Chapter 9

GREEN'S FUNCTIONS

This chapter discusses the Green's functions for various systems. A Green's function is similar to the impulse response in the system theory. In electromagnetic problems, it corresponds to the solution of the partial differential equation(s) with an impulse (Dirac delta) source as the driving function. For example, for a scalar wave equation

$$\nabla^2\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = f(\mathbf{r}) \tag{9.1}$$

where the wave field $\phi(\mathbf{r})$ is due to the source $f(\mathbf{r})$ in the medium, with appropriate boundary conditions. The corresponding Green's function $G(\mathbf{r},\mathbf{r}')$ can be define as

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + k^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{9.2}$$

with some (not necessarily the same) boundary conditions. It will become clear that once the solution to the Green's function is known, the solution for the wave field can be obtained quite easily.

In this chapter, we will be mainly concerned with scalar Green's functions. At the end, we will briefly discuss the dyadic Green's functions for vector wave equations.

9.1 Green's Identities and Methods

Let's consider the scalar Helmholtz equation in (9.1) with the homogeneous boundary condition on the enclosed surface S

$$\alpha_1\phi(\mathbf{r}_s) + \alpha_2\frac{\partial\phi(\mathbf{r}_s)}{\partial n} = 0 \tag{9.3}$$

or the radiation condition if $S \to S_\infty$

$$\lim_{r_s\to\infty} r_s\left[\frac{\partial\phi(\mathbf{r}_s)}{\partial r_s} + jk\phi(\mathbf{r}_s)\right] = 0. \tag{9.4}$$

For the system (9.1) and (9.3) or (9.4), the corresponding Green's function $G(\mathbf{r}, \mathbf{r}')$ is defined by (9.2) and the following boundary condition

$$\alpha_1 G(\mathbf{r}_s, \mathbf{r}') + \alpha_2 \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} = 0 \tag{9.5}$$

or the radiation condition if $S \to S_\infty$

$$\lim_{r_s \to \infty} r_s \Big[\frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial r_s} + jkG(\mathbf{r}_s, \mathbf{r}') \Big] = 0. \tag{9.6}$$

In the above, $\hat{n}$ is the unit outward normal direction of the surface S.

9.1.1 Green's First and Second Identities

If two functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ have continuous first and second derivatives on S, then

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \tag{9.7}$$

Integrating (9.7) and using the divergence theorem yields

$$\oint_S (\phi \nabla \psi) \cdot d\mathbf{s} = \int_V \phi \nabla^2 \psi dv + \int_V (\nabla \phi \cdot \nabla \psi) dv \tag{9.8a}$$

or

$$\oint_S (\phi \frac{\partial \psi}{\partial n}) ds = \int_V \phi \nabla^2 \psi dv + \int_V (\nabla \phi \cdot \nabla \psi) dv \tag{9.8b}$$

which is called the Green's first identity.

Similarly, we have

$$\oint_S (\psi \nabla \phi) \cdot d\mathbf{s} = \int_V \psi \nabla^2 \phi dv + \int_V (\nabla \psi \cdot \nabla \phi) dv \tag{9.9}$$

Combining (9.8a) and (9.9), we have

$$\oint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{s} = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv \tag{9.10a}$$

or

$$\oint_S (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) ds = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv \tag{9.10b}$$

which is called the Green's second identity.

9.1.2 Green's Function Methods

Subtracting (9.1) multiplied by G from (9.2) multiplied by ϕ, we have

$$\phi\nabla^2 G - G\nabla^2\phi = \phi\delta(\mathbf{r}-\mathbf{r}') - fG,$$

which after volume integration yields

$$\begin{aligned}\phi(\mathbf{r}') &= \int_V f(\mathbf{r})G(\mathbf{r},\mathbf{r}')dv + \int_V [\phi(\mathbf{r})\nabla^2 G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla^2\phi]dv \\ &= \int_V f(\mathbf{r})G(\mathbf{r},\mathbf{r}')dv + \oint_S [\phi(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla\phi]\cdot d\mathbf{s} \qquad (9.11)\end{aligned}$$

where the Green's second identity has been used. Exchanging $\mathbf{r}$ and $\mathbf{r}'$ and using the reciprocity $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$ in (9.11) yields

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')dv' + \oint_S [\phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'\phi(\mathbf{r}')]\cdot d\mathbf{s}' \qquad (9.12)$$

where the prime denotes the derivative with respect to $\mathbf{r}'$.

Note that so far we have not yet specified the boundary conditions, which will contribute to the second term in (9.12). Equation (9.12) expresses the field $\phi(\mathbf{r})$ in terms of the Green's function and the boundary conditions.

(a) Homogeneous Dirichlet boundary conditions:

If $\alpha_2 = 0$ in (9.3) and (9.5), we have the Dirichlet boundary conditions

$$\phi(\mathbf{r}) = 0, \quad G(\mathbf{r},\mathbf{r}') = 0 \qquad \mathbf{r}\in S \qquad (9.13)$$

Then (9.12) becomes

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')dv' \qquad (9.14)$$

(b) Nonhomogeneous Dirichlet boundary conditions:

If the boundary conditions are

$$\phi(\mathbf{r}) = \phi_0(\mathbf{r}), \quad G(\mathbf{r},\mathbf{r}') = 0 \qquad \mathbf{r}\in S \qquad (9.15)$$

Then (9.12) becomes

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')dv' + \oint_S \phi_0(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\cdot d\mathbf{s}' \qquad (9.16)$$

(c) Homogeneous Neumann boundary conditions:

If the boundary conditions are

$$\frac{\partial\phi(\mathbf{r})}{\partial n}=0,\quad \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}=\frac{1}{S_0}\qquad \mathbf{r}\in S \tag{9.17}$$

Then (9.12) becomes

$$\phi(\mathbf{r})=\int_V G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')dv'+\frac{1}{S_0}\oint_S \phi_0(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\cdot d\mathbf{s}' \tag{9.18}$$

However, the second term can be dropped since $\phi(\mathbf{r}_s)$ is undetermined by an additive constant.

