

*Fully characterizing the set of time consistent discount functions.*

Zachary S. Brown

Nicholas School of the Environment

Duke University

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#### ABSTRACT

This paper reexamines the time consistency of discount functions as analyzed by Strotz (1956) and Koopmans (1960). Following on an observation by Heal (1998) regarding the time consistency of logarithmic discounting, I fully characterize the class of time consistent—but not necessarily autonomous—discount functions. I use my findings to explain an apparent “paradox” in the use of the gamma discounting method formulated by Weitzman (2001). One conclusion from the analysis is that a policymaker, when valuing intertemporal projects, can select a discount function which exhibits at most two of the following three properties: (1) time consistency, (2) autonomy, and/or (3) declining discount rates. The results of the paper suggest that the discounting debate be reframed to consider the costliness of time inconsistency against the benefits of employing declining discount rates and retaining autonomy. An interesting mathematical result of the paper is that time consistent discounting is equivalent to creating a metric space which is isometric to the real numbers.

## Introduction

In 1956, Strotz demonstrated that the classical “discounted utilitarian” preference structure for deterministic intertemporal planning problems yields time consistent optimal paths if and only if the rate of discount for utility streams is constant over time. In recent years there has been renewed interest and research in the formal derivation and use of alternative intertemporal preference structures, including the use of time varying discount rates. The reasons for this interest are very broad, ranging from a desire for preference structures that better describe individuals’ behavior to the policy and welfare implications of employing such alternative structures in planning for long-term social projects.

The basic purpose of this paper is to fully characterize the set of time-consistent discounting functions that can be used in the discounted utilitarian preference structure. While other authors have identified non-exponential discounting functions which are time consistent (1), as shown here, it turns out that the entire set of such functions is uncountably infinite. This result does not, as some might initially suspect, contradict Strotz’s proof. The reasons for this, as has been studied in various contexts (1, 2), stem from the differences between the *autonomy* and *time-consistency* of intertemporal plans. The exponential discounting function is the only one which yields optimal plans for infinite-horizon problems which are both autonomous and time-consistent. Thus, for instance, time consistent discount functions do not generally ensure that the Hamilton-Jacobi-Bellman partial differential equation for infinite-horizon problems is satisfied. To use this equation (e.g., in dynamic programming), exponential discounting is indeed required. This fact, as is the link between autonomy, time consistency, and constant discount rates, is intimately related to Koopman’s (3) axiomatization of the discounted utilitarian preference structure under a constant discount rate. It is in particular the stationarity assumption of his axiomitization that confines utility streams to be discounted at a constant rate.

The plan for this paper is therefore as follows: I first state my working definitions and characterize the class of time consistent discount functions, discuss why the infinite-horizon Hamilton-Jacobi-Bellman partial differential equation holds only for constant exponential discounting, and discuss precisely how this fact is linked to Koopman’s axiomatic approach. Importantly, time consistency is

derived entirely outside of an optimal control framework (or its discrete time analogue). I then apply the primary result of this paper to a class of discount functions first proposed by Weitzman (4), and which has been the topic of much debate and apparent confusion in the environmental economics literature (5, 6). I demonstrate, using my main result, that such discount functions are all time *inconsistent* and, more importantly, how this relates to the apparent confusion in the literature. I conclude with a discussion of some aspects of the problem that I do not address formally. I should make clear that I am drawing almost entirely on a number of results from the authors I cite, aiming merely to consolidate some of these results in a coherent and novel framework using the distinct concepts of time consistency and autonomy. However, I have not seen elsewhere the full characterization and equivalence theorems of the class of time consistent discount functions, especially without the use of optimal control theory.

### **Optimal Timing and the Class of Time Consistent Discount Factors**

I conduct the analysis in a continuous time framework, and use the notation  $f_i$  to denote the partial derivative of a function  $f$  with respect to its  $i^{\text{th}}$  argument. I define discount functions as follows:

*Definition*— A function  $\Delta: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is a *discount function* if it is continuously differentiable and:

- i.  $\Delta_1 > 0$  and  $\Delta_2 < 0$  (impatience)
- ii.  $\Delta(t, \tau) = 1 \forall t \in \mathbb{R}_+$  (normalize by the present) ■

Thus, for a given  $\Delta$  and two dates  $t < \tau$ , the value  $\Delta(t, \tau)$  indicates how much a given amount of utility attained at date  $\tau$  is worth at date  $t$ . I call the date  $t$  the *present date* and  $\tau$  the *valuation date*. Condition i is a *partially symmetric* condition, in that utility at a given valuation date is worth more the closer the present date is to it ( $\Delta_1 > 0$ ), while, given a present date, the weight placed on future utility decreases the farther in the future the valuation date is ( $\Delta_2 < 0$ ). A *fully symmetric* discount factor would further satisfy:

- iii.  $\Delta_1 = -\Delta_2$  (full symmetry)

Condition iii says that moving the present date into the past has precisely the same effect as moving the valuation date into the future. This condition is a simple first-order partial differential equation which

when solved implies that  $\Delta(t, \tau) = v(t - \tau)$  for some (strictly increasing) function  $v$ . That is, we can interpret condition iii as a requirement that we measure time in a perfectly linear fashion, as most dynamic economic analyses do. A central point of this paper, however, is that we can consistently measure time in an infinite number of other ways.

