To calculate the induced electric current density on the PEC surface, we need to computer the tangential magnetic field components. In this case, the ϕ component is

$$H_\phi = \frac{kE_0}{j\omega\mu}\sum_{n=-\infty}^{\infty} j^{-n}\left[J_n'(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)H_n^{(2)'}}(k\rho)\right]e^{jn\phi} \tag{7.69}$$

At $\rho = a$,

$$\begin{aligned} H_\phi^t(\rho=a) = \frac{kE_0}{j\omega\mu}\sum_{n=-\infty}^{\infty} j^{-n}\frac{J_n'(k\rho)H_n^{(2)}(ka) - J_n(ka)H_n^{(2)'}(ka)}{H_n^{(2)}(ka)}e^{jn\phi} \\ = \frac{2E_0}{\pi a\omega\mu}\sum_{n=-\infty}^{\infty} j^{-n}\frac{e^{jn\phi}}{H_n^{(2)}(ka)} \end{aligned} \tag{7.70}$$

The electric current density on the PEC is

$$\mathbf{J}_s = \hat{n}\times\mathbf{H}|_{\rho=a}^t = \hat{z}\frac{2E_0}{\pi a\omega\mu}\sum_{n=-\infty}^{\infty} j^{-n}\frac{e^{jn\phi}}{H_n^{(2)}(ka)} \tag{7.71}$$

where we have used the Wronskian relation.

To calculate the far-zone scattering field, we use the asymptotic relation for $k\rho \gg 1$:

$$H_n^{(2)}(k\rho) \longrightarrow \sqrt{\frac{2j}{\pi k\rho}}j^n e^{-jk\rho} \tag{7.72}$$

Therefore the field-zone scattered electric field is

$$E_z^s \longrightarrow -E_0\sqrt{\frac{2j}{\pi k}}\frac{e^{-jk\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{\infty}\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{jn\phi} \tag{7.73}$$

The 2-D scattering width is

$$\delta_{2-D} = lim_{\rho\to\infty}\Big] = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty}(2-\delta_n o)\frac{J_n(ka)}{H_n^{(2)}(ka)}\cos n\phi\right|^2 \tag{7.74}$$

In particular, if the radius of the PEC is much smaller than the wavelength, or $a \ll \lambda$, the scattering width can be approximated as

$$\delta_{2-D} \overset{a\ll\lambda}{\approx} \frac{\pi\lambda}{2}\left|\frac{1}{\ln(0.89ka)}\right|^2 \tag{7.75}$$

7.4.2 Normal Incidence TE$_z$ Plane Wave Scattering by Circular PEC Cylinder

For a plane TE$_z$ incident wave, the magnetic field is

$$\mathbf{H}^i = \hat{z}H_0 e^{-jkx} = \hat{z}H_0 e^{-jk\rho\cos\phi} = \hat{z}H_0\sum_{n=-\infty}^{\infty} j^{-n}J_n(k\rho)e^{jn\phi} \tag{7.76}$$

The scattered magnetic field can be expanded as

$$H_z^s = \hat{z}H_0\sum_{n=-\infty}^{\infty} d_n H_n^{(2)}(k\rho)$$

Then the total electric field can be written as

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H} \tag{7.77}$$

The ϕ component of the electric field is

$$E_\phi = -\frac{1}{j\omega\epsilon}\frac{\partial H_z}{\partial\rho} = -\frac{kH_0}{j\omega\epsilon}\sum_{n=-\infty}^{\infty}[j^{-n}J_n'(k\rho)e^{jn\phi} + d_n H_n^{(2)'}(k\rho) \tag{7.78}$$

At $\rho = a$, $E_\phi^i + E_\phi^s = 0$ Therefore,

$$d_n = -j^{-n}\frac{J_n'(ka)}{H_n^{(2)'}(ka)}e^{jn\phi} \tag{7.79}$$

For far field, $k\rho \longrightarrow \infty$,

$$H_{s,\phi} \longrightarrow -H_0\sqrt{\frac{2j}{\pi k}}\frac{e^{-jk\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{\infty} j^{-n}\frac{J_n'(ka)}{H_n^{(2)'}(ka)}e^{jn\phi} \tag{7.80}$$

The scattering width is

$$\sigma_{2D} = \lim_{\rho\to\infty}\left[2\pi\rho\frac{|H_z^s|^2}{|H_z^i|^2}\right] = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty}\epsilon_n\frac{J_n'(k\rho)}{H_n^{(2)'}(k\rho)}\cos n\phi\right|^2 \tag{7.81}$$

For $a \ll \lambda$, we keep up to $n = \pm 1$ terms ($n = 0$ term is zero) to arrive at the approximate SW

$$\delta_{2-D} \stackrel{a \ll \lambda}{\approx} \frac{\pi\lambda}{8}(ka)^4[1 - 2\cos\phi]^2 \tag{7.82}$$

This asymptotic behavior is very different from that for the TM_z incidence.