(d) Mixed boundary conditions:

If the boundary conditions are

$$\left\{\begin{array}{l}\phi(\mathbf{r})=0\\ G(\mathbf{r},\mathbf{r}')=0\end{array}\right.\ \mathbf{r}\in S_1,\qquad \left\{\begin{array}{l}\frac{\partial\phi(\mathbf{r})}{\partial n}=0\\ \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}=0\end{array}\right.\qquad \mathbf{r}\in S-S_1 \tag{9.19}$$

Then (9.12) becomes

$$\phi(\mathbf{r})=\int_V G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')dv' \tag{9.20}$$

(e) Radiation boundary conditions:

When $S\to S_\infty$, and ϕ and G both satisfy the radiation boundary condition (9.4) and (9.6), the surface integral

$$\begin{aligned}&\oint_S[\phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')-G(\mathbf{r},\mathbf{r}')\nabla'\phi(\mathbf{r}')]\cdot d\mathbf{s}'\\ &\quad\sim r^2\Big\{\phi[-jkG+O(r^{-2})-G[-jk\phi+O(r^{-2})]\Big\}\\ &\quad\sim r^2(\phi-G)O(r^{-2})\\ &\quad\sim O(r^{-1})\to 0\qquad \text{as } r\to\infty\end{aligned} \tag{9.21}$$

Therefore, (4.12) becomes

$$\phi(\mathbf{r})=\int_V G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')dv' \tag{9.22}$$

9.2 Sturm-Liouville Problems

In this section we study a 1-D problem that is more general than the Helmholtz equation. It is the so-called Sturm-Liouville problem

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] - q(x)y = f(x), \qquad a \le x \le b \tag{9.23}$$

Note that a general 1-D PDE of the form

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = S(x) \tag{9.24}$$

can be converted to the form in (9.22). To accomplish this, we convert (9.23) and (9.24) into

$$\frac{d^2y}{dx^2} + \frac{1}{p(x)}\frac{dp}{dx}\frac{dy}{dx} - \frac{q(x)}{p(x)}y = \frac{f(x)}{p(x)} \tag{9.25}$$

$$\frac{d^2y}{dx^2} + \frac{B(x)}{A(x)}\frac{dy}{dx} + \frac{C(x)}{A(x)}y = \frac{S(x)}{A(x)} \tag{9.26}$$

Comparing (9.25) and (9.26) we obtain

$$\begin{cases} p(x) & = \exp\left[\int\limits_0^x \frac{B(t)}{A(t)}dt\right] \\ q(x) & = -\frac{C(x)}{A(x)}p(x) \\ f(x) & = \frac{S(x)}{A(x)}p(x) \end{cases} \tag{9.27}$$

An **example** of this conversion is the Bessel differential equation

$$x^2\frac{dy}{dx^2} + x\frac{dy}{dx} + (x^2 - \lambda^2)y = 0$$

which can be converted to SL form:

$$\begin{cases} p(x) & = \exp\left[\int\limits_0^x \frac{t}{t^2}dt\right] = x \\ q(x) & = -\frac{(x^2-\lambda^2)}{x^2}x = -\frac{x^2-\lambda^2}{x} \\ f(x) & = 0 \end{cases}$$

Therefore, the SL form is

$$\frac{d}{dx}x\frac{dy}{dx} + \frac{x^2-\lambda^2}{x}y = 0$$

In the following, we will derive the Green's function for the SL problem in three different forms.

9.2.1 Green's Function in Closed Form

A more general SL problem including an eigenvalue λ can be written as

$$\frac{d}{dx}p(x)\frac{dy}{dx} - q(x)y + \lambda r(x)y = f(x), \qquad a \le x \le b \tag{9.28a}$$

or

$$[L + \lambda r(x)]y = f(x) \tag{9.28b}$$

where L is the SL operator, and $r(x)$ is assumed piecewise continuous in the region of interest $a \le x \le b$. The equation (4.28) is of course subject to some homogeneous boundary conditions.

Equation (9.28) will have a Green's function for all values of λ except those satisfying

$$[L + \lambda r(x)]y = 0 \tag{9.29}$$

If the Green's function is defined as

$$[L + \lambda r(x)]G(x, x') = \delta(x - x') \tag{9.30}$$

which is also subject to the same homogeneous boundary conditions, then

$$y(x) = \int_a^b f(x')G(x, x')dx' \tag{9.31}$$

Equation (9.31) can be derived from (9.28) and (15.30) by taking into account the boundary conditions:

$$\left\{ y(x')p(x')\frac{dG(x', x)}{dx'} - G(x', x)p(x')\frac{dy(x')}{dx'} \right\}_a^b = 0 \tag{9.32}$$

Formally from (9.28b), we have

$$y(x) = [L + \lambda r(x)]^{-1} f(x) \tag{9.33}$$

Hence $G(x, x')$ can be thought of as the inverse of the operator $[L + \lambda r(x)]$ is λ is not an eigenvalue satisfying (9.29).

The following items are some properties of Green's functions:

(i) $G(x, x')$ satisfies the homogeneous differential equation except at the source point $x = x'$.

(ii) $G(x, x')$ is symmetric, i.e., $G(x, x') = G(x', x)$. This is the result of the reciprocity.

(iii) $G(x, x')$ is continuous everywhere, even at $x = x'$. This can be proven by reciprocity (ii) and perturbation.

(iv) The derivative of Green's function is discontinuous

$$\frac{dG(x=x'+0,x')}{dx} - \frac{dG(x=x'-0,x')}{dx} = \frac{1}{p(x')} \tag{9.34}$$

This can be proven by integrating (9.30) over $(x'-\epsilon, x'+\epsilon)$,

$$\lim_{\epsilon\to 0}\left\{p(x)\frac{dG(x,x')}{dx}\bigg|_{x'-\epsilon}^{x'+\epsilon} + \int_{x'-\epsilon}^{x'+\epsilon}[-q(x)+\lambda r(x)]G(x,x')dx\right\} = 1$$

and recognizing that the second term is zero because $q(x)$, $r(x)$, and $G(x,x')$ are at least piecewise continuous.

Using the above properties, we can construct the Green's function. First, assume that $y_1(x)$ and $y_2(x)$ are solutions satisfying the homogeneous differential equation [i.e., $f=0$ in (9.28)] for $a \le x < x'$ and for $x' \le x < b$ respectively. Then

$$G(x,x') = \begin{cases} A_1 y_1(x), & \text{for } a \le x < x' \\ A_2 y_2(x), & \text{for } x' \le x < b \end{cases} \tag{9.35}$$

The unknown constants A_1 and A_2 can be determined from properties (iii) and (iv) above, since

$$\begin{cases} A_1 y_1(x') = A_2 y_2(x') \\ -A_1 y_1'(x') + A_2 y_2'(x') = \frac{1}{p(x')} \end{cases}$$

which gives

$$A_1 = \frac{y_2(x')}{p(x')W(x')}, \quad A_2 = \frac{y_1(x')}{p(x')W(x')} \tag{9.36}$$

where

$$W(x') = y_1(x')y_2'(x') - y_2(x')y_1'(x') \tag{9.37}$$

is called the Wronskian of y_1 and y_2 at $x = x'$.

Therefore, the Green's function for (9.30) is

$$G(x,x') = \begin{cases} \frac{y_2(x')}{p(x')W(x')} y_1(x), & a \le x < x' \\ \frac{y_1(x')}{p(x')W(x')} y_2(x), & x' \le x < b \end{cases} \tag{9.38}$$

or in a shorthand notation as

$$G(x,x') = \frac{y_1(x_<)y_2(x_>)}{p(x')W(x')} \tag{9.39}$$

where $x_> = \max(x,x')$ and $x_< = \min(x,x')$. If the solutions to the homogeneous differential equation are known, then G can be obtained by (9.39).

Note that $G(x,x')$ has singularities when $W(x') = 0$.