There are a number of ways to understand what it means to measure time consistently. At a basic level, it should mean that, in and of itself, the passage of time holds no information content, and that only “real” changes in states of the world should impact our perceptions—values and beliefs—of it. This, of course, reveals one of the central difficulties in evaluating positively whether some observed behavior is time consistent: Is a decision maker’s apparently inconsistent behavior instead the result of a perceived state change about which the observer remains unaware? In a normative context—one in which we effectively endow a model decision maker with values and beliefs—it is easier to rigorously specify time consistent behavior.

The standard treatment of the time consistency problem is thus usually presented in an optimal control framework. This is the case in Strotz (7) and in Caputo (8). Such a framework will be addressed further in more detail below, but a more fundamental understanding of time consistency can be gleaned from examination of a much more basic problem in intertemporal choice: one of optimal timing. Here are some working definitions:

*Definition*— Given an *information set*  $I$  belonging to a space  $X$ , a *current value function*  $V: \mathbb{R}^+ \times X \rightarrow \mathbb{R}$ , a present date  $t \geq 0$ , and a discount function  $\Delta$ , then the *optimal timing problem*  $OT[t, V, \Delta, I]$  is defined as:

$$OT[t, V, \Delta, I] \equiv \inf \arg \max_{\tau \geq 0} \Delta(t, \tau) V(\tau, I)$$

■

Note that the inf operator only serves to keep  $OT[t, V, \Delta, I]$  single-valued, which simplifies the rest of the analysis. I am also not concerned here with conditions under which  $OT[t, V, \Delta, I]$  exists; this is straightforward to analyze, but not relevant here. Examples of this general type of problem—or problems that can often be re-specified in this form—arise frequently in economics and finance, and include finding the optimal timing of developing a parcel of land (9-11), deciding when to sell a stock or option (12),

when to change a bus engine (13), and when to retire (14), just to name a few. Note that the general form of the specified problem allows for the possibility of uncertainty if we so desire; I do not elaborate on that possibility further here. The nature of the space  $X$  of information sets  $I$  also need not be specified further: It is included in the definition just to make explicit the difference between changing a decision because of time elapsing versus changing a decision because the information set changes. Note lastly that we usually take as granted that we can re-specify optimal timing problems as binary decision problems. I call this latter class of problems ones of *optimal stopping*, which abuses the conventional terminology in the literature; define such a problem as a decision rule mapping an optimal timing problem to the binary set:

*Definition*— Given  $I \subseteq X$ ,  $V: \mathbb{R}_+ \times X \rightarrow \mathbb{R}$ ,  $t \geq 0$ , and  $\Delta$ , then the *optimal stopping problem*  $OS[t, V, \Delta, I]$  is defined by the decision rule:

$$OS[t, V, \Delta, I] \equiv \mathbf{1}[OT[t, V, \Delta, I] \leq t]$$

where  $\mathbf{1}[\cdot]$  is the indicator function. ■

The interpretation of this definition is that if the solution of the optimal timing problem is in the past then the best response is to “stop the process,” if it has not been stopped already ( $OS[t, V, \Delta, I] = 1$ ). If the solution to the optimal timing problem remains in the future, then the best response is to allow the process to continue ( $OS[t, V, \Delta, I] = 0$ ). It turns out that whether  $OS[t, V, \Delta, I]$  is sensibly posed depends—in a manner to be made precise—on whether  $\Delta$  is time consistent. To that end, define time consistency formally as follows:

*Definition*— The discount function  $\Delta$  is *time consistent* if for any value function  $V$ , information set  $I$ , and present dates  $t', t \in \mathbb{R}^+$ , we have  $OT[t, V, \Delta, I] = OT[t', V, \Delta, I]$ , whenever  $OT[\cdot, V, \Delta, I]$  is well-defined. ■

That is,  $\Delta$  is time consistent if the solution to any well-posed optimal timing problem is the same regardless of the present date. We can now state the main result of this paper:

*Theorem*— Given a discount function  $\Delta$ , the following statements are equivalent:

- (a)  $\Delta$  is time consistent.

(b) There exists a function  $\delta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $-\frac{\Delta_2}{\Delta} \Big|_{t,\tau} = \delta(\tau) \quad \forall t, \tau \in \mathbb{R}_+$ . That is, the *marginal discount rate*  $\delta(\tau)$  does not depend on the present date  $t$ .