7.4.3 Oblique TM_z Scattering

Now let's consider the oblique incidence of a plane TM_z wave where the magnetic field for the incident field only has a nonzero y component. The incidence vector is $\hat{k}_i = \hat{x}\sin\theta_i - \hat{z}\cos\theta_i$. The incident magnetic field is $\mathbf{H}_i = \hat{y}\frac{E_0}{\eta}\exp(-jk\hat{k}_i \cdot \mathbf{r})$ and the electric field is

$$\mathbf{E}^i = E_0(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-jkx\sin\theta_i}e^{jkz\cos\theta_i} \tag{7.83}$$

The z component of electric field can be expanded as

$$E_{i,z} = E_0\sin\theta_i e^{jkz\cos\theta_i}\sum_{n=-\infty}^{\infty} j^{-n}J_n(k\rho\sin\theta_i)e^{jn\phi} \tag{7.84}$$

$$E_{s,z} = E_0\sin\theta e^{-jkz\cos\theta}\sum_{n=-\infty}^{\infty} j^{-n}a_n H_n^{(2)}(k\rho\sin\theta)e^{jn\phi} \tag{7.85}$$

At $\rho = a$, the boundary condition requires $E_z = 0$, or

$$\sum_{n=-\infty}^{\infty}[\sin\theta_i e^{jkz\cos\theta_i}j^{-n}J_n(ka\sin\theta_i) + sin\theta eo^{-jkz\cos\theta}j^{-n}a_n H_n^{(2)}(k\rho\sin\theta)] = 0$$

From phase matching of $e^{jkz\cos\theta_i}$ and $e^{-jkz\cos\theta}$ we obtain $\cos\theta = -\cos\theta_i$, or

$$\theta_s = \pi - \theta_i \tag{7.86}$$

Hence, we obtain the unknown expansion coefficient

$$a_n = -\frac{J_n(ka\sin\theta_i)}{H_n^{(2)}(ka\sin\theta_i)} \tag{7.87}$$

The scattered electric field is

$$E_{s,z} = E_0\sin\theta_i e^{jkz\cos\theta_i}\sum_{n=-\infty}^{\infty} j^{-n}a_n H_n^{(2)}(k\rho\sin\theta_i)e^{jn\phi} \tag{7.88}$$

The other component of electric field can be obtained similarly, but the procedures are more involved because the x component involves both ϕ and ρ components in cylindrical coordinates. For the sake of obtaining the RCS, we only need the scattered magnetic field, which is simpler than the scattered electric field as it does not have the z component. Therefore, we will derive the scattered magnetic field below.

Since $\mathbf{E}_s = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H}_s$ and $H_{s,z} = 0$, we have

$$E_{s,\rho} = -\frac{1}{j\omega\epsilon}\frac{\partial H_{s,\phi}}{\partial z} \tag{7.89}$$

$$E_{s,\phi} = \frac{1}{j\omega\epsilon}\frac{\partial H_{s,\rho}}{\partial z} \tag{7.90}$$

On the other hand, since all z dependence is in the exponential form $e^{jkz\cos\theta_i}$ for the scattered field, we have

$$\frac{\partial}{\partial z} \longrightarrow jk\cos\theta_i \tag{7.91}$$

Furthermore, using

$$\mathbf{H}_s = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E}_s \tag{7.92}$$

together with (7.90) and (7.90) we obtain

$$\frac{1}{k^2}(k^2 + \frac{\partial^2}{\partial z^2})H_\rho^s = -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z^s}{\partial \phi} \tag{7.93}$$

$$\frac{1}{k^2}(k^2 + \frac{\partial^2}{\partial z^2})H_\phi^s = \frac{1}{j\omega\mu}\frac{\partial E_z^s}{\partial \rho} \tag{7.94}$$

Using (7.91) in (7.93) and (7.94) yields

$$\begin{aligned} H_{s,\rho} &= jE_0\frac{e^{jkz\cos\theta_i}}{\omega\mu\rho\sin\theta_i}\sum_{n=-\infty}^{\infty} nj^{-n}a_n H_n^{(2)}(k\rho\sin\theta_i)e^{jn\phi} \\ H_{s,\phi} &= -j\frac{E_0}{\eta}e^{jkz\cos\theta_i}\sum_{n=-\infty}^{\infty} j^{-n}a_n H_n^{(2)'}(k\rho\sin\theta_i)e^{jn\phi} \\ H_{s,z} &= 0 \end{aligned} \tag{7.95}$$

For the far field $\rho \longrightarrow \infty$, $H_\rho^s \propto 1/\rho^{\frac{3}{2}}$, $H_\phi^s \propto 1/\rho^{\frac{1}{2}}$. Therefore, with $|\mathbf{H}_i| = |E_0/\eta|$, the scattering width is

$$\sigma_{2D} = \lim_{\rho\to\infty}\left(2\pi\rho\frac{|H_\phi^s|^2}{|\mathbf{H}^i|^2}\right) = \frac{2\lambda}{\pi\sin\theta_i}\left|\sum_{n=0}^{\infty}(2-\delta_{n0})a_n\cos n\phi\right|^2 \tag{7.96}$$

where a_n is given by (7.86). In particular, for small $a \ll \lambda$, this can be approximated as

$$\sigma_{2D} \overset{a\ll\lambda}{\approx} \frac{\pi\lambda}{2\sin\theta_i}\left|\frac{1}{\ln(0.89ka\sin\theta_i)}\right|^2 \tag{7.97}$$

and is independent of ϕ.