Example: Sturm-Liouville operator L exhibits Hermitian (symmetrical) properties. Assuming that the B.C. is

$$\begin{cases} \alpha_1 y_i(a) + \alpha_2 y_i'(a) & = z_a \\ \beta_1 y_i(b) + \beta_2 y_i'(b) & = z_b \end{cases} \tag{9.40}$$

then one can easily derive

$$\begin{aligned}&\int_a^b (y_2Ly_1 - y_1Ly_2)dx = \int_a^b [(y_2py_1')' - (y_1Ly_2')']dx \\ &= [y_2py_1' - y_1Ly_2']_a^b \\ &= [y_2(b)p(b)y_1'(b) - y_1(b)p(b)y_2'(b)] - [y_2(a)p(a)y_1'(a) - y_1()p(a)y_2'(a)] \\ &= \frac{p(b)}{\beta_2}[y_2(b) - y_1(b)]z_b - \frac{p(a)}{\alpha_2}[y_2(a) - y_1(a)]z_a \\ &= 0 \qquad \text{if } z_a = z_b = 0 \end{aligned} \tag{9.41}$$

Hence for a homogeneous B.C.,

$$\int_a^b (y_2Ly_1)dx = \int_a^b (y_1Ly_2)dx. \tag{9.42}$$

Example: 1-D electrostatic problem. The system is described by

$$\frac{d^2V}{dx^2} = -\frac{1}{\epsilon_0}\rho(x), \quad a \le x \le b$$

with Dirichlet boundary conditions $V(a) = V(b) = 0$. Obviously,

$$y_1 = (x-a), \quad y_2 = (b-x), \quad W(x') = -(b-a).$$

Therefore,

$$G(x,x') = -\frac{(x_< - a)(x_> - b)}{(b-a)}.$$

There is no singularity for this Green's function.

9.2.2 Green's Function in Series Form

Now we derive the Green's function in series form. Again we consider the SL problem

$$\left\{\begin{array}{ll} [L+\lambda r(x)]G(x,x') & = \delta(x-x') \\ \alpha_1 G(a,x') + \alpha_2 G'(a,x') & = 0 \\ \beta_1 G(b,x') + \beta_2 G'(b,x') & = 0 \end{array}\right. \tag{9.43}$$

If $\{\psi_n(x)\}$ is a complete set of orthonormal eigenfunction for the SL operator L such that

$$[L+\lambda_n r(x)]\psi_n(x) = 0 \tag{9.44}$$

subject to the same B.C.'s as in (9.43), with

$$\int_a^b r(x)\psi_n(x)\psi_m(x)dx = \delta_{mn} \tag{9.45}$$

then is the Green's function exists, it can be written as

$$G(x,x') = \sum_n a_n(x')\psi_n(x) \tag{9.46}$$

where

$$a_n(x') = \int_a^b G(x,x')\psi_n(x)r(x)dx \tag{9.47}$$

From (9.43) and (9.44) we have

$$\int_a^b [\psi_n(x)LG(x,x') - G(x,x')L\psi_n(x)]dx = -(\lambda - \lambda_n)\int_a^b G(x,x')\psi_n(x)r(x)dx + \psi_n(x)$$

That is

$$0 = -(\lambda - \lambda_n)a_n(x') + \psi_n(x')$$

or

$$a_n(x') = \frac{\psi(x')}{\lambda - \lambda_n)}$$

Finally, the Green's function can be written as

$$G(x,x') = \sum_n \frac{\psi_n(x')\psi_n(x)}{(\lambda - \lambda_n)} \tag{9.48}$$

This is the bilinear formula for the Green's function. Note that it has singularities at $\lambda = \lambda_n$, i.e., when the parameter λ coincides with the eigenvalues λ_n.

Example: The transmission line problem. The equation for a transmission line problem for $0 \le x \le l$ can be written as

$$\begin{cases} \frac{d^2\phi(x)}{dx^2} + k^2\phi(x) = f(x) \\ \phi(0) = \phi(l) = 0 \end{cases} \tag{9.49}$$

where $k^2 = \omega^2\mu\epsilon$.

Solution. Written in the SL form, we find

$$p(x) = 1, \quad q(x) = 0, \quad r(x) = 1, \quad \lambda = k^2. \tag{9.50}$$

(a) Closed form solution. For the homogeneous differential equation, we have the solutions

$$\begin{cases} \phi_1(x) = \sin(kx), & 0 \le x \le x' \\ \phi_2(x) = \sin k(l-x), & x' \le x \le l \end{cases}$$

Therefore,

$$W(x') = -k\sin kx' \cos k(l-x') - k\cos kx' \sin k(l-x') = -k\sin kl$$

Hence,

$$G(x,x') = \begin{cases} -\frac{\sin[k(l-x')]\sin kx}{k\sin kl}, & 0 \le x \le x' \\ -\frac{\sin kx' \sin[k(l-x)]}{k\sin kl}, & x' \le x \le l \end{cases}$$

Note that G is singular at $kl = \frac{n\pi}{l}$, or for $f_r = \frac{n}{2l\sqrt{\mu\epsilon}}, n = 1, 2, \cdots$ where f_r is the resonant frequency.

(b) Series form. The eigenvalue problem is

$$\begin{cases} \frac{d^2\psi_n(x)}{dx^2} + k^2\psi_n(x) = 0 \\ \psi_n(0) = \psi_n(l) = 0 \end{cases}$$

Hence $\psi_n(x) = B\sin(\frac{n\pi x}{l})$ where

$$B^2 \int_0^l \sin^2(\frac{n\pi x}{l})dx = 1, \qquad B = \sqrt{\frac{2}{l}}$$

Thus

$$\psi_n(x) = \sqrt{\frac{2}{l}}\sin(\frac{n\pi x}{l})$$

$$\lambda = k^2 = \omega^2\mu\epsilon, \quad \lambda_n = k_n^2 = (\frac{n\pi}{l})^2$$

Finally

$$G(x,x') = \frac{2}{l}\sum_{n=1}^{\infty}\frac{\sin(\frac{n\pi}{l})x'\sin(\frac{n\pi}{l}x)}{k^2 - (\frac{n\pi}{l})^2}$$

Again the singularities occur at $k = \frac{n\pi}{l}$ or $f_r = \frac{n}{2l\sqrt{\mu\epsilon}}$.

9.2.3 Green's Function in Integral Form

This is a special case when the domain of the Green's function tends to infinity, i.e., a or/and b approaches infinity. In that case, the eigenvalue λ_n becomes continuous. Then the series representation (9.48) becomes an integral. We discuss this with an example.