(c) There exists a strictly increasing function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \setminus \{0\}$  such that:

$$\Delta(t, \tau) = \frac{f(t)}{f(\tau)} \quad \forall t, \tau \in \mathbb{R}_+$$

(d)  $\Delta(t, t')\Delta(t', \tau) = \Delta(t, \tau) \quad \forall t, t', \tau \in \mathbb{R}_+$

(e)  $(\mathbb{R}_+, |\log \Delta|)$  is a *metric space* and is *isometric* to  $(A, |\cdot|)$  for some  $A \subseteq \mathbb{R}$ .

(f) If  $OT[\cdot, V, \Delta, I]$  is well-defined, then  $\inf\{s \mid OS[s, V, \Delta, I] = 1\} = OT[t, V, \Delta, I] \quad \forall t \in \mathbb{R}_+$ . ■

*Proof—*

We cycle through these equivalences as follows: First show that (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d)  $\Rightarrow$

(a). Then show that (c)  $\Leftrightarrow$  (e), and that (a)  $\Leftrightarrow$  (f)

To see that (a) implies (b), choose  $V(\tau, I) \equiv \tau(1 - \tau)$ ,  $\forall \tau \geq 0, I \subseteq X$ . Observe that because  $\Delta > 0$  and is finite, then  $\Delta V$  is finite and  $\Delta(t, \tau)V(\tau, I) > 0$  iff  $\tau \in (0, 1)$ . Because  $\Delta V$  is continuous in  $\tau$ , the Extreme Value Theorem then implies that  $\Delta V$  attains a maximum on the closed interval  $[0, 1]$ , and the previous fact implies that this maximum is in fact global and occurs in the *open* interval  $(0, 1)$ .

That is,  $OT[t, V, \Delta, I]$  exists, and  $OT[t, V, \Delta, I] \in (0, 1)$  for all  $t \in \mathbb{R}_+$  (and all  $I$ ).

For concision, suppress  $V, \Delta, I$  as arguments and define  $\tau^*(t) \equiv OT[t, V, \Delta, I]$ . Also define  $\delta(t, \tau) \equiv -\frac{\Delta_2(t, \tau)}{\Delta(t, \tau)}$ , which is well-defined since  $\Delta$  is differentiable. We wish to show that  $\delta_2 = 0$ . Since our choice of  $V$  is differentiable, then  $\tau^*(t)$  must satisfy the following first-order condition:

$$\delta(t, \tau^*) = \frac{V'(\tau^*)}{V(\tau^*)}$$

The definition of time consistency says we must have  $\frac{\partial \tau^*}{\partial t} = 0$ . As such, note that a total differentiation of the above expression (which is valid for the same reasons the first-order condition is valid) implies:

$$\delta_2 = \left(1 + \frac{\Delta_{12}}{\Delta} - \frac{\Delta_1 \Delta_2}{\Delta^2}\right) \frac{\partial \tau^*}{\partial t} = 0$$

Thus (a) implies (b).

Now assume (b) holds, and define  $G \equiv \log \Delta$ . Then we have  $G_{12} = 0$ . When combined with the partial symmetry condition (i) and the normalization condition (ii), this implies that  $G(t, \tau) = F(t) - F(\tau)$  for some strictly increasing and continuously differentiable function  $F$ . Defining  $f \equiv \exp(F)$  establishes (c). To see that (c) implies (d) follows from direct calculation.

Now let  $V$  be an arbitrary value function, with associated space  $X$  of information sets  $I$ . Let  $t, t' \in \mathbb{R}$  be two distinct present dates. Then (d) implies that  $\Delta(t, t')\Delta(t', \tau)V(\tau) = \Delta(t, \tau)V(\tau)$ . Assuming that  $OT[\cdot, V, \Delta, I]$  is well-defined, then  $OT[t, V, \Delta, I]$  by definition maximizes  $\Delta(t, \tau)V(\tau)$  with respect to  $\tau$ . Moreover, since  $OT[t', V, \Delta, I]$  maximizes  $\Delta(t', \tau)V(\tau)$  with respect to  $\tau$ , it must do the same for  $\Delta(t, t')\Delta(t', \tau)V(\tau)$ , since  $\Delta(t, t')$  is positive and fixed with respect to  $\tau$ . This establishes that (d) implies (a).