In the case of a finite cylinder of length ℓ, the radar cross section RCS σ_{3D} have to be obtained by 3-D numerical or approximate methods. However, in the special case where the length $\ell \gg \lambda$, we can obtain an approximate relation between σ_{2D} and σ_{3D} for the finite cylinder of length ℓ

$$\sigma_{3D} \approx \sigma_{2D} \left\{ \frac{2\ell^2}{\lambda} \sin^2\theta_p \left[\frac{\sin^2 \frac{k\omega}{2}(\cos\phi + \cos\phi_i)}{\frac{k\omega}{2}(\cos\phi + \cos\phi_i)^2} \right]^2 \right\}, \tag{7.98}$$

where

$$\theta_p = \begin{cases} \theta & \text{for TM}_z \text{ wave} \\ \theta_i & \text{for TE}_z \text{ wave} \end{cases} \tag{7.99}$$

However, note that this equation is approximate for $\ell \gg \lambda$, and is not valid for θ_i close to zero (or axial incidence).

7.4.4 Oblique TE$_z$ Scattering

Similarly, if the incident plane wave is an oblique TE$_z$ wave, we have

$$\mathbf{E}_i = -\hat{y} H_0 \eta e^{-jk\hat{k}_i\cdot\mathbf{r}}, \quad \hat{k}_i = \hat{x}\sin\theta_i - \hat{z}\cos\theta_i$$

$$\begin{aligned} \mathbf{H}_i &= H_0(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i) e^{jkz\cos\theta_i} e^{-jkx\sin\theta_i} \\ &= H_0(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i) e^{jkz\cos\theta_i} \sum_{n=-\infty}^{\infty} j^{-n} J_n(k\rho\sin\theta_i) e^{jn\phi} \end{aligned} \tag{7.100}$$

Transformed in the cylindrical coordinates, this gives

$$H_{i,\rho} = H_{i,x}\cos\phi \tag{7.101}$$

$$H_{i,\phi} = -H_{i,x}\sin\phi \tag{7.102}$$

Since $E_z^i = 0$, we again obtain

$$\left(1 - \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) E_{i,\rho} = \frac{1}{j\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z^i}{\partial\phi} \tag{7.103}$$

$$\left(1 - \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) E_{i,\phi} = -\frac{1}{j\omega\epsilon}\frac{\partial H_{i,z}}{\partial\rho} \tag{7.104}$$

Using the phase-matching condition and $\frac{\partial}{\partial z} \longrightarrow jk\cos\theta_i$, we have

$$H_{s,z} = H_0 \sin\theta_i e^{jkz\cos\theta_i} \sum_{n=-\infty}^{\infty} j^{-n} b_n H_n^{(2)}(k\rho\sin\theta_i) e^{jn\phi} \tag{7.105}$$

From the boundary condition $H_\rho = 0$ at $\rho = a$, we find

$$b_n = -\frac{J_n'(ka\sin\theta_i)}{H_n^{(2)'}(k\rho\sin\theta_i)} \tag{7.106}$$

Finally, we obtain the scattering width

$$\sigma_{2D} = \lim_{\rho\to\infty}\left(2\pi\rho\frac{|E_\phi^s|^2}{|\mathbf{E}^i|^2}\right) = \frac{2\lambda}{\pi\sin\theta_i}\left|\sum_{n=0}^{\infty}(2-\delta_{n0})b_n\cos n\phi\right|^2 \tag{7.107}$$

For small $a \ll \lambda$, this can be approximated as

$$\sigma_{2D} \overset{a\ll\lambda}{\approx} \frac{\pi\lambda(ka\sin\theta_i)^4}{8\sin\theta_i}[1-2\cos\phi]^2 \tag{7.108}$$

7.4.5 Line-Source Scattering (Line parallel to z)

A. Electric Line Source (TM$_z$)

For an electric line source, the incident field is

$$E_z^i = -\frac{k^2 I_e}{4\omega\epsilon}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) = -\frac{k^2 I_e}{4\omega\epsilon}\sum_{n=-\infty}^{\infty}J_n(k\rho_<)H_n^{(2)}(k\rho_>)e^{jn(\phi-\phi')} \tag{7.109}$$

where the addition theorem has been used.

The scattered field can be also expanded as

$$E_{s,z} = -\frac{k^2 I_e}{4\omega\epsilon}\sum_{n=-\infty}^{\infty}c_n H_n^{(2)}(k\rho) \qquad \text{for } \rho \gg a \tag{7.110}$$

On the circular PEC surface at $\rho = a$, the boundary condition requires $E_z = 0$, or

$$c_n = -H_n^{(2)}(k\rho')\frac{J_n(ka)}{H_n^{(2)}(ka)}e^{jn(\phi-\phi')} \tag{7.111}$$

Therefore the scattered electric field is

$$E_{s,z} = \frac{k^2 I_e}{4\omega\epsilon}\sum_{n=-\infty}^{\infty}H_n^{(2)}(k\rho')\frac{J_n(ka)}{H_n^{(2)}(ka)}H_n^{(2)}(k\rho)e^{jn(\phi-\phi')} \tag{7.112}$$