Example. A system is described as

$$\begin{cases} \frac{d^2\phi}{dx^2} + k_0^2\phi(x) = f(x), & -\infty < x < \infty \\ \phi(x) \sim e^{-jk_0|x|}, & \text{for } |x| \to \infty \end{cases} \tag{9.51}$$

where the radiation is appropriate for this problem. The Green's function is defined as

$$\frac{d^2G(x,x')}{dx^2} + k_0^2G(x,x') = \delta(x-x'), \quad -\infty < x < \infty \tag{9.52}$$

with the same radiation condition. We take a different approach than the textbook to find this Green's function. Let

$$G(x, x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k, x') e^{-jkx} dk \tag{9.53}$$

Using (9.53) and

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-jk(x-x')} \tag{9.54}$$

in (15.42), we have

$$g(k, x') = \frac{1}{\sqrt{2\pi}} \frac{e^{jkx'}}{(k_0^2 - k^2)}. \tag{9.55}$$

Therefore, the integral form of the Green's function is

$$G(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{jkx'}}{(k_0^2 - k^2)} dk. \tag{9.56}$$

However, it turn out that (9.56) can be simplified to the closed form. Note there are poles at $k = \pm k_0$. Using residue calculus,

$$G(x, x') = \frac{-j}{2k_0} e^{-jk_0|x-x'|} \tag{9.57}$$

9.3 2-D Green's Functions

9.3.1 Static Fields in a Box

For DC problems, the PDE for the electrostatic problem is the Poisson's equation. Since $\mathbf{E} = -\nabla V$, from $\nabla \cdot \mathbf{D} = \rho$ we have

$$\nabla \cdot \epsilon \nabla V = -\rho(\mathbf{r}) \tag{9.58}$$

where ρ is the electric charge density. For a homogeneous medium, we have

$$\nabla^2 V = \frac{1}{\epsilon_0} \rho(\mathbf{r}) \tag{9.59}$$

Here we consider the 2-D problem

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = f(x, y) = -\frac{1}{\epsilon_0} \rho(x, y), \quad 0 \le x \le a, 0 \le y \le b \tag{9.60}$$

with the Dirichlet boundary conditions at $x = 0, a$ and $y = 0, b$. Now we define the Green's function

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) G(x, y; x', y') = \delta(x - x')\delta(y - y'), \quad 0 \le x \le a, 0 \le y \le b \tag{9.61}$$

with the same boundary conditions. Then by the Green's function method

$$V(x,y) = -\frac{1}{\epsilon_0}\int_0^b\int_0^a \rho(x',y')G(x,y;x',y')dx'dy' \tag{9.62}$$

(a). Closed Form

First we choose functions satisfying boundary conditions in x direction so that

$$G(x,y;x',y') = \sum_{m=1}^{\infty} g_m(y;x',y')\sin(\frac{m\pi}{a}x) \tag{9.63}$$

Substituting (9.63) into (9.61) and multiplying the equation by $\sin(\frac{n\pi}{a}x)$ and integrating over x, we have

$$\left\{\begin{array}{l} [\frac{d^2}{dy^2} - (\frac{m\pi}{a})^2]g_m(y;x',y') = \frac{2}{a}\sin(\frac{n\pi}{a}x')\delta(y-y') = C_0\delta(y-y') \\ g_m(0;x',y') = g_m(b;x',y') = 0 \end{array}\right. \tag{9.64}$$

where the orthogonality of sinusoidal functions has been used, and $C_0 = \frac{2}{a}\sin(\frac{n\pi}{a}x')$. By the principle of superposition, we can solve (9.64) for $C_0 = 1$ and then multiply the result by the actual C_0. The system in (9.64) is a 1-D problem. The homogeneous solutions are

$$\left\{\begin{array}{ll} y_1 = \sinh(\frac{m\pi}{a}y), & 0 \le y < y' \\ y_2 = \sinh(\frac{m\pi}{a}(b-y)), & y' \le y < b \end{array}\right.$$

The Wronskian is

$$\begin{aligned} W(y';x',y') &= -\frac{m\pi}{a}\Big\{\sinh(\frac{m\pi}{a}y')\cosh[\frac{m\pi}{a}(b-y')] + \cosh(\frac{m\pi}{a}y')\sinh[\frac{m\pi}{a}(b-y')]\Big\} \\ &= -\frac{m\pi}{a}\sinh(\frac{m\pi}{a}b) \end{aligned}$$

Hence,

$$g_m(0;x',y') = C_0\cdot(-\frac{a}{m\pi})\frac{\sinh(\frac{m\pi}{a}y_<)\sinh[\frac{m\pi}{a}(b-y_>)]}{\sinh(\frac{m\pi}{a}b)}$$

$$G(x,y;x',y') = -\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin(\frac{m\pi}{a}x')\sin(\frac{m\pi}{a}x)\sinh(\frac{m\pi}{a}y_<)\sinh[\frac{m\pi}{a}(b-y_>)]}{m\sinh(\frac{m\pi}{a}b)} \tag{9.65}$$

Of course, one can alternatively solve for the eigenvalue problem in y first. Then the Green's function can be written equivalently as

$$G(x,y;x',y') = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\frac{n\pi}{b}y')\sin(\frac{n\pi}{b}y)\sinh(\frac{n\pi}{b}x_<)\sinh[\frac{n\pi}{b}(a-x_>)]}{n\sinh(\frac{n\pi}{b}a)} \tag{9.66}$$

Example: Closed-form Green's function for a 2-D cylindrical shell. The system is described as

$$\begin{cases} \nabla^2 V(\rho,\phi,z) = -\frac{1}{\epsilon} q(\rho,\phi) \\ V(a,\phi) = 0 \end{cases}$$

where neither the source nor the geometry is a function of z. Therefore the problem is 2-D. The Green's function is thus defined as

$$\begin{cases} \nabla^2 G(\rho,\phi;\rho',\phi') = \delta(\boldsymbol{\rho}-\boldsymbol{\rho}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \\ G(a,\phi;\rho',\phi') = 0 \end{cases} \tag{9.67}$$

where $\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial G}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi}$. Since $\delta(\phi-\phi')$ is an even function of $(\phi-\phi')$, we can expand

$$G(\rho,\phi;\rho',\phi') = \sum_{m=0}^{\infty} g_m(\rho;\rho',\phi')\cos m(\phi-\phi') \tag{9.68}$$

Substituting (9.68) in (9.67), and using

$$\delta(\phi-\phi') = \sum_{m=0}^{\infty} \frac{1}{(1+\delta_{m0})\pi}\cos m(\phi-\phi') \tag{9.69}$$

$$[\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho}]g_m = \frac{1}{(1+\delta_{m0})\pi}\delta(\rho-\rho') \tag{9.70}$$

which is already in the SL form with

$$p(\rho)=\rho, \quad q(\rho)=\frac{m^2}{\rho}, \quad \lambda=0, \quad r(\rho)=1 \tag{9.71}$$

Note that the homogeneous solutions to (9.70) depends on the value of m. To satisfy the boundary condition at $\rho=a$ and the boundedness at $\rho=0$, we choose

$$y_1 = \rho^m, \qquad 0 \le \rho \le \rho'$$

$$y_2 = \begin{cases} \rho^m - \frac{a^{2m}}{\rho^m}, & \text{for } m \neq 0 \\ \ln(\rho/a), & \text{for } m = 0 \end{cases}, \qquad \rho' \le \rho \le a$$

Hence the Wronskian $W = y_1 y_2' - y_1' y_2$ can be calculated as

$$W_m(\rho') = \begin{cases} \frac{2ma^{2m}}{\rho'}, & \text{for } m \neq 0 \\ \frac{1}{\rho'}, & \text{for } m = 0 \end{cases}$$