Suppose that  $(\mathbb{R}_+, |\log \Delta|)$  is isometric to  $(A, |\cdot|)$  for some  $A \subseteq \mathbb{R}$ . Then we can select an isometry  $F: \mathbb{R}_+ \rightarrow A$  such that  $\forall t, \tau \in \mathbb{R}_+$  we have  $|\log \Delta(t, \tau)| = |F(t) - F(\tau)|$ . Note that  $F$  is by definition one-to-one: Suppose first that  $F$  is strictly increasing, and define  $f \equiv \exp(F)$ . We can easily verify, using the properties of  $|\cdot|$  that  $f$  satisfies the conditions of statement (c). If  $F$  is instead strictly decreasing, define  $f \equiv \exp(-F)$ , which again establishes (c). To see that (c) implies (e), take any  $f$  satisfying (c) and first check that  $|\log \Delta|$  is indeed a metric. Then define  $F \equiv \log f$  and  $A \equiv F(\mathbb{R}_+) \subseteq \mathbb{R}$ . Showing that  $F$  is an isometry from  $(\mathbb{R}_+, |\log \Delta|)$  onto  $(A, |\cdot|)$  is straightforward.

Define  $T \equiv \{s \mid OS[s, V, \Delta, I] = 1\} = \{s \mid OT[s, V, \Delta, I] \leq s\}$  and set  $\hat{\tau} \equiv \inf T$ . Since  $\Delta$  is time consistent,  $OT[t, V, \Delta, I]$  does not vary with  $t$ , so set  $\tau^* \equiv OT[t, V, \Delta, I] \forall t$ . In particular,  $\tau^* = OT[\tau^*, V, \Delta, I]$  so that  $\tau^* \in T$ . But of course  $t < \tau^* = OT[t, V, \Delta, I]$  implies that  $t \notin T$ , so that  $\tau^* = \min T = \hat{\tau}$ . Thus (a) implies (f). To see that (f) implies (a), observe that statement (f) implies that  $OT[\cdot, V, \Delta, I]$  is time invariant whenever it is well-defined, which is the definition of time consistency. ■

Showing that (a) implies (b) relies on essentially the same logic as that used by Strotz (7). The exercise is, however, much simpler in this context, having utilized nothing related to optimal control theory or Pontryagin's Maximum Principle. When the full symmetry condition iii holds, it immediately follows

that we are back in the familiar case of constant discount rates. Perhaps the most fundamental way to interpret the above claim is through statement (d), which establishes that time consistency is simply a requirement that, on some scale, time is being measured additively (*i.e.* using the absolute value metric).

The most interesting equivalence from the above regards statement (f). This says that only with time consistent discount factors does the optimal stopping problem behave as we would expect it to: If we view the graph of the optimal stopping problem as a record of the decisions whether to continue or whether to stop at each moment in time, then statement (f) says that the first stop is the optimal one.

Lastly, note that the information set  $I$  has evidently played no role in the analysis. This is precisely because  $I$  was held constant: An arbitrary time varying information *process*  $I(t)$  would generally yield a time varying function  $OT[\cdot, V, \Delta, I(t)]$ , regardless of whether  $\Delta$  is time consistent. However, allowing for time variation but keeping  $\Delta$  time consistent, it would be interesting to examine how much could be inferred about an *implicit* information process  $I(t)$  based simply on the observed variation of the optimal time  $OT$ . This, of course, ventures into the realm of positive theory, and the overarching nature of this paper is expressly normative: If we wish, for example, to choose a discount function which is time consistent but which exhibits a strictly declining marginal discount rate, the above theorem provides us with an infinite variety of such discount functions. However, only the canonical exponential discount function yields solutions to infinite horizon optimal control problems which are time autonomous.

## Optimal Control Problems and Autonomy

Time consistency is usually discussed in the context of dynamic optimization, either in discrete or continuous time. The original analysis of time consistency by Strotz (7) is conducted in terms of a continuous time deterministic optimal control problem, and I remain in that framework for this section. I define a generic deterministic, infinite horizon optimal control problem as follows:

*Definition*— Given a discount function  $\Delta$ , an *initial state*  $x \in \mathbb{R}^n$ , a *law of motion*  $Q: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ , an *instantaneous utility function*  $u: \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ , and a present time  $t \geq 0$ , the (*current value infinite horizon*) *optimal control*  $OC(\tau, x, t, \Delta, u, Q)$  is defined as:

$$OC(\tau, x, t, \Delta, u, Q) \equiv \arg \max_{C(\cdot)} \int_t^{\infty} u(C(\tau), X(\tau)) \Delta(t, \tau) d\tau$$

such that:  $\dot{X} = Q(X(\tau), C(\tau))$  and  $X(0) = x$  ■

Additionally, if  $Z(\tau, x, t, \Delta, v, G)$  is the function that solves the ordinary differential equation  $\dot{Z} = Q(Z(t), OC(\tau, x, t, \Delta, v, Q))$  and  $Z(0) = x$ , then the *current value function* is defined as:

$$V(x, t, \Delta, u, Q) \equiv \int_t^{\infty} u(OC(\tau, x, t, \Delta, u, Q), Z(\tau, x, t, \Delta, u, Q)) \Delta(t, \tau) d\tau$$

I do not concern myself with the conditions under which  $OC(\tau, x, t, \Delta, u, Q)$  and  $V(x, t, \Delta, u, Q)$  are well-defined here; see (8) for an exhaustive discussion. Note that I do not allow the instantaneous utility function  $u$  or the law of motion  $Q$  to depend on time directly. The utility of this definition here is to distinguish between time consistent and autonomous plans.