The total electromagnetic fields are

$$\mathbf{E} = -\hat{z}\frac{k^2 I_e}{4\omega\epsilon}\sum_{n=-\infty}^{\infty}H_n^{(2)}(k\rho_>)\left[\frac{J_n(k\rho_<)H_n^{(2)}(ka) - J_n(ka)H_n^{(2)}(k\rho_<)}{H_n^{(2)}(ka)}\right]e^{jn(\phi-\phi')} \tag{7.113}$$

$$\begin{aligned}\mathbf{H} &= \hat{\rho}\frac{I_e}{4\rho}\sum_{n=-\infty}^{\infty}nH_n^{(2)}(k\rho_>)\left[\frac{J_n(k\rho_<)H_n^{(2)}(ka) - J_n(ka)H_n^{(2)}(k\rho_<)}{H_n^{(2)}(ka)}\right]e^{jn(\phi-\phi'} \\ &\quad +\hat{\phi}\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial\rho}\end{aligned} \tag{7.114}$$

Far field,

$$E_z \overset{k\rho\gg 1}{\approx} -\frac{k^2 I_e}{4\omega\epsilon}\sqrt{\frac{2j}{\pi k}}\frac{e^{-jk\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{\infty} j^n \left[\frac{J_n(k\rho')H_n^{(2)}(ka) - J_n(ka)H_n^{(2)}(k\rho')}{H_n^{(2)}(ka)}\right] e^{jn(\phi-\phi')} \tag{7.115}$$

Obviously, at the shadow zone (the back of the cylinder where the observation point is $\pi/2 \le \phi - \phi' \le 3\pi/2$), the fields are not zero. These are known as the "creeping" waves. If the cylinder is infinite in radius (i.e., a planar PEC), there would be no waves in this shadow zone.

B. Magnetic Line Source (TE$_z$)

For a magnetic line source parallel to z, the incident wave is

$$\begin{aligned} H_{i,z} &= -\frac{k^2 I_m}{4\omega\mu} H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) \\ &= -\frac{k^2 I_m}{4\omega\mu}\sum_{n=-\infty}^{\infty} J_n(k\rho_<)H_n^{(2)}(k\rho_>)e^{jn(\phi-\phi')} \end{aligned} \tag{7.116}$$

The scattered magnetic field can also be written as

$$H_{s,z} = -\frac{k^2 I_m}{4\omega\mu}\sum_{n=-\infty}^{\infty} d_n H_n^{(2)}(k\rho)e^{jn(\phi-\phi')} \tag{7.117}$$

To find the unknown expansion coefficient d_n, we apply the boundary condition for the tangential electric field component E_ϕ. Specifying (7.116) for $\rho \le \rho'$, we have the total magnetic field as

$$H_z = -\frac{k^2 I_m}{4\omega\mu}\sum_{n=-\infty}^{\infty}\left[J_n(k\rho)H_n^{(2)}(k\rho') + d_n H_n^{(2)}(k\rho)\right]e^{jn(\phi-\phi')} \tag{7.118}$$

From the magnetic field, we can find

$$E_\phi = \frac{1}{j\omega\mu}\frac{\partial H_z}{\partial\rho} = -\frac{jkI_m}{4}\sum_{n=-\infty}^{\infty}\left[J_n'(k\rho)H_n^{(2)}(k\rho') + d_n H_n^{(2)'}(k\rho)\right]e^{jn(\phi-\phi')} \tag{7.119}$$

At the PEC surface $\rho = a$, the boundary condition $E_\phi = 0$ yields

$$d_n = -H_n^{(2)}(k\rho')\frac{J_n'(ka)}{H_n^{(2)'}(ka)} \tag{7.120}$$

For the far-zone $k\rho \gg 1$, we obtain the total magnetic field

$$H_z \overset{k\rho\gg 1}{\approx} -\frac{k^2 I_m}{4\omega\mu}\sqrt{\frac{2j}{\pi k}}\frac{e^{-jk\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{\infty} j^n\left[J_n(k\rho') - \frac{J_n'(ka)}{H_n^{(2)'}(ka)}H_n^{(2)}(k\rho')\right]e^{jn(\phi-\phi')} \tag{7.121}$$

7.5 Scattering by a Conducting Wedge

We now consider the scattering by a conducting wedge with its faces between $-\alpha \leq \phi \leq \alpha$ and extends to $\rho \to \infty$, and the wedge is infinite in z direction. The scattering of incident waves from line sources in z direction will be considered here.