The solution to (9.70) is then

$$g_m(\rho,\rho') = \frac{1}{(1+\delta_{m0})\pi}\begin{cases} \frac{1}{2ma^{2m}}\rho_<^m(\rho_>^m - \frac{a^{2m}}{\rho_>^m}), & \text{for } m \neq 0 \\ \ln(\rho_>/a), & \text{for } m = 0 \end{cases} \tag{9.72}$$

Therefore, using (9.68), the Green's function in closed form is

$$G(\rho,\phi;\rho',\phi') = \frac{1}{2\pi}\ln(\rho_>/a) + \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho_<}{a})^m[(\frac{\rho_>}{a})^m - (\frac{a}{\rho_>})^m] \tag{9.73}$$

(b). Series Form

To find the series form, we first solve the following eigenvalue problem:

$$\begin{cases} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\psi_{mn} = -\lambda_{mn}\psi_{mn} \\ \psi_{mn}(0,y) = \psi_{mn}(a,y) = \psi_{mn}(x,0) = \psi_{mn}(x,b) = 0 \end{cases} \tag{9.74}$$

Using the separation of variables, $\psi_{mn}(x,y) = f(x)g(y)$, we arrive at

$$\begin{cases} (\frac{d^2}{dx^2} + p^2)f = 0 \\ (\frac{d^2}{dy^2} + q^2)g = 0 \\ \lambda_{mn} = p^2 + q^2 \end{cases} \tag{9.75}$$

Applying the boundary conditions, we then arrive at

$$\psi_{mn} = B_{mn} \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y) \tag{9.76}$$

where from the normalization we have $B_{mn} = 2/\sqrt{ab}$. Now recognizing that $\lambda = 0$ for the Sturm-Liouville problem, we finally have

$$G(x,y;x',y') = -\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(\frac{m\pi}{a}x')\sin(\frac{n\pi}{b}y')\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)}{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}. \tag{9.77}$$

Example: 2-D potential in cylindrical coordinates. The PDE is the same as

$$\begin{cases} [\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}]G(\rho,\phi;\rho',\phi') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \\ G(a,\phi;\rho',\phi') = 0 \end{cases} \tag{9.78}$$

The corresponding eigenvalue problems is

$$\begin{cases} [\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}]\psi_{mn}(\rho,\phi) = -\lambda_{mn}\psi_{mn}(\rho,\phi) \\ \psi_{mn}(a,\phi) = 0 \end{cases} \tag{9.79}$$

Using the separation of variables, $\psi_{mn} = f(\rho)g(\phi)$, we have

$$\begin{cases} \rho f'' + \rho f' + (\lambda_{mn}\rho^2 - m^2)f = 0 \\ \frac{d^2g}{d\phi^2} = -m^2 g \end{cases} \tag{9.80}$$

Considering the boundary conditions at $\rho = 0$ and $\rho = a$, as well as the symmetric condition of $\delta(\phi - \phi')$, we have $\lambda_{mn} = \chi_{mn}^2/a^2$ (where χ_{mn} is the n-th zero of $J_m(x)$, and the solutions

$$f(\rho) = AJ_m(\frac{\chi_{mn}}{a}\rho), \quad g(\phi) = C\cos(\phi - \phi')$$

Hence,

$$\psi_{mn} = A_{mn}J_m(\sqrt{\lambda_{mn}}\rho)\cos(\phi - \phi').$$

Using the integrals

$$I_1 = \int_0^a \rho J_m(\chi_{mn}\rho/a) J_p(\chi_{pq}/a) d\rho = \frac{a^2}{2}[J'_m(\chi_{mn})]^2 \delta_{nq}$$

$$I_2 = \int_0^{2\pi} \cos m(\phi - \phi') \cos p(\phi - \phi') d\phi = (1 + \delta_{m0})\pi\delta_{mp}$$

and the requirement that ψ_{mn} be orthonormal, we have

$$A_{mn} = \frac{1}{\sqrt{(1+\delta_{m0})\pi/2}aJ'_m(\chi_{mn})}. \tag{9.81}$$

Hence

$$\psi_{mn} = \frac{1}{\sqrt{(1+\delta_{m0})\pi/2}aJ'_m(\chi_{mn})} J_m(\chi_{mn}\rho/a) \cos m(\phi - \phi') \tag{9.82}$$

$$G(\rho, \phi; \rho', \phi') = -\frac{J_0(\chi_{0n}\rho'/a)J_0(\chi_{0n}\rho/a)}{\pi a \chi_{0n}[J'_0(\chi_{0n})]^2} - 2\sum_{m=1}^{\infty} \frac{J_m(\chi_{mn}\rho'/a)J_m(\chi_{mn}\rho/a)}{\pi a \chi_{mn}[J'_m(\chi_{mn})]^2} \cos m(\phi - \phi'). \tag{9.83}$$

9.3.2 Time-Harmonic Fields

For TM^z waves, one can derive a system

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2)E_z = j\omega\mu J_z(x, y)4 \quad 0 \le x \le a, 0 \le y \le b \tag{9.84}$$

and a Dirichlet boundary condition is assume at $x = 0, a$ and $y = 0, b$. Here let's consider the Green's function in series form.

The eigenvalue problem is

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_{mn}^2)\psi_{mn} = 0, \qquad 0 \le x \le a, 0 \le y \le b \tag{9.85}$$

with the Dirichlet boundary condition. In the above $\lambda = k^2$, and $\lambda_{mn} = k_{mn}^2$. Again using the separation of variables and orthonormality we have

$$\psi_{mn} = \frac{2}{\sqrt{ab}} \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi}{b}y). \tag{9.86}$$

Then

$$G(x, y; x', y') = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(\frac{m\pi}{a}x') \sin(\frac{n\pi}{b}y') \sin(\frac{m\pi}{a}x) \sin(\frac{n\pi}{b}y)}{k^2 - [(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]} \tag{9.87}$$

$$E_z(x,y) = j\omega\mu \int_0^b \int_0^a G(x,y;x',y')J_z(x',y')dx'dy' \tag{9.88}$$

Note that at the resonant frequency $k = k_r = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$ the Green's function does not exist. This arise because of the nonuniqueness of the resonance problem.