*Definition*— A (well-defined) optimal control  $OC(\cdot, x, t, \Delta, u, Q)$  is *time consistent* if  $\forall t' \geq t \geq 0$   
 $OC(\tau, x, t, \Delta, u, Q) = OC(\tau, Z(t'), t', \Delta, u, Q)$  for all  $\tau \geq t'$ . ■

As Strotz (7) pointed out, if we impose condition (iii) and hence can write the discount function symmetrically as  $\Delta(t, \tau) \equiv v(t - \tau)$ , then the only such function that makes an arbitrary optimal control time consistent is of the following form:

*Theorem (Strotz, 1956)*— A symmetric discount function  $v$  satisfying condition (iii) yields a time consistent optimal control  $OC(\cdot, \cdot, \cdot, v, u, Q)$  for arbitrary  $u, Q$  iff  $v(t - \tau) \equiv \exp[-r(\tau - t)]$  for some  $r > 0$ . ■

If we do not impose the full symmetry condition iii on the discount function, then a weaker result obtains, which is obvious, given the previous section's results:

*Claim*— A discount function  $\Delta$  yields a time consistent optimal control  $OC(\cdot, \cdot, \cdot, \Delta, u, Q)$  for arbitrary  $u, Q$  if and only if  $\Delta$  is time consistent. ■

*Proof*— Assuming an arbitrary time consistent optimal control, one can follow Caputo (8, pp. 324) almost entirely to arrive at statement (b) of the Theorem in the previous section. The other direction of the above equivalence follows from invoking statement (c) in the above Theorem along with the Principal of Optimality for optimal control problems (e.g., 8, pp. 81). ■

Thus far no additional understanding has been gained from considering optimal control problems in our reexamination of time consistency. However, perhaps the most frequently used result in dynamic economic analysis, the Hamilton-Jacobi-Bellman (HJB) partial differential equation, is derived under the assumption that the integrand of the value function and law of motion are time autonomous (which the definition above imposes) and the discount function is time consistent. However, it is not true that any time consistent discount function, coupled with the above optimal control problem, implies that the HJB equation must hold: Among time consistent discount functions, only the standard exponential discount function satisfies this relationship.

There are multiple ways to state the HJB equation (e.g., 8, pp. 402, 513), depending on what method is being utilized. To avoid having to restate the entire Maximum Principle, I do not go into the HJB equation itself in detail here. Nevertheless, one conclusion that can be drawn from this equation is that the current value function  $V(x, t, \Delta, u, G)$  is invariant with the present time. In particular, it must be that  $V(x, 0, \Delta, u, Q) = V(x, t, \Delta, u, Q) \forall t \geq 0$  if the HJB equation is to hold. To demonstrate this, the standard “trick” is to use a change of variables. I demonstrate here what we require of a discount function for this trick to suffice generally.

To this end, let  $\Delta$  be an arbitrary discount function and change variables from  $\tau$  to  $s$ , where  $s$  satisfies  $\Delta(0, s) \equiv \Delta(t, \tau)$ . Because discount functions are strictly monotonic in each of their arguments according to the above definition, then  $s = s(t, \tau)$  is well-defined, and is an implicit function of  $t$  and  $\tau$ . Note that because the optimal control and state variables *are determined by the problem*, we can express them as functions of any one-to-one mapping of  $\tau$  and can hence write the current value function at time  $t$  as follows, suppressing all atemporal arguments:

$$V(t) \equiv \int_t^\infty u(OC(-\log \Delta(t, \tau)), Z(-\log \Delta(t, \tau))) \Delta(t, \tau) d\tau$$

The intuition for why I use the term  $(-\log \Delta(t, \tau))$  as the argument in the optimal control and statement variable is that when the discount function is time consistent, this term can, via statement (e), be properly interpreted as the distance in time. Because this term will be used further below, define  $G(t, \tau) \equiv -\log \Delta(t, \tau)$ . To change variables to  $s$ , first define  $\check{\tau}(s, t)$  implicitly by  $\Delta(0, s) \equiv \Delta(t, \check{\tau}(s, t))$ . We thus can write:

$$V(t) = \int_0^\infty u(OC(G(0, s)), Z(G(0, s))) \Delta(t, \check{\tau}(s, t)) \frac{\delta(0, s)}{\delta(t, \check{\tau}(s, t))} ds \quad (1)$$

where  $\delta \equiv -\frac{\Delta_2}{\Delta} = G_2$  is the marginal discount rate. Note that we have not yet imposed time consistency.