7.5.1 Electric Line Source

For an electric line source in z direction, the incident electric field is

$$E_z^i = -\frac{k^2 I_e}{4\omega\epsilon} \sum_{n=-\infty}^{\infty} J_n(k\rho_<)(k\rho)H_n^{(2)}(k\rho_>)e^{jn(\phi-\phi')} \tag{7.122}$$

Note that this expansion is for a periodic problem in ϕ. Now we have a non-periodic problem, so we seek an alternative form for the total field. Instead of using the integer index n, we now use a noninteger index v so that the total electric field for $\alpha \leq \phi/2\pi - \alpha$ is written as

$$E_z = \begin{cases} \sum_v c_v f(\rho') J_v(k\rho) \sin[v(\phi'-\alpha)] \sin[v(\phi-\alpha)] & \text{for } \rho \leq \rho' \\ \sum_v d_v g(\rho') H_v^{(2)}(k\rho) \sin[v(\phi'-\alpha)] \sin[v(\phi-\alpha)] & \text{for } \rho \geq \rho' \end{cases} \tag{7.123}$$

At $\rho = \rho'$, the continuity of E_z^t requires

$$c_v = d_v \equiv a_v, \quad f(\rho') = H_n^{(2)}(k\rho'), \quad g(\rho') = J_n^{(2)}(k\rho')$$

At $\phi = \alpha$ and $\phi = 2\pi - \alpha$, $E_z^t = 0$

$$v = \frac{m\pi}{2(\pi-\alpha)}, \qquad m = 1, 2, 3, \cdots \tag{7.124}$$

Thus, the total electric field is

$$E_z = \sum_v a_v J_v(k\rho_<) H_v^{(2)}(k\rho'_>) \sin[v(\phi'-\alpha)] \sin[v(\phi-\alpha)] \tag{7.125}$$

The magnetic field can be obtained by

$$H_\rho = -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi}, \quad H_\phi = \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial \rho} \tag{7.126}$$

The unknown coefficient a_v can be obtained by the boundary condition at the source location $\rho = \rho'$. This boundary condition requires that

$$\mathbf{J}_z = \hat{\rho} \times [\mathbf{H}(\rho = \rho' + 0) - \mathbf{H}(\rho = \rho' - 0)]$$

where the surface current $\mathbf{J}_s = \hat{z} I_e \frac{2\pi}{\rho'}\delta(\phi - \phi')$.

$$I_e \frac{2\pi}{\rho'}\delta(\phi-\phi') = [H_\phi(\rho = \rho'+0) - H_\phi(\rho = \rho'-0)] = \frac{2}{\pi\omega\mu\rho'} \sum_v a_v \sin[v(\phi'-\alpha)] \sin[v(\phi-\alpha)] \tag{7.127}$$

where we have made use of the Wronskian $W[J_v, H_v^{(2)}] = -2j/\pi x$. But since

$$J_z = -\frac{I_e}{(\pi-\alpha)\rho'}\sum_v \sin[v(\phi'-\alpha)]\sin[v(\phi-\alpha)] \tag{7.128}$$

we finally obtain

$$a_v = -\frac{\pi\omega\mu I_e}{2(\pi-\alpha)} \tag{7.129}$$

Far field,

$$E_z \overset{k\rho\to\infty}{\approx} f_e(\rho)\sum_v j^v J_v(k\rho')\sin[v(\phi'-\alpha)]\sin[v(\phi-\alpha)] \tag{7.130}$$

$$f_e(\rho) = -I_e\sqrt{\frac{\pi j}{2k}}\frac{\omega\mu}{\pi-\alpha}\frac{e^{-jk\rho}}{\sqrt{\rho}} \tag{7.131}$$

Note that if expanded in terms of these sine series, a delta function is

$$\delta(\phi-\phi') = \frac{1}{(\pi-\alpha)}\sum_v \sin[v(\phi'-\alpha)]\sin[v(\phi-\alpha)] \tag{7.132}$$

This can be proved by integration. Similarly,

$$\delta(\phi-\phi') = \frac{1}{2(\pi-\alpha)}\sum_s (2-\delta_{s0})\cos[s(\phi'-\alpha)]\cos[s(\phi-\alpha)] \tag{7.133}$$

Equations (7.132) and (7.133) can be easily derived by multiplying these equation with $\sin[w(\phi-\alpha)]$ and $\cos[w(\phi-\alpha)]$ respectively and integrating from α to $(2\pi-\alpha)$.

7.5.2 Magnetic Line Source

For a magnetic line source, the total magnetic field is now expanded in terms of cosine series

$$H_z = \begin{cases} \sum_s b_s J_s(k\rho) H_s(\rho')\cos[s(\phi'-\alpha)]\cos[s(\phi-\alpha)] & \text{for } \rho \le \rho' \\ \sum_s b_s J_s(k\rho') H_s(\rho)\cos[s(\phi'-\alpha)]\cos[s(\phi-\alpha)] & \text{for } \rho \ge \rho' \end{cases} \tag{7.134}$$

The electric field can be written in terms of **H** as

$$E_\rho = \frac{1}{j\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z^t}{\partial\phi} \tag{7.135}$$

To satisfy boundary condition, it is required that

$$s = \frac{m\pi}{2(\pi-\alpha)} \qquad m = 0, 1, 2, 3, \cdots \tag{7.136}$$

From $\mathbf{M}_s = -\hat{n} \times (\mathbf{E}_+^t - \mathbf{E}_-^t)$ we obtain

$$b_s = (2 - \delta_{s0}) \frac{\pi \omega \epsilon I_m}{4(\pi - \alpha)} \tag{7.137}$$