Example: 2-D polar coordinates. Again we consider the system in polar (ρ, ϕ) coordinates

$$(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_0^2)E_z = j\omega\mu J_z(\rho,\phi)$$

with the Dirichlet boundary condition at $\rho = a$. The corresponding Green's function is defined as

$$(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_0^2)G(\rho,\phi;\rho',\phi') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \tag{9.89}$$

with the same boundary condition. Then

$$E_z(\rho,\phi) = j\omega\mu \int_S J_z(\rho',\phi')G(\rho,\phi;\rho',\phi')\rho' d\rho' d\phi' \tag{9.90}$$

In the closed form, we express

$$G(\rho,\phi;\rho',\phi') = \sum_{m=0}^{\infty} g_m(\rho;\rho',\phi')\cos m(\phi-\phi').$$

By using this and the delta function in (9.69) in (9.90), we have

$$[\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + k_0^2\rho - \frac{m^2}{\rho})]g_m = \frac{1}{(1+\delta_{m0})\pi}\delta(\rho-\rho')$$

which is in the standard SL form with

$$p(\rho) = \rho, \quad q(\rho) = \frac{m^2}{\rho}, \quad r(\rho) = \rho, \quad \lambda = k_0^2$$

The two solutions to the the homogeneous differential equations satisfying boundary conditions are

$$y_1 = J_m(k_0\rho), \quad 0 \le \rho \le \rho'$$

$$y_2 = J_m(k_0\rho) - \frac{J_m(k_0 a)}{Y_m(k_0 a)}Y_m(k_0\rho), \quad \rho' \le \rho \le a$$

Using the Wronskian of J_m and Y_m

$$J_m(x)Y_m'(x) - Y_m(x)J_m'(x) = \frac{2}{\pi x}$$

we have

$$W(\rho') = -\frac{2}{\pi\rho}\frac{J_m(k_0a)}{Y_m(k_0a)}$$

Hence

$$g_m = -\frac{1}{2(1+\delta_{m0})}\frac{J_m(k_0\rho_<)}{J_m(k_0a)}[J_m(k_0\rho_>)Y_m(k_0a) - J_m(k_0a)Y_m(k_0\rho_>)$$

$$\begin{aligned} G(\rho,\phi;\rho',\phi') = -\frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{(1+\delta_{m0})}\frac{J_m(k_0\rho_<)}{J_m(k_0a)} \\ [J_m(k_0\rho_>)Y_m(k_0a) - J_m(k_0a)Y_m(k_0\rho_>)]\cos m(\phi-\phi'). \quad (9.91)\end{aligned}$$

Note that the Green's function is singular at $J_m(k_0a) = 0$ which is the resonant frequencies.

Example 2: Green's function for a line source in 2-D polar coordinates for a open domain. The PDE is

$$[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2]G(\rho,\phi;\rho',\phi') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \qquad (9.92)$$

where the Green's function $G(a,\phi;\rho',\phi')$ satisfies the radiation boundary condition at $\rho\to\infty$. Similar to the static problem, we expand

$$G = \sum_{m=0}^{\infty} g_m(\rho;\rho',\phi')\cos m(\phi-\phi') \qquad (9.93)$$

Then (9.92) gives

$$\frac{d}{d\rho}\rho\frac{dg_m}{d\rho} + (k^2\rho - \frac{m^2}{\rho})g_m = \frac{1}{(1+\delta_{m0})\pi}\delta(\rho-\rho') \qquad (9.94)$$

which has the SL form with $p=\rho$, $q = m^2/\rho$, $r=\rho$, $\lambda = k^2$. Because of the radiation condition, we have

$$\begin{cases} y_1 &= J_m(k\rho), \qquad \rho\le\rho' \\ y_2 &= H_m^{(2)}(k\rho), \qquad \rho\ge\rho' \end{cases} \qquad (9.95)$$

Then

$$W(\rho') = -jk[J_m(k\rho')H_m^{(2)\prime}(k\rho') - J_m'(k\rho')H_m^{(2)}(k\rho')] = -\frac{2j}{\pi\rho'}$$

Therefore,

$$g_m = \frac{j}{2(1+\delta_{m0})}J_m(k\rho_<)H_m^{(2)}(k\rho_>) \qquad (9.96)$$

$$G(\rho,\phi;\rho',\phi') = \sum_{m=0}^{\infty}\frac{j}{2(1+\delta_{m0})}J_m(k\rho_<)H_m^{(2)}(k\rho_>)\cos m(\phi-\phi')$$

However, by the addition theorem, we get

$$G(\rho,\phi;\rho',\phi') = \frac{j}{4}H_0^{(2)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|) \qquad (9.97)$$

9.4 3-D Scalar Green's Functions

9.4.1 Cartesian Coordinates

We try to solve a rectangular waveguide problem with the EM fields excited by a $\hat{y}$ direction probe located at $y = b$. From Chapter 3 we know that **E** field is governed by the following PDE

$$(\nabla^2 + k^2)\mathbf{E} = \nabla \times \mathbf{M}_i + j\omega\mu\mathbf{J}_i + \frac{\nabla\rho_{ei}}{\tilde{\epsilon}}. \tag{3.22}$$

Therefore, if $\rho_{ei} = 0$, i.e., $\nabla \cdot \mathbf{J} = 0$, and $\mathbf{M}_i = 0$, the PDE for the E_y component is

$$\nabla^2 E_y + k^2 E_y = j\omega\mu J_y \tag{9.98}$$

with the tangential components of the electric field being zero on the waveguide walls at $x = 0, a$ and $y = 0, b$. At $x = 0$ and $x = a$, these B.C.s' are

$$E_y = 0, \qquad \text{for } x = 0, a \tag{9.99}$$

Because of $\nabla \cdot \mathbf{E} = 0$ and $E_x = E_z = 0$ at $y = 0, b$, we have

$$\frac{\partial E_y}{\partial y} = -(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z}) = 0, \qquad \text{for } y = 0, b \tag{9.100}$$

Therefore, the Green's function is defined as

$$\begin{cases} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})G(x, y, z; x'y', z') = \delta(x - x')\delta(y - y')\delta(z - z') \\ G = 0, \qquad \text{at } x = 0, a \\ \frac{\partial G}{\partial y} = 0, \quad \text{at } y = 0, b \end{cases} \tag{9.101}$$

It is obvious that this problem cannot be written in the series form as the dimension in z is infinite.

(a). Closed Form

We first use Fourier series to reduce the 3-D problem into a 1-D problem

$$G(x, y, z; x', y', z') = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} g_{mn}(z; x', y', z') \sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y) \tag{9.102}$$

Noting that

$$\delta(x - x') = \frac{2}{a}\sum_{m=1}^{\infty} \sin(\frac{m\pi}{a}x')\sin(\frac{m\pi}{a}x) \tag{9.103}$$

$$\delta(y - y') = \frac{2}{b}\sum_{n=0}^{\infty} \frac{1}{\epsilon_n}\cos(\frac{n\pi}{b}y')\cos(\frac{n\pi}{b}y) \tag{9.104}$$

where $\epsilon_n = 1 + \delta_{n0}$, we have

$$(\frac{d^2}{dz^2}+k_z^2)g_{mn} = \frac{4}{ab\epsilon_n}\sin(\frac{m\pi}{a}x')\cos(\frac{n\pi}{b}y')\delta(z-z'), \qquad k_z^2 = k^2-(\frac{m\pi}{a})^2-(\frac{n\pi}{b})^2 \tag{9.105}$$

This has a closed solution of

$$g_{mn} = \frac{j2}{abk_z\epsilon_n}\sin(\frac{m\pi}{a}x')\cos(\frac{n\pi}{b}y')e^{-jk_z|z-z'|} \tag{9.106}$$

Hence

$$G(x,y,z;x',y',z') = \frac{j2}{ab}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\epsilon_n}\frac{\sin(\frac{m\pi}{a}x')\cos(\frac{n\pi}{b}y')}{k_z}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-jk_z|z-z'|} \tag{9.107}$$

Note that at the resonant frequency $k_z = 0$, the Green's function is singular.