On the other hand, we can write  $V(0)$  as:

$$V(0) \equiv \int_0^\infty u(OC(G(0, s)), Z(G(0, s))) \Delta(0, s) ds \quad (2)$$

Comparing (1) and (2) it is clear that  $V(0) = V(t) \forall t \geq 0$  for an otherwise arbitrary specification only if  $\delta(0, s) = \delta(t, \tau) \forall t, \tau \geq 0$ . Fully differentiating this identity in both  $t$  and  $\tau$ , and substituting in for  $G$

and  $ds = \frac{[G_1|_{t,\tau} dt + G_1|_{t,\tau} d\tau]}{G_2|_{0,s}}$ , we have:

$$0 = \left( G_{21}|_{t,\tau} - \frac{G_{22}}{G_2} \Big|_{0,s} G_1|_{t,\tau} \right) dt + \left( G_{22}|_{t,\tau} - \frac{G_{22}}{G_2} \Big|_{0,s} G_2|_{t,\tau} \right) d\tau$$

First set  $dt = 0$  and  $d\tau = 1$ . Then we see that  $\frac{G_{22}}{G_2} \Big|_{0,s} = \frac{G_{22}}{G_2} \Big|_{t,\tau}$ . Now set  $dt = 1$  and  $d\tau = 0$ , and

substitute in the previous result to eliminate the variable  $s$  and thus obtain:

$$\frac{G_{21}}{G_1} = \frac{G_{22}}{G_2} \quad \Leftrightarrow \quad \Delta_1 = -k\Delta_2 \quad \text{for some } k \in \mathbb{R}$$

Solving this simple partial differential equation reveals that we must have  $\Delta(t, \tau) \equiv v(kt - \tau)$  for some strictly increasing function  $v$ . Because  $\Delta$  is a discount function (*i.e.* conditions i and ii apply), it must be that  $k = 1$ : By the normalization condition, at  $t = \tau = 0$  we have  $v(0) = 1$ . Since  $v$  is strictly increasing, then the normalization condition again implies that  $k = 1$ . Therefore, we can summarize this autonomy prerequisite as follows:

$$\Delta(t, \tau) = v(t - \tau) \quad \text{for some } v: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \setminus \{0\} \quad (3)$$

This is a startling result: Equation (3) is equivalent to the full symmetry condition (iii) above. We can hence define a discount function  $\Delta$  to be *autonomous* if it satisfies the full symmetry condition (iii), or equation (3). Because we know that the standard exponential discount function indeed satisfies the HJB equation, the above immediately establishes the following:

*Claim—* A time consistent discount function  $\Delta$  is autonomous iff  $\Delta(t, \tau) \equiv \exp[-r(\tau - t)]$  for some  $r > 0$ . ■

While this is at first-glance no different than the standard result we are familiar with in dynamic economic theory—that the only discount function of the form in (3) which is time consistent is the exponential discount function—there is a deeper interpretation here, given the above development of time consistency. Specifically, the reason the exponential discount function is so special is precisely because it is the only such function which is both time consistent *and* autonomous.

One could view this assertion simply as another (perhaps more convoluted) way of stating of Koopman's (3) axiomatization of exponential discounting. Informally, his axioms consist of continuity, independence, and stationarity. See Meyer (15) for a good summary of these assumptions and their implications. For our purposes (*i.e.* in a continuous time framework and where the objective is assumed to be time separable), the stationarity assumption takes the following form:<sup>1</sup>

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<sup>1</sup> This is my own idiosyncratic presentation of this assumption, give the previous discussion in this paper. See Koopmans [pp. 294] for the original assumption off of which I am basing this. In particular, in this paper I assume

*Definition*— A discount function  $\Delta$  is *stationary* if for any  $W: \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $\Delta$  is  $W$ -integrable we have, for all  $t > 0$ :

$$\int_t^\infty \Delta(t, \tau) dW(\tau - t) \geq 0 \quad \Leftrightarrow \quad \int_0^\infty \Delta(0, \tau) dW(\tau) \geq 0$$