In the far-field zone,

$$H_z \overset{k\rho \to \infty}{\approx} I_m \sqrt{\frac{\pi j}{8k}} \frac{\omega \epsilon}{\pi - \alpha} \frac{e^{-jk\rho}}{\sqrt{\rho}} \sum_s (2 - \delta_{s0}) j^s J_s(k\rho') \cos[s(\phi' - \alpha)] \cos[s(\phi - \alpha)] \tag{7.138}$$

For a source at long distance ($k\rho' \gg 1, \rho' > 1$)

$$H_z \overset{k\rho' \to \infty}{\approx} H_0 \sum_s (2 - \delta_{s,0} j^s J_s(k\rho) \cos[s(\phi' - \alpha)] \cos[s(\phi - \alpha)] \tag{7.139}$$

where

$$H_0 = g_n(\rho') = I_m \sqrt{\frac{\pi j}{8k}} \frac{\omega \epsilon}{\pi - \alpha} \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \tag{7.140}$$

For a uniform TE^z plane wave of strength H_0, this is also the total field.

A special case is a half plane ($\alpha = 0$). In this case

$$s = \frac{m\pi}{2\pi} = \frac{m}{2}, \qquad m = 0, 1, 2, 3, \cdots \tag{7.141}$$

$$H_z \overset{k\rho' \to \infty}{\approx} H_0 \sum_{m=0}^{\infty} (2 - \delta_{\frac{m}{2},0}) j^{\frac{m}{2}} J_{\frac{m}{2}}(k\rho) \cos(m\phi'/2) \cos(m\phi/2) \tag{7.142}$$

7.5.3 A Combined Formula for TE_z and TM_z

If a new reference $\psi = \phi - \alpha$ is chosen, then $\psi' = \phi' - \alpha$. Furthermore, let $2\alpha = (2 - n)\pi$ or $n = 2 - \frac{2\alpha}{\pi}$, then

$$s = \frac{m\pi}{2(\pi - \alpha)} = \frac{m}{n} \qquad m = 0, 1, 2, 3, \cdots \tag{7.143}$$

$$a_v = -\frac{\omega \mu I_e}{2(2/n)} \tag{7.144}$$

$$b_s = \frac{\omega \epsilon I_m}{2} (\epsilon_s / n) \tag{7.145}$$

Then,

$$E_z^t = -\frac{\omega \mu I_e}{4} G(\rho, \rho', \psi, \psi', n) \qquad \text{for TM}_z \tag{7.146}$$

$$H_z^t = \frac{\omega \epsilon I_m}{4} G(\rho, \rho', \psi, \psi', n) \qquad \text{for TE}_z \tag{7.147}$$

where

$$G = \frac{1}{n}\sum_{m=0}^{\infty}(2-\delta_{m,0})J_{m/n}(k\rho_<)H^{(2)}_{m/n}(k\rho_>) \cdot\cos[\frac{m}{n}(\psi-\psi')]\pm\cos[\frac{m}{n}(\psi+\psi')], \quad \begin{cases} + \text{ for } \mathrm{TM}_z \\ - \text{ for } \mathrm{TE}_z \end{cases} \quad (7.148)$$

7.6 Spherical Wave Functions

7.6.1 Dipole Fields

For infinitesimal electric dipole $\mathbf{J} = \hat{a}I_e\ell\delta(\mathbf{r}-\mathbf{r}')$ and magnetic dipole $\mathbf{M} = \hat{a}I_m\ell\delta(\mathbf{r}-\mathbf{r}')$ produce magnetic and electric vector potentials

$$\mathbf{A} = \hat{a}\frac{\mu I_e\ell}{4\pi}\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = -\hat{z}j\frac{\mu k I_e\ell}{4\pi}h_0^{(2)}(k|\mathbf{r}-\mathbf{r}'|) \quad (7.149)$$

$$\mathbf{F} = -\hat{a}j\frac{\epsilon k I_m\ell}{4\pi}h_0^{(2)}(k|\mathbf{r}-\mathbf{r}'|) \quad (7.150)$$

respectively, where $\mathbf{r}'$ is the location of the dipoles. These potential expressions are written in terms of spherical wave functions. Similar to cylindrical coordinates, here we will present wave transformation relations in spherical coordinates.

7.6.2 Orthogonality Relationships

Legendre Polynomials (also called zonal harmonics) $P_n(cos\theta)$ satisfy the following orthogonality relationship

$$\int_0^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1}\delta_{mn} \quad (7.151)$$

Because of this orthogonality, a function of θ can be expanded as a Fourier-Legendre series

$$f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \qquad \text{for } 0\le\theta\le\pi \quad (7.152)$$

$$a_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta d\theta \quad (7.153)$$

The even and odd tesseral harmonics are defined as

$$T^e_{mn}(\theta,\phi) = P_n^m(\cos\theta)\cos(m\phi) \quad (7.154)$$

$$T^0_{mn}(\theta,\phi) = P_n^m(\cos\theta)\sin(m\phi) \quad (7.155)$$

which satisfy the following orthogonality relationships

$$\int_0^{2\pi}\int_0^{\pi} T^i_{mn}(\theta,\phi)T^j_{pq}(\theta,\phi)\sin\theta d\theta d\phi = \frac{2\pi}{2n+1}(1+\delta_{m0})\delta_{ij}\delta_{mp}\delta_{nq} \qquad \text{where } i,j = e \text{ or } o \quad (7.156)$$