9.4.2 3-D Cylindrical Coordinates

We consider a TM^z wave

$$\begin{cases} (\nabla^2 + k^2)E_z = j\omega\mu J_z(\rho,\phi,z) \\ E_z = 0, \qquad \text{for } \rho = a \end{cases} \tag{9.108}$$

Note that

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

We can derive the Green's function G in closed form as

$$G(\rho,\phi,z;\rho',\phi',z') = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}g_{mn}(z;z')J_m(\frac{\chi_{mn}}{a}\rho')J_m(\frac{\chi_{mn}}{a}\rho)\cos m(\phi-\phi') \tag{9.109}$$

where χ_{mn} is the zeros of $J_m(x)$. Noting that

$$\delta(\phi-\phi') = \sum_{m=0}^{\infty}\frac{1}{\pi\epsilon_m}\cos m(\phi-\phi') \tag{9.110}$$

$$\frac{1}{\rho}\delta(\rho-\rho') = \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_m(\frac{\chi_{mn}}{a}\rho')J_m(\frac{\chi_{mn}}{a}\rho)}{[J_m'(\chi_{mn}]^2} \tag{9.111}$$

we have

$$(\frac{d^2}{dz^2}+k_z^2)g_{mn} = \frac{2}{\pi\epsilon_m a^2}\frac{\delta(z-z')}{[J_m'(\chi_{mn})]^2}, \quad k_z^2 = k^2-(\frac{\chi_{mn}}{a})^2 \tag{9.112}$$

Hence,

$$g_{mn} = \frac{j}{\pi \epsilon_m a^2 k_z [J'_m(\chi_{mn}]^2} e^{-jk_z|z-z'|} \tag{9.113}$$

$$G(\rho, \phi, z; \rho', \phi, z') = \frac{j}{\pi a^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{J_m(\frac{\chi_{mn}}{a}\rho') J_m(\frac{\chi_{mn}}{a}\rho)}{\epsilon_m k_z [J'_m(\chi_{mn}]^2} \cos m(\phi - \phi') e^{-jk_z|z-z'|} \tag{9.114}$$

Note that the Green's function is singular at $k_z = 0$, i.e., the resonant frequencies of the waveguide.

9.4.3 Spherical Coordinates

The scalar Green's function in spherical coordinates is defined as

$$(\nabla^2 + k_0^2) G(r, \theta, \phi; r', \theta', \phi') = \delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2 \sin\theta} \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi') \tag{9.115}$$

and satisfies the Dirichlet boundary condition at $r = a$

$$G(a, \theta, \phi; r', \theta', \phi') = 0 \tag{9.116}$$

Note that in spherical coordinates

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{r^2 \sin\theta} \frac{\partial^2}{\partial\phi^2} \tag{9.117}$$

The Green's function can be written in terms of the tesseral harmonics

$$G(r, \theta, \phi; r', \theta', \phi') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_{mn}(r; r', \theta', \phi') T^*_{mn}(\theta', \phi') T_{mn}(\theta, \phi) \tag{9.118}$$

where the tesseral harmonics are given as

$$\begin{cases} T_{mn}(\theta, \phi) & = C_{mn} P_n^m(\cos\theta) e^{jm\phi} \\ T^*_{mn}(\theta', \phi') & = C_{mn} P_n^m(\cos\theta') e^{-jm\phi'} \end{cases} \tag{9.119}$$

$$C_{mn} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \tag{9.120}$$

and $P_n^m(\cos\theta)$ are the associated Legendre polynomials. The tesseral harmonics satisfy

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial T_{mn}}{\partial\theta} + [n(n+1) - \frac{m^2}{\sin^2\theta}] T_{mn} = 0 \tag{9.121}$$

The tesseral harmonics have the orthonormal relation

$$\int_0^{2\pi} \int_0^{\pi} T_{mn}(\theta, \phi) T^*_{pq}(\theta, \phi) \sin\theta d\theta d\phi = \delta_{mp} \delta_{nq} \tag{9.122}$$

Note that we can write

$$\frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} T^*_{mn}(\theta',\phi')T_{mn}(\theta,\phi) \tag{9.123}$$

Then substituting (9.118) and (9.123) into (9.115) yields

$$\frac{\partial}{\partial r}r^2\frac{\partial g_{mn}}{\partial r} + [k_0^2r^2 - n(n+1)]g_{mn} = \delta(r-r') \tag{9.124}$$

which is already in the SL form with $p = r^2$, $q = -n(n+1)$, $r = r^2$, $\lambda = k^2$. Recognizing that (9.124) is the spherical Bessel differential equation, we can write down two solutions to the homogeneous equation

$$\begin{cases} y_1 = j_n(k_0r), & 0 \le r \le r' \\ y_2 = j_n(k_0r) - \frac{j_n(k_0a)}{y_n(k_0a)}y_n(k_0r), & r' \le r \le a \end{cases}$$

Then Wronskian is

$$W(r') = k_0\frac{j_n(k_0a)}{y_n(k_0a)}[j_n'(k_0r')y_n(k_0r') - j(k_0r')y_n'(k_0r')] = -\frac{1}{r'^2}\frac{j_n(k_0a)}{y_n(k_0a)}$$

Then

$$g_{mn} = -k_0\frac{j_n(k_0r_<)}{j_n(k_0a)}[j_n(k_0r_>)y_n(k_0a) - j(k_0a)y_n(k_0r_>)]$$

Hence

$$\begin{aligned} G(r,\theta,\phi;r',\theta',\phi') = \; & -k_0\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{j_n(k_0r_<)}{j_n(k_0a)}[j_n(k_0r_>)y_n(k_0a) - j(k_0a)y_n(k_0r_>)] \\ & \cdot T^*_{mn}(\theta',\phi')T_{mn}(\theta,\phi) \end{aligned} \tag{9.125}$$

Since

$$T^*_{pq}(\theta,\phi) = (-1)^pT_{(-p)q}(\theta,\phi) \tag{9.126}$$

we have

$$\begin{aligned} G(r, \; & \theta,\phi;r',\theta',\phi') = -k_0\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^mC^2_{mn}\frac{j_n(k_0r_<)}{j_n(k_0a)} \\ & \cdot[j_n(k_0r_>)y_n(k_0a) - j(k_0a)y_n(k_0r_>)]P_n^m(\cos\theta)P_n^{-m}(\cos\theta')e^{jm(\phi-\phi')} \end{aligned} \tag{9.127}$$

9.5 Dyadic Green's Functions

9.5.1 Dyadics

(a) Dyad

A dyad is the juxtaposition of two vectors $\overline{\mathbf{d}} = \mathbf{ab}$ defined by

$$\overline{\mathbf{d}}\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \tag{9.128}$$

$$\mathbf{c} \cdot \overline{\mathbf{d}} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \tag{9.129}$$

Or in matrix form,

$$\overline{\mathbf{d}} = \mathbf{ab} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix} \tag{9.130}$$

or $d_{ij} = a_i b_j \ (i, j = x, y, z)$.