■

While this property is reminiscent of autonomy, it is in fact much stronger, as it implies time consistency *and* autonomy, meaning that  $\Delta$  must be exponential: To see that stationarity implies autonomy, fix  $s > 0$  and define  $W(\tau) \equiv \mathbf{1}[\tau \in (0, s)] + [1 - \Delta(0, s)^{-1}] \mathbf{1}[\tau > s]$ . Then  $\int_0^\infty \Delta(0, \tau) dW(\tau) = 0$ , and stationarity then implies that  $0 = \int_t^\infty \Delta(t, \tau) dW(\tau - t)$ , which can be rearranged to yield  $\Delta(t, s + t) = \Delta(0, s)$  for all  $s, t > 0$ . This in turn implies that  $\Delta_1 + \Delta_2 = 0$ , which is the definition of autonomy. It is then straightforward to see that time consistency follows: By autonomy, we can write  $\Delta(t, \tau) = v(-(\tau - t))$ , and consider finding the  $u: \mathbb{R}_+ \rightarrow \mathbb{R}$  which maximizes  $\int_0^\infty v(-\tau) u(\tau) d\tau$  subject to  $u \in U$ , for some space of functions  $U$ . Any maximal  $u$  by definition satisfies  $\int_0^\infty v(-\tau) [u(\tau) - \tilde{u}(\tau)] d\tau \geq 0$  for all  $\tilde{u} \in U$ , and stationarity implies that  $\int_t^\infty v(-(\tau - t)) [u(\tau - t) - \tilde{u}(\tau - t)] d\tau \geq 0$  for all  $\tilde{u} \in U$ . Thus maximal  $u$  remain so at all future dates, implying the time consistency of  $\Delta$ .

I would argue, perhaps not so objectively, that, if we are in a situation where we assume we can characterize time preference (either normatively or positively) via discount functions, then the simultaneous satisfaction of the full symmetry and time consistency properties are what distinguish the exponential discount function from others, and that these are more fundamental concepts than stationarity: Full symmetry requires that moving the decision maker's present self backwards in time has precisely the same effect as moving the valuation date forward in time. Time consistency requires that, *on some scale*, we measure distances in time additively. One reason these properties of the exponential discount factor are more fundamental is that they are essentially divorced from optimal control theory. Time consistency was derived entirely outside of an optimal control framework, and the full symmetry condition, as

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the presence of a discounted utilitarian preference structure, leaving arbitrary only the nature of the discount function. Koopmans *generates* such a preference structure (in discrete time) from his axioms.

presented in condition iii, certainly has deeper meaning than its effect on optimal control problems. Indeed, it would be an interesting exercise to evaluate autonomy outside of an optimal control framework, similar to the above analysis of time consistency.

### **Application: Gamma Discounting and Gollier’s “Future Value”**

Before closing, I would like to briefly apply the above results concerning time consistency to a recent debate that has been ongoing in the literature on valuing long-term projects when there is uncertainty *in the discount function*. This debate was in fact the original impetus for this paper.

The seminal work on discounting under this variety of uncertainty is Weitzman (4), in which, for the standard exponential setting, the essential point is made that it is the discount *function* that we should be taking expectations over, not the discount *rate*. Weitzman, following convention, utilizes the expected net present value (ENPV) criterion for valuing projects; however, the uncertainty analyzed is coming from the discount function as opposed to the project-specific variables. This sets the stage for the “paradox” posed by Gollier (5), in which projects that are cost effective under the ENPV criterion can fail to be so under the expected net future value (ENFV) criterion, when uncertainty exists in the discount rates. A third paper, by Hepburn and Groom (6), that pursues this line of research analyzes this resulting inconsistency much as I do here, by explicitly incorporating a present date  $t$ , for which  $ENV(t)$  is defined as the expected net value of the future project when evaluated at date  $t$ . Hepburn and Groom, as well as Gollier, focus on projects with a discrete payout in the future and a cost in the present (termed “Gollier” projects). Hepburn and Groom then proceed to characterize how  $ENV(t)$  varies with  $t$ , finding, for instance, that  $ENV(t) > 0$  implies  $ENV(s) > 0$  for all  $s < t$ . This is easy enough to show in the present context, using the properties of discount functions.

What is underlying the apparent inconsistency yielded by Weitzman’s discounting method, and analyzed in detail by Gollier and Hepburn and Groom, is in the mathematical sense at least, time inconsistency. In none of the articles mentioned above does a direct acknowledgement of this exist: Weitzman leaves the time consistency issue for later research, Gollier does not, as far as I am aware,

address it directly, and Hepburn and Groom allude only in a footnote to time consistency, claiming that it is related to but distinct from the topic they are examining.

While my main point can be established here without the aid of any of the above new results, understanding why this evidently obvious fact was overlooked is aided by the equivalence theorem for time consistency. Let  $E$  denote the expectations operator with respect to the discount *rate*  $r$ , which has some known distribution. Then the general form of Weitman's discount function is  $\Delta(t, \tau) \equiv E e^{-r(\tau-t)}$ . Since this function satisfies the full symmetry condition (iii), it is easy to show that such a function is time consistent if and only if  $Vr = 0$ , where  $V$  is the variance operator. This point plays no explicit role, in particular, in Hepburn and Groom's analysis. What does play a much more obvious role is the failure of statement (d) of the above equivalence theorem for time consistency. That is, that there exist  $t, t', \tau \in \mathbb{R}_+$  with  $\Delta(t, t')\Delta(t', \tau) \neq \Delta(t, \tau)$  is what "causes all the trouble."