7.6.3 Wave Transformations and Theorems

Similar to cylindrical coordinates, a plane wave can be represented in terms of spherical wave functions:

$$e^{-jkz} = e^{-jkr\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta) \tag{7.157}$$

To find a_n we multiply both side by P_m and integrate to obtain

$$\int_0^{\pi} e^{-jkr\cos\theta} P_m(\cos\theta)\sin\theta d\theta = \frac{2a_m}{2m+1} j_m(kr) \tag{7.158}$$

However, we can use an identity

$$\boxed{\int_0^{\pi} e^{-jkr\cos\theta} P_m(\cos\theta)\sin\theta d\theta = 2j^{-m} j_m(kr)} \tag{7.159}$$

to arrive at

$$a_m = j^{-m}(2m+1) \tag{7.160}$$

$$\boxed{e^{-jkz} = e^{-jkr\cos\theta} = \sum_{n=0}^{\infty} j^{-n}(2n+1) j_n(kr) P_n(\cos\theta)} \tag{7.161}$$

Similarly,

$$\boxed{E_x^- = e^{jkz} = e^{jkr\cos\theta} = \sum_{n=0}^{\infty} j^{n}(2n+1) j_n(kr) P_n(\cos\theta)} \tag{7.162}$$

The addition theorem for spherical wave functions are

$$\boxed{h_0^{(2)}(k|\mathbf{r}-\mathbf{r}'|) = \sum_{n=0}^{\infty} (2n+1) j_n(kr_<) h_n^{(2)}(kr_>) P_n(\cos\theta)} \tag{7.163}$$

$$\boxed{h_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|) = \sum_{n=0}^{\infty} (2n+1) j_n(kr_<) h_n^{(2)}(kr_>) P_n(\cos\theta)} \tag{7.164}$$

7.7 Scattering by a PEC Sphere

Assuming an incident TM_x plane wave propagating along the z axis. The incident electric field can be written as

$$\mathbf{E}_i = \hat{x}E_0 e^{-jkr\cos\theta} = \hat{r}E_r^i + \hat{\theta}E_\theta^i + \hat{\phi}E_\phi^i \tag{7.165}$$

In spherical coordinates, the r component is

$$\begin{aligned} E_{i,r} &= \hat{r}\cdot\mathbf{E}^i = \hat{r}\cdot\hat{x}E_0 e^{-jkr\cos\theta} = E_x^i \sin\theta\cos\phi = E_0\frac{\cos\phi}{jkr}\frac{\partial}{\partial\theta}(e^{-jkr\cos\theta}) \\ &= E_0\frac{\cos\phi}{jkr}\sum_{n=0}^{\infty} j^{-n}(2n+1) j_n(kr)\frac{\partial}{\partial\theta}P_n(\cos\theta) \end{aligned} \tag{7.166}$$

$$\begin{aligned} E_{i,\theta} &= \hat{\theta}\cdot\hat{x}E_0 e^{-jkr\cos\theta} = E_0\cos\theta\cos\phi e^{-jkr\cos\theta} \\ &= E_0\cos\theta\cos\phi\sum_{n=0}^{\infty} j^{-n}(2n+1)j_n(kr)P_n(\cos\theta) \end{aligned} \tag{7.167}$$

$$\begin{aligned} E_{i,\phi} &= \hat{\phi}\hat{x}E_0 e^{-jkr\cos\theta} = -E_0\sin\phi e^{-jkr\cos\theta} \\ &= -E_0\sin\phi\sum_{n=0}^{\infty} j^{-n}(2n+1)j_n(kr)P_n(\cos\theta) \end{aligned} \tag{7.168}$$

Using Schelkunoff spherical Bessel and Hankel functions

$$\hat{B}_n(kr) = krb_n(kr) \tag{7.169}$$

and the relation that

$$\frac{\partial P_n}{\partial\theta} = P_n^1(\cos\theta), \qquad P_0^1 = 0 \tag{7.170}$$

We have

$$E_{i,r} = -jE_0\frac{\cos\phi}{(kr)^2}\sum_{n=1}^{\infty} j^{-n}(2n+1)\hat{J}_n(kr)P_n^1(\cos\theta) \tag{7.171}$$

$$E_{i,\theta} = E_0\frac{\cos\theta\cos\phi}{kr}\sum_{n=0}^{\infty} j^{-n}(2n+1)\hat{J}_n(kr)P_n^0(\cos\theta) \tag{7.172}$$

$$E_{i,\phi} = -E_0\frac{\sin\phi}{kr}\sum_{n=0}^{\infty} j^{-n}(2n+1)\hat{J}_n(kr)P_n^0(\cos\theta) \tag{7.173}$$

In spherical coordinates, waves can be written as the superposition of TM_r (A_r) and TE_r (F_r), which are always coupled. For TM_r wave, $H_r = 0$ and

$$E_r = \frac{1}{j\omega\mu\epsilon}(\frac{\partial^2}{\partial r^2} + k^2)A_r \tag{7.174}$$