(b) Dyadic

A dyadic is a linear combination of dyads

$$\overline{\mathbf{D}} = \sum_{n=1}^{N} \mathbf{A}^n \mathbf{B}^n \tag{9.131}$$

where in general $N = 3$ since $\overline{\mathbf{D}}^n = \mathbf{A}^n \mathbf{B}^n$ is a special second rank tensor having a null space of rank 2 (since in three dimensions there exists a two-dimensional plane orthogonal to $\mathbf{B}^n$). In general we can write

$$\overline{\mathbf{D}} = \hat{x}\mathbf{D}_1 + \hat{y}\mathbf{D}_2 + \hat{z}\mathbf{D}_3 = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \\ \mathbf{D}_3 \end{bmatrix} \tag{9.132}$$

where for $i = 1, 2, 3$

$$\mathbf{D}_i = \hat{x} D_{i1} + \hat{y} \mathbf{D}_{i2} + \hat{z} \mathbf{D}_{i3} \tag{9.133}$$

(c) Unit dyadic

A unit dyadic $\overline{\mathbf{I}}$ is defined as $\overline{\mathbf{I}} \cdot \mathbf{c} = \mathbf{c} \cdot \overline{\mathbf{I}} = \mathbf{c}$. Obviously

$$\overline{\mathbf{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{9.134}$$

(d) Differentiation

$$\nabla \cdot \overline{\mathbf{I}} g(\mathbf{r}) = (\hat{x}\frac{\partial}{\partial x} + \hat{z}\frac{\partial}{\partial z} + \hat{z}\frac{\partial}{\partial z}) \cdot (\hat{x}\hat{x}g + \hat{y}\hat{y}g + \hat{z}\hat{z}g) = \nabla g(\mathbf{r}) \tag{9.135}$$

$$\nabla \times \nabla \times \overline{\mathbf{I}} g(\mathbf{r}) = \nabla[\nabla \cdot \overline{\mathbf{I}} g] - \overline{\mathbf{I}} \nabla^2 g = \nabla\nabla g - \overline{\mathbf{I}} \nabla^2 g \tag{9.136}$$

$$\nabla \times \nabla\nabla g = (\nabla \times \nabla)\nabla g = 0 \tag{9.137}$$

If $\mathbf{a}$ is a constant vector,

$$\begin{aligned} \nabla \times [\mathbf{a} \cdot \overline{\mathbf{D}}(\mathbf{r})] &= \nabla \times [(\hat{x}a_x + \hat{y}a_y + \hat{z}a_z) \cdot (\hat{x}\mathbf{D}_1 + \hat{y}\mathbf{D}_2 + \hat{z}\mathbf{D}_3)] \\ &= \nabla \times [a_x \mathbf{D}_1 + a_y \mathbf{D}_2 + a_z \mathbf{D}_3] \\ &= [a_x \nabla \times \mathbf{D}_1 + a_y \mathbf{D}_2 + a_z \mathbf{D}_3] = \mathbf{a} \cdot (\nabla \times \overline{\mathbf{D}}) \end{aligned} \tag{9.138}$$

$$\begin{aligned} \nabla\nabla \cdot \mathbf{a} g(\mathbf{r}) &= \nabla[a_x \frac{\partial g}{\partial x} + a_y \frac{\partial g}{\partial y} + a_z \frac{\partial g}{\partial z}] \\ &= (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})[a_x \frac{\partial g}{\partial x} + a_y \frac{\partial g}{\partial y} + a_z \frac{\partial g}{\partial z}] \\ &= \mathbf{a} \cdot [\nabla\nabla g(\mathbf{r})] \end{aligned} \tag{9.139}$$

9.5.2 Dyadic Green's Function for Homogeneous Space

For a homogeneous medium we can derive the PDE from Maxwell's equations as

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = -j\omega\mu\mathbf{J} - \nabla \times \mathbf{M} \tag{9.140}$$

This is a vector equation. So we define a dyadic Green's function $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ such that

$$\mathbf{E} = \int\limits_V dv'[-j\omega\mu\mathbf{J}(\mathbf{r}') - \nabla \times \mathbf{M}(\mathbf{r}')] \cdot \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \tag{9.141}$$

However, we first write

$$(-j\omega\mu\mathbf{J} - \nabla \times \mathbf{M}) = \mathbf{E} = \int\limits_V dv'[-j\omega\mu\mathbf{J}(\mathbf{r}') - \nabla \times \mathbf{M}(\mathbf{r}')] \cdot \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \tag{9.142}$$

From (9.140)–(9.142) we have

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - k^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \tag{9.143}$$

Note that the interchange of differential operator $\nabla\times\nabla\times$ *and the volume integral has serious implications when* $\mathbf{r} \in V$. *This is related to the singularity of the dyadic Green's function, and will not be discussed further in this chapter.*

A heuristic way to derive $\overline{\mathbf{G}}$ is to let

$$\overline{\mathbf{G}} = [\overline{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla]g(\mathbf{r},\mathbf{r}') \tag{9.144}$$

Then by using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\cdot(\mathbf{a}\cdot\mathbf{b})$, we have

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \nabla \times \nabla \times [\overline{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla]g(\mathbf{r},\mathbf{r}' = (\nabla\nabla g - \overline{\mathbf{I}}\nabla^2 g) \tag{9.145}$$

Hence ((9.143) becomes

$$-\overline{\mathbf{I}}(\nabla^2 + k^2)g = \overline{\mathbf{I}}\delta(\mathbf{r},\mathbf{r}')$$

or

$$(\nabla^2 + k^2)g = -\delta(\mathbf{r},\mathbf{r}') \tag{9.146}$$

which has a solution of

$$g(\mathbf{r},\mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{9.147}$$

Finally,

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = [\overline{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla]\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{9.148}$$

An alternative derivation is to use the vector potential

$$\mathbf{E} = -j\omega\mathbf{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) \tag{9.149}$$

where

$$\mathbf{A} = \frac{\mu}{4\pi} \int\limits_V \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{R} dv' = \mu \int\limits_V \mathbf{J}(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') dv' \tag{9.150}$$

Hence,

$$\begin{aligned}
\mathbf{E} \quad &= [-j\omega\mu - \tfrac{j}{\omega\epsilon} \nabla\nabla\cdot] \int\limits_V \mathbf{J}(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') dv' \\
&= -j\omega\mu [1 + \tfrac{1}{k^2} \nabla\nabla\cdot] \int\limits_V \mathbf{J}(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') dv' \\
&= -j\omega\mu \int\limits_V \mathbf{J}(\mathbf{r}') [\overline{\mathbf{I}} + \tfrac{1}{k^2} \nabla\nabla] g(\mathbf{r}, \mathbf{r}') dv'
\end{aligned} \tag{9.151}$$

Therefore,

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = [\overline{\mathbf{I}} + \frac{1}{k^2} \nabla\nabla] g(\mathbf{r}, \mathbf{r}') \tag{9.152}$$

Again, the interchange of the differential and integral operators introduces the singularities in the dyadic Green's function.