The identification of time consistency as the root of why the net present and future value criteria differ can thus be seen to actually enhance the relevance of Hepburn and Groom's results: Their analysis of how the expected net value varies with the present date can really be seen as a classification of the *extent* of inconsistency in at least one variety of time *inconsistent* discount function.

## **Discussion**

This paper presented a number of technical results concerning time consistency and autonomy in discounting. Most of these results have already been established in various forms elsewhere. One of my aims here was to present these previous results in a novel fashion. Another aim was to extend these results slightly by characterizing the entire set of time consistent discount factors in as many ways as possible, and to subsequently demonstrate why autonomy and time consistency are two distinct concepts. A final, related aim was to disassociate these properties of discount functions as much as possible from optimal control problems, with which such concerns have generally been associated, and to demonstrate the wider relevance of time consistency concerns in intertemporal problems. A demonstration of this

latter point can be seen in the Weitzman's gamma discounting method and the subsequent analysis by Gollier, followed by Hepburn and Groom, of its inconsistency.

The major topic left unaddressed is whether there is any practical utility from the above analysis. One central motivating interest for this paper was to find other discount functions that could be used by a planner in evaluating long-term projects. This kind of concern directly follows inquiry by Heal (1), and his informative consideration of the problem. He derives one non-exponential type of discount function, of the logarithmic variety, which is demonstrably time consistent. My analysis exhausts the possibilities of similar logic: The set of such discount functions is extremely large (roughly speaking, on the order of the space of strictly increasing univariate functions).

Of particular interest, in terms of current policy relevance, are discount functions that exhibit *declining* discount rates (16). Newell and Pizer (17) have argued that discount rates can be seen empirically to decline over long timescales, and the government of the United Kingdom apparently uses a declining discount rate schedule for some long-term project valuation (see 16). Moreover, there a number of alternative preference structures, lying outside of the standard discounted utilitarian framework, which have been formalized—some axiomatized; many of these can be shown to be equivalent to a discounted utilitarian preference structure using declining (though time inconsistent) discount rates (e.g., 18, 19).

In this framework, an interesting, well-defined choice can be presented to the policymaker who must value projects of an intertemporal nature: The above analysis makes clear that we cannot have a discount function which is time consistent, autonomous, and exhibits declining discount rates. We can, however, select discount functions with any two of these three properties. If we consider all of these properties to be desirable, then the policymaker is faced with a trade-off: For instance, how costly is time inconsistency and is it outweighed by the benefits of employing declining discount rates and retaining autonomy? Are we instead willing to lose autonomy to select a time consistent discount function exhibiting declining discount rates? These are questions of potential interest for future work.

## Bibliography

1. G. Heal, *Valuing the Future: Economic Theory and Sustainability*. (Columbia University Press, New York, 1998).
2. K. J. Arrow, P. Dasgupta, K.-G. Mäler, in *The Economics of Non-Convex Ecosystems*, P. Dasgupta, Ed. (Springer, Dordrecht, The Netherlands, 2004).
3. T. C. Koopmans, *Econometrica* **28**, 287 (1960).
4. M. L. Weitzman, *American Economic Review* **91**, 261 (2001).
5. C. Gollier, *Finance Research Letters*, (2004).
6. C. Hepburn, B. Groom, *Journal of Environmental Economics and Management* **53**, 99 (2007).
7. R. H. Strotz, *The Review of Economic Studies* **23**, 165 (1956).
8. M. R. Caputo, *Foundations of Dynamic Economic Analysis*. (Cambridge University Press, New York, 2005).
9. R. J. Arnott, F. D. Lewis, *The Journal of Political Economy* **87**, (1979).
10. D. R. Capozza, R. W. Helsey, *Journal of Urban Economics* **26**, (1989).
11. G. Turnbull, *Journal of Urban Economics* **55**, 215 (2004).
12. B. Oksendal, *Stochastic Differential Equations: Introduction with Applications*. (Springer, The Netherlands, ed. 6th edition, 2003).
13. J. Rust, in *Handbook of Econometrics*, R. F. E. D. L. McFadden, Ed. (1994), vol. 4.
14. R. L. Rumsdaine, J. H. Stock, D. A. Wise, in *NBER Working Paper Series*. (National Bureau of Economic Research, 1990).
15. R. F. Meyer, in *Decisions with Multiple Objectives: Preferences and Value Trade-Offs*, R. L. Keeney, H. Raiffa, Eds. (Cambridge University Press, New York, 1976).
16. B. Groom, C. Hepburn, P. Koundouri, D. Pearce, *Environmental and Resource Economics* **32**, 445 (2005).
17. R. Newell, W. Pizer, *Journal of Environmental Economics and Management*, (2003).
18. G. Chichilnisky, *Social Choice and Welfare* **13**, (1996).
19. C. Z. Li, K. G. Löfgren, *Journal of Environmental Economics and Management* **40**, (2000).