Comparing (7.171) and (7.174), we arrive at

$$A_{i,r} = E_0\frac{\cos\phi}{\omega}\sum_{n=\phi}^{\infty} a_n\hat{J}_n(kr)P_n^1(\cos\theta) \tag{7.175}$$

where

$$a_n = j^{-n}\frac{(2n+1)}{n(n+1)} \tag{7.176}$$

Similarly for TE^r,

$$F_{i,r} = E_0 \frac{\sin\phi}{\omega\eta} \sum_{n=1}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta) \tag{7.177}$$

Similarly, for the scattered fields, the vector potentials can expanded as

$$A_{s,r} = E_0 \frac{\cos\phi}{\omega} \sum_{n=1}^{\infty} b_n \hat{H}_n^{(2)}(kr) P_n^1(\cos\theta) \tag{7.178}$$

$$F_{s,r} = E_0 \frac{\sin\phi}{\omega\eta} \sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)}(kr) P_n^1(\cos\theta) \tag{7.179}$$

The reason for the ϕ dependence of A_r and F_r to differ is make the two terms in E_θ to have the same ϕ dependence that will make the electric field to be an even function of x as in the incident field. Thus the total fields are

$$E_r = \frac{1}{j\omega\mu\epsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right) A_r \tag{7.180}$$

$$E_\theta = \frac{1}{j\omega\mu\epsilon} \frac{1}{r} \frac{\partial^2 A_r}{\partial r \partial\theta} - \frac{1}{\epsilon} \frac{1}{r\sin\theta} \frac{\partial F_r}{\partial\phi} \tag{7.181}$$

$$E_\phi = \frac{1}{j\omega\mu\epsilon} \frac{1}{r\sin\theta} \frac{\partial^2 A_r}{\partial r \partial\phi} + \frac{1}{\epsilon} \frac{1}{r} \frac{\partial F_r}{\partial\theta} \tag{7.182}$$

In particular, the θ component of electric field is

$$\begin{aligned} E_\theta = & -j\tfrac{E_0}{\omega\mu\epsilon r}\left\{\tfrac{k}{\omega}\cos\phi \sum_{n=1}^{\infty}\left[a_n \hat{J}_n'(kr) + b_n \hat{H}_n^{(2)'}(kr)\right] P_n'^1(\cos\theta)\right\} \\ & -\tfrac{E_0}{\epsilon r \sin\theta}\left\{\tfrac{1}{\omega\eta}\cos\phi \sum_{n=1}^{\infty}\left[a_n \hat{J}_n(kr) + c_n \hat{H}_n^{(2)}(kr)\right] P_n'(\cos\theta)\right\} \end{aligned} \tag{7.183}$$

The boundary conditions at $r = a$ requires that $E_\theta = 0$, or

$$b_n = -a_n \frac{\hat{J}_n'(kr)}{\hat{H}_n^{(2)'}(kr)} \tag{7.184}$$

$$c_n = -a_n \frac{\hat{J}_n(kr)}{\hat{H}_n^{(2)}(kr)} \tag{7.185}$$

To calculate the far-zone scattered field, note that asymptotic forms for $\hat{B}_n(kr) = \sqrt{\frac{\pi kr}{2}} B_{n+1/2}(kr)$ are

$$\hat{H}_n^{(2)}(kr) \stackrel{kr\to\infty}{\longrightarrow} j^{n+1} e^{-jkr} \tag{7.186}$$

$$\hat{H}_n^{(2)'}(kr) \stackrel{kr\to\infty}{\longrightarrow} j^{n} e^{-jkr} \tag{7.187}$$

$$\hat{H}_n^{(2)''}(kr) \stackrel{kr\to\infty}{\longrightarrow} -j^{n+1} e^{-jkr} \tag{7.188}$$

Thus, the far-zone scattered fields are

$$\hat{E}_{s,r} \longrightarrow 0 \tag{7.189}$$

$$\hat{E}_{s,\theta} \longrightarrow jE_0 \frac{e^{-jkr}}{kr} \cos\theta f_\theta \tag{7.190}$$

$$\hat{E}_{s,\phi} \longrightarrow jE_0 \frac{e^{-jkr}}{kr} \sin\theta f_\phi \tag{7.191}$$

where

$$f_\theta = \sum_{n=1}^{\infty} \left[b_n \sin\theta P_n''(\cos\theta) - c_n \frac{P_n'(\cos\theta)}{\sin\theta} \right]$$

$$f_\phi = \sum_{n=1}^{\infty} \left[b_n \frac{P_n'(\cos\theta)}{\sin\theta} - c_n \sin\theta P_n'(\cos\theta) \right]$$

The radar cross section for the PEC sphere is

$$\sigma_{3D}(\text{bistatic}) = \lim_{\rho\to\infty} \left(4\pi r^2 \frac{|\mathbf{E}_s|^2}{|\mathbf{E}_i|^2} \right) = \frac{\lambda^2}{\pi} [\cos^2\phi |f_\theta|^2 + \sin^2\phi |f_\phi|^2] \tag{7.192}$$

The monostatic RCS is

$$\sigma_{3D}(\text{monostatic}) = \frac{\lambda^2}{4\pi} \left| \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{\hat{H}_n^{(2)'}(ka) \hat{H}_n^{(2)}(ka)} \right|^2 \tag{7.193}